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ON A CUSPED DOUBLE-LAYERED PRISMATIC SHELL

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Abstract. The present paper is devoted to the system of degenerate partial differential equations that arise from the investigation of elastic two layered prismatic shells. The well-posedness of the boundary value problems (BVPs) under the reasonable boundary conditions at the cusped edge and given displacements at the non-cusped edge is studied. The classical and weak setting of the BVPs in the case of the zero approximation of hierarchical models is considered.

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Introduction

In the 1950s I. Vekua recommended to investigate cusped prismatic shells ([8], [9], [10]), i.e., plates whose thickness vanishes on some part or on the whole boundary of the shell projection. One can find a survey concerning cusped prismatic shells in [3], [4].

The present paper is devoted to the system of degenerate partial differential equations that arise from the investigation of elastic two layered prismatic shells. Using I. Vekua's dimension reduction method hierarchical models for elastic layered prismatic shells are constructed in [5]. For each layer were constructed hierarchical models assuming to be known stress vector components on the face surfaces of the layered body (structure) under consideration, while the values of X_{ij} and u_i on the interfaces are calculated from their Fourier-Legendre expansions. For the sake of simplicity we consider the case of the zeroth approximation (hierarchical model).

Let as in [5] a layered prismatic shell consist of two prismatic shells P_γ, $\gamma = 1,2$, as plies with the upper $\overset{(+)}{h}_\gamma(x_1,x_2)$, $\gamma = 1,2$, and lower $\overset{(-)}{h}_\gamma(x_1,x_2)$, $\gamma = 1,2$, face surfaces, herewith,

$$\overset{(+)}{h}_2(x_1,x_2) \equiv \overset{(-)}{h}_1(x_1,x_2), \qquad (x_1,x_2) \in \omega,$$

where ω is the common for the both prismatic shells projection on the plane $x_3 = 0$. S denotes the joined lateral cylindrical surface parallel to the x_3-axis according to the definition of prismatic shells.

Let us denote the thickness of the layered prismatic shell by

$$\begin{aligned} 2h(x_1,x_2) &:= \overset{(+)}{h}_1(x_1,x_2) - \overset{(-)}{h}_2(x_1,x_2) \\ &= \overset{(+)}{h}_1(x_1,x_2) - \overset{(-)}{h}_1(x_1,x_2) + \overset{(-)}{h}_1(x_1,x_2) - \overset{(-)}{h}_2(x_1,x_2) = 2h_1 + 2h_2 \geq 0, \end{aligned}$$

where

$$2h_\gamma(x_1,x_2) := \overset{(+)}{h}_\gamma(x_1,x_2) - \overset{(-)}{h}_\gamma(x_1,x_2) \geq 0, \quad \gamma = 1,2,$$

are the thickness of the plies.

Under well-known restrictions the following Fourier-Legendre expansions are convergent

$$\left(X_{ij}^\gamma, e_{ij}^\gamma, u_i^\gamma\right)\ (x_1,x_2,x_3,t) = \sum_{l=0}^{\infty} a_{\underline{\gamma}} \left(l+\frac{1}{2}\right)$$

$$\times \left(X_{ijl}^\gamma, e_{ijl}^\gamma, u_{il}^\gamma\right)\ (x_1,x_2,t) P_l(a_\gamma x_3 - b_\gamma) dx_3,$$

where

$$\left(X_{ijl}^\gamma, e_{ijl}^\gamma, u_{il}^\gamma\right)\ (x_1,x_2,t)$$

$$= \int_{\overset{(-)}{h}}^{\overset{(+)}{h}} \left(X_{ij}^\gamma, e_{ij}^\gamma, u_i^\gamma\right)\ (x_1,x_2,x_3,t) P_l(a_{\underline{\gamma}} x_3 - b_{\underline{\gamma}}) dx_3, \quad l = 0,1,2,...,$$

$$a_\gamma := \frac{1}{h_\gamma}, \quad b_\gamma := \frac{\tilde{h}_\gamma}{h_{\underline{\gamma}}}, \quad 2\tilde{h}_\gamma := \overset{(+)}{h}_\gamma + \overset{(-)}{h}_\gamma, \quad \gamma = 1,2.$$

A bar under one of repeated indices means that in this case we do not use Einstein's summation convention. Latin and Greek indices take values $1,2,3$ and $1,2$, correspondingly.

Let

$$X_{ij0}^\gamma(x_1,x_2,t), \ \ e_{ij0}^\gamma(x_1,x_2,t), \ \ u_{i0}^\gamma(x_1,x_2,t), \ i,j = 1,2,3, \ \gamma = 1,2,$$

be zero-th order moments of the stress X_{ij}^γ and strain e_{ij}^γ tensors and displacement vector u_i^γ components of the plies.

Problem. Determine the stress-strain state of the elastic layered prismatic shell considered as a three-dimensional (3D) body (plies and the whole body may occupy non-Lipschitz domains, see [3]), when on the face surfaces of the body stress-vectors

$$Q_{\overset{(+)}{\nu}_1 i}(x_1,x_2, \overset{(+)}{h}_1(x_1,x_2),t) \text{ and } Q_{\overset{(-)}{\nu}_2 i}(x_1,x_2, \overset{(-)}{h}_2(x_1,x_2),t) \text{ are known,}$$

where $\overset{(+)}{\nu}_1$ and $\overset{(-)}{\nu}_2$ are outward to the body normals to the upper (for the first ply) and lower (for the second ply) face surfaces ($\overset{(+)}{\nu}_2 \equiv -\overset{(-)}{\nu}_1$), on the lateral surfaces arbitrary admissible boundary conditions (BC) are prescribed, and on the interface the following conditions are fulfilled:

$$\left(X_{ji}^1 \overset{(+)}{\nu}_{2i}, u_j^1\right) \left(x_1,x_2,x_3 = \overset{(-)}{h}_1(x_1,x_2) \equiv \overset{(+)}{h}_2(x_1,x_2), t\right)$$

$$= \left(X_{ji}^2 \overset{(+)}{\nu}_{2i}, u_j^2\right) \left(x_1,x_2,x_3 = \overset{(+)}{h}_2(x_1,x_2) \equiv \overset{(-)}{h}_1(x_1,x_2), t\right), \ j = 1,2,3.$$

For clearness, on the lateral surface $S = S_1 \bigcup S_2$ we confine ourselves to boundary conditions (BCs) in displacements, i.e., for each lateral boundary of plies

$$u_i^\gamma \Big|_{S_{\underline{\gamma}}}, \quad \gamma = 1, 2, \ \text{ are prescribed.} \tag{1}$$

The system of equations for the displacement in the first fly looks like [5]

$$\begin{aligned} &\mu \left[\left(h_1 v^1_{\alpha 0,\beta} \right)_{,\beta} + \left(h_1 v^1_{\beta 0,\alpha} \right)_{,\beta} \right] + \lambda \left(h_1 v^1_{\beta 0,\beta} \right)_{,\alpha} \\ &+\tfrac{1}{2} \left\{ \overset{(-)}{h}_{1,\beta} \left[\lambda v^1_{\gamma,\gamma} \delta_{\alpha\beta} + \mu \left(v^1_{\alpha 0,\beta} + v^1_{\beta 0,\alpha} \right) \right] - \mu v^1_{30,\alpha} \right\} \\ &+ Q_{\overset{(+)}{\nu_1}\alpha} \sqrt{\left(\overset{(+)}{h}_{1,1} \right)^2 + \left(\overset{(+)}{h}_{1,2} \right)^2 + 1} + \Phi^1_{\alpha 0} = 0, \quad \alpha = 1, 2; \end{aligned} \tag{2}$$

$$\begin{aligned} &\mu \left(h_1 v^1_{30,\beta} \right)_{,\beta} + \frac{1}{2} \left(\mu \overset{(-)}{h}_{1,\beta} v^1_{30,\beta} - \lambda v^1_{\beta 0,\beta} \right) \\ &+ Q_{\overset{(+)}{\nu_1}3} \sqrt{\left(\overset{(+)}{h}_{1,1} \right)^2 + \left(\overset{(+)}{h}_{1,2} \right)^2 + 1} + \Phi^1_{30} = 0, \end{aligned} \tag{3}$$

while the system of equations for the displacement in the second fly can be written as follows

$$\begin{aligned} &\mu \left[\left(h_2 v^2_{\alpha 0,\beta} \right)_{,\beta} + \left(h_2 v^2_{\beta 0,\alpha} \right)_{,\beta} \right] + \lambda \left(h_2 v^2_{\beta 0,\beta} \right)_{,\alpha} \\ &+ \frac{1}{2h_1} Q^1_{\overset{(+)}{\nu_2}\alpha} \sqrt{\left(\overset{(+)}{h}_{2,1} \right)^2 + \left(\overset{(+)}{h}_{2,2} \right)^2 + 1} \\ &+ Q_{\overset{(-)}{\nu_2}\alpha} \sqrt{\left(\overset{(-)}{h}_{2,1} \right)^2 + \left(\overset{(-)}{h}_{2,2} \right)^2 + 1} + \Phi^2_{\alpha 0} = 0, \quad \alpha = 1, 2; \end{aligned} \tag{4}$$

$$\begin{aligned} &\mu \left(h_2 v^2_{30,\beta} \right)_{,\beta} + \frac{1}{2h_1} Q^1_{\overset{(+)}{\nu_2}3} \sqrt{\left(\overset{(+)}{h}_{2,1} \right)^2 + \left(\overset{(+)}{h}_{2,2} \right)^2 + 1} \\ &+ Q_{\overset{(-)}{\nu_2}3} \sqrt{\left(\overset{(-)}{h}_{2,1} \right)^2 + \left(\overset{(-)}{h}_{2,2} \right)^2 + 1} + \Phi^2_{30} = 0, \end{aligned} \tag{5}$$

where by Φ^γ_{j0}, $j = 1, 2, 3$, $\gamma = 1, 2$ we denote zero moments of the volume forces acting on each fly,

$$v^\gamma_{j0} := \frac{u^\gamma_{j0}}{h_\gamma}, \quad j = 1, 2, 3.$$

Problem (4), (5), (1) coincide with the BVP for zero approximation of Vekuas hierarchical models for a single layered prismatic shells and is studied in [1].

In the next section problem (2), (3), (1) is considered.

2. Variational formulation of the problem (2), (3), (1)

Let the thickness of the first fly be

$$\overset{(-)}{h}_1 = 0, \quad \overset{(+)}{h}_1 = h_0 x_2^{\kappa}, \quad h_0, \kappa = const, \quad h_0 > 0, \quad \kappa \geq 0.$$

Denoting by $L^{(0)}(x, \partial)$ the 3×3 matrix differential operator generated by the left-hand side expressions of system (2), (3). We can rewrite (2), (3) in the following vector form

$$L^1(x, \partial) v^1(x) = F^1(x), \quad x \in \omega, \tag{6}$$

where

$$L^1(x, \partial) := \begin{Vmatrix} L_{11} & L_{12} & L_{13} \\ L_{21} & L_{22} & L_{23} \\ L_{31} & L_{32} & L_{33} \end{Vmatrix},$$

$$L_{11} := -h_1(2\mu + \lambda)\frac{\partial^2}{\partial x_1^2} - h_1\mu\frac{\partial^2}{\partial x_2^2} - h_{1,2}\,\mu\frac{\partial}{\partial x_2},$$

$$L_{12} := -h_1(\mu + \lambda)\frac{\partial^2}{\partial x_1 \partial x_2} - h_{1,2}\,\mu\frac{\partial}{\partial x_1}, \quad L_{13} = \frac{1}{2}\mu\frac{\partial}{\partial x_1},$$

$$L_{21} := -h_1(\mu + \lambda)\frac{\partial^2}{\partial x_1 \partial x_2} - h_{1,2}\,\mu\frac{\partial}{\partial x_1},$$

$$L_{22} := -h_(2\mu + \lambda)\frac{\partial^2}{\partial x_2^2} - h_\mu\frac{\partial^2}{\partial x_1^2} - h_{1,2}\,(\mu + \lambda)\frac{\partial}{\partial x_2},$$

$$L_{23} = \frac{1}{2}\mu\frac{\partial}{\partial x_2}, \quad L_{31} = \frac{1}{2}\lambda\frac{\partial}{\partial x_1}, \quad L_{32} = \frac{1}{2}\lambda\frac{\partial}{\partial x_2},$$

$$L_{33} := -h_1\mu(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}) + h_{1,2}\,\mu\frac{\partial}{\partial x_2},$$

$$F_j^1 := Q_{\overset{(+)}{\nu_1} j}\sqrt{\left(\overset{(+)}{h}_{1,2}\right)^2 + 1} + \Phi_{j0}^1, \quad j = 1, 2, 3,$$

$$v^1 := (v_{10}^1, v_{20}^1, v_{30}^1)^\top,$$

the symbol $(\cdot)^\top$ means transposition.

Let

$$v^1, v^* \in C^2(\omega) \cap C^1(\overline{\omega}), \quad v^* := (v_{10}^*, v_{20}^*, v_{30}^*)^\top,$$

where v^1 and v^* are arbitrary vectors of the above class. We obtain the following Green's formula

$$\int_\omega L^1 v^1 \cdot v^* d\omega = B^1(v^1; v^*) - \int_{\partial\omega} T_n^1 v^1 \cdot v^* d\partial\omega = \int_\omega F^1 \cdot v^* d\omega. \tag{7}$$

Here and in what follows the $\cdot$ denotes the scalar product of two vectors, $n := (n_1, n_2)$ is the inward normal to $\partial\omega$:

$$T^1_n = \Big\{X^1_{n10}, X^1_{n20}, X^1_{n30}\Big\},$$

with

$$X^1_{\alpha\beta 0} = \lambda h_1 v^1_{\delta 0,\delta}\delta_{\alpha\beta} + \mu h_1 (v^1_{\alpha 0,\beta} + v^1_{\beta 0,\alpha}),\;\; X^1_{3\beta 0} = \mu h_1 v^1_{30,\beta},\;\; X^1_{33} = \lambda b_1 v^1_{\beta 0,\beta},$$

where

$$B^1(v^1, v^*) = \int_\omega \Big\{\mu h_1 v^1_{j0,\beta} v^*_{j0,\beta} + \mu h_1 v^1_{\beta 0,j} v^*_{j0,\beta} + \lambda h_1 v^1_{i0,i} v^*_{j0,j} + \frac{\mu}{2} v^1_{30,\beta} v^*_{\beta 0} + \frac{\lambda}{2} v^1_{\beta 0,\beta} v^*_{30}\Big\} d\omega. \tag{8}$$

If we consider BVPs for system (6) with homogeneous boundary conditions for which the curvilinear integralalong $\partial\omega$ in (7) disappears, we arrive at the equation

$$B^1(v^1, v^*) = \int_\omega F^1 \cdot v^* d\omega.$$

Let us consider the following Dirichlet problem in the classical setting: Find a 3-dimensional vector

$$v^1 = (v^1_{10}, v^1_{20}, v^1_{30})^\top$$

in ω satisfying the system of differential equations (6) in ω and the homogeneous Dirichlet boundary condition on

$$[v^1(x)]^+ = 0, \quad x \in \partial\omega. \tag{9}$$

Note that throughout the paper, for smooth classical solutions, equation (6) and boundary condition (9) are understood in the classical point-wise sense, while for generalized weak solutions equation (6) is understood in the distributional sense and boundary condition (9) understood in the usual trace sense. To derive the weak setting of the above problem, we have to apply Green‘s formulas (7). We arrive at the variational equation:

$$B^1(v, v^*) = \langle F^1, v^*\rangle, \tag{10}$$

where the bilinear form $B^1(v^1, v^*)$ is defined by (8) and

$$\langle F^1, v^*\rangle = \int_\omega \Big(Q_{\overset{(+)}{\nu_1} j}\sqrt{\Big(\overset{(+)}{h}_{1,2}\Big)^2 + 1} + \Phi^1_{j0}\Big) v^*_{j0} d\omega. \tag{11}$$

Note that the bilinear form (8) can be represented as follows

$$B^1(v, v^*) := \frac{1}{2}\int_\omega \Big\{a[\lambda e^1_{kko}(v^1) e^1_{ii0}(v^*) + 2\mu e^1_{ij0}(v^1) e^1_{ij}(v^*)]$$

$$+\frac{\mu}{2}v^1_{30,j}v^*_{j0}+\frac{\lambda}{2}v^1_{\beta 0,\beta}v^*_{30}\Big\}d\omega,$$

where e^1_{ij0} is given by the following following expression (see [5])

$$e^1_{\alpha\beta 0}(v^1)=\frac{1}{2}h_1(v^1_{\alpha 0,\beta}+v^1_{\beta 0,\alpha}),\quad e^1_{\beta 30}=\frac{1}{2}h_1v^1_{30,\beta},\quad e^1_{\alpha\alpha 0}=h_1v^1_{\alpha 0,\alpha}.$$

Further,we construct the vectors in $\Omega:=\{(x;x_3):x\in\omega,-h_1(x)<x_3<h_1(x)\}:$

$$w_i(x,x_3)=\frac{1}{2}v^1_{i0}(x),\quad i=1,2,3,$$

$$w^*_i(x,x_3)=\frac{1}{2}v^*_{i0}(x),\quad i=1,2,3.$$

It can be shown that

$$\begin{aligned} B(w,w^*) &:= \int_\Omega\Big[X^1_{ij}(w)e^1_{ij}(w^*)+\frac{\mu}{h_1}w_{3,\beta}w^*_\beta+\frac{\lambda}{h_1}w_{\beta,\beta}w^*_3\Big]d\Omega \\ &= B^1(v^1,v^*), \end{aligned} \tag{12}$$

where $w(x,x_3):=(w_1,w_2,w_3)$ and $w^*(x,x_3):=(w^*_1,w^*_2,w^*_3)$ are vectors and $B(w,w^*)$ is the bilinear form corresponding to the three-dimensional potential energy for the displacement vector w. Owing to the positive definiteness of the potential energy for $2\lambda+3\mu>0$ and $\mu>0$

$$\begin{aligned} &B(w,w)\geq c_2\sum_{i,j=1}^{3}\int_\Omega[e^1_{ij}(w)]^2d\Omega+\int_\Omega\frac{\lambda-\mu}{h_1}w_{\beta,\beta}w_3d\Omega \\ &=c_2\int_\omega d\omega\int_{-h_1}^{h_1}\frac{1}{2}ae^1_{ij0}(v^1)\cdot\frac{1}{2}ae^1_{ij0}(v^1)dx_3+\frac{\lambda-\mu}{2}\int_\omega v^1_{\beta 0,\beta}v^1_{30}d\omega \\ &=c_1\sum_{i,j=1}^{3}\int_\omega(e^1_{ij0}(v^1))^2\frac{d\omega}{h_1}, \end{aligned} \tag{13}$$

here we have taken into account the following properties
for $v(x_1,x_2)|_{x\in\partial\omega}=0$,

$$\int_\omega(\nabla_2v)vd\omega=0,\quad \nabla_2:=\partial_1+\partial_2,$$

on the other hand in our case

$$\partial_iv_{i0}=\partial_\alpha v_{\alpha 0}.$$

The positive constants c_1 and c_2 depend only on the material parameters λ and μ.

Remark 1. $B^1(v^1,v^1)=0\Longrightarrow v^1=0$. Indeed, if $B^1(v^1,v^1)=0$, then $B(w,w)=0$ by (13).

Denote by $D(\omega)$ a space of infinity differentiable functions with compact support in ω and introduce the linear form $[D(\omega)]^3$ by the formula:

$$(v^1, v^*)_{X_1^\kappa} = \int_\omega e_{ij0}(v^1)e_{ij0}(v^*)\frac{d\omega}{h_1} + \frac{1}{4}\int_\omega v^1_{j0,j}v^*_{j0,j}d\omega + \frac{1}{4}\int_\omega v^1_{j0}v^*_{j0}d\omega$$
$$= \int_\omega \frac{1}{4}[h_1(v^1_{i0,j} + v^1_{j0,i})][h_1(v^*_{i0,j} + v^*_{j0,i})]\frac{d\omega}{h_1}$$
$$+\frac{1}{4}\int_\omega v^1_{j0,j}v^*_{j0,j}d\omega + \frac{1}{4}\int_\omega v^1_{j0}v^*_{j0}d\omega. \tag{14}$$

Denote by $X_1^\kappa := X_1^\kappa(\omega)$ the completion of the space $\big[D(\omega)\big]^3$ with the help of the norm:

$$\|v^1\|^2_{X_1^\kappa} = \frac{1}{4}\int_\omega [h_1(v^1_{i0,j} + v^1_{j0,i})]^2\frac{d\omega}{h_1} + \frac{1}{4}\int_\omega \Big[\Big(v^1_{j0,j}\Big)^2 + \Big(v^1_{j0}\Big)^2\Big]d\omega \tag{15}$$

corresponding to the inner product (14) X_1^κ is a Hilbert space.

Now we can formulate the weak setting of the homogeneous Dirichlet problem (9), (10):

Find a vector $v^1 = (v^1_{10}, v^1_{20}, v^1_{30})^\top \in X_1^\kappa$ satisfying the equality

$$B^1(v^1, v^*) = \langle F^1, v^*\rangle \quad \text{for all} \quad v^* \in X_1^\kappa. \tag{16}$$

Here, the vector F^1 belongs to the adjoint space $[X_1^\kappa]^*$, in general, and $\langle\cdot,\cdot\rangle$ denotes duality brackets between the spaces $[X_1^\kappa]^*$ and X_1^κ.

Lemma 2.1. *The bilinear form $B^1(\cdot,\cdot)$ is bounded and strictly coercive in the space $X_1^\kappa(\omega)$, i.e., there are positive constant C_0 and C_1 such that*

$$|B^1(v^1, v^*)| \le C_1\|v^1\|_{X_1^\kappa}\|v^*\|_{X_1^\kappa}, \tag{17}$$
$$B^1(v^1, v^1) \ge C_0\|v^1\|^2_{X_1^\kappa} \tag{18}$$

for all $v^1, v^* \in X_1^\kappa$.

Proof.

$$|B^1(v^1, v^*)|^2 = |B^1(w, w^*)|^2 = \Big[\int_\Omega \Big\{(2\mu e^1_{ij}(w) + \lambda\delta_{ij}e^1_{kk}(w))e^1_{ij}(w^*)$$
$$+\frac{\mu}{h_1}w_{3,j}w^*_j + \frac{\lambda}{h_1}w_{\beta,\beta}w^*_3\Big\}d\Omega\Big]^2$$
$$\le C_1\int_\omega \frac{1}{2}\sum_{i,j=1}^3 (e^1_{ij0}(v^1))^2\frac{d\omega}{h_1}\int_\omega \frac{1}{2}\sum_{i,j=1}^3 (e^1_{ij0}(v^*))^2\frac{d\omega}{h_1}$$
$$+\frac{\mu^2}{4}\int_\omega \Big(v^1_{30,j}\Big)^2\int_\omega \Big(v^*_{j0}\Big)^2 d\omega + \frac{\lambda}{4}\int_\omega \Big(v^1_{j0}\Big)^2\int_\omega \Big(v^*_{30,j}\Big)^2 d\omega$$
$$\le C_2\|v^1\|^2_{X_1^\kappa}\|v^*\|^2_{X_1^\kappa}.$$

Whence (17) follows. Inequality (18) immediately follows from (12) and (13). ■

Theorem 2.2. *Let $F^1 \in [X_1^\kappa]^*$. Then the variational problem (16) has a unique solution $v^1 \in X_1^\kappa$ for an arbitrary value of the parameter κ and*

$$\|v^1\|_{X_1^\kappa} \leq \frac{1}{C_0}\|F\|_{[X_1^\kappa]^*}.$$

Proof. The proof directly follows from the Lax-Milgram theorem (see Appendix, Theorem A.1). ■

It can be easily shown that if $F^1 \in [L(\omega)]^3$ and $\operatorname{supp} F^1 \cap \overline{\gamma}_0 = \emptyset$, then $F^1 \in [X_1^\kappa]^*$ and

$$\langle F^1 \,,\, v^* \rangle = \int\limits_\omega F^1(x)\, v^*(x)\, d\omega,$$

since $v^* \in [H^1(\omega_\varepsilon)]^3$, where ε is sufficiently small positive number such that $\operatorname{supp} F \subset \omega_\varepsilon = \omega \cap \{x_2 > \varepsilon\}$. Therefore,

$$|\langle F^1 \,,\, v^* \rangle| = \Big| \int\limits_\omega F^1(x)\, v^*(x)\, d\omega \Big| \leq ||F^1||_{[L_2(\omega)]^3}\; ||v^*||_{[L_2(\omega_\varepsilon)]^3}$$

$$\leq ||F^1||_{[L_2(\omega)]^3}\; ||v^*||_{[H^1(\omega_\varepsilon)]^3} \leq\; C_\varepsilon\, ||F^1||_{[L_2(\omega)]^3}\; ||v^*||_{X_1^\kappa}\,.$$

In this case, we obtain the estimate

$$||v^1||_{X_1^\kappa} \leq \frac{C_\varepsilon}{C_0}\, ||F^1||_{[L_2(\omega)]^3}\,.$$

Now we establish a representation of the space X_1^κ as a weighted Sobolev space. To this end, we introduce the following space:

$$Y_0^\kappa := \Big[\overset{0}{W}{}^1_{2\,,\kappa}(\omega)\Big]^3,$$

where $\overset{0}{W}{}^1_{2\,,\kappa}(\omega)$ is a completion $\mathcal{D}(\omega)$ by means of the norm

$$\|f\|^2_{\overset{0}{W}{}^1_{2\,,\kappa}(\omega)} := \int_\omega x_2^\kappa\Big(|\nabla f|^2\Big) d\omega, \quad \nabla f = (f_{,1}\,, f_{,2}\,).$$

The norm in the space Y_0^κ for a vector (v_{10}, v_{20}, v_{30}) reads as

$$\|v\|^2_{Y_0^\kappa} := \int_\omega x_2^\kappa\Big(\sum_{j=1}^{3} |\nabla v_{j0}|^2\Big) d\omega.$$

Using Korn's and Hardy's inequalities (see Appendix) it (similarly, to the Theorem 5.1 of [1]) the following theorem can be proved

Theorem 2.3. *Let $\kappa < 1$. Then the linear spaces X_1^κ and Y_0^κ as sets of vector functions coincide and the norms $\|\cdot\|_{X_1^\kappa}$, $\|\cdot\|_{Y_0^\kappa}$ are equivalent.*

Remark 2. From the trace theorem (see Appendix, Theorem A.4) it follows that the components v^1_{j0} of the vector $v^1 \in X^\kappa_1$ have the zero traces on $\partial\omega$ if $\kappa < 1$.

Lemma 2.4. *Let $\kappa < 1$ and $x_2^{1-\kappa/2} F^1_j \in L_2(\omega)$, $j = 1,2,3$. Then the linear functional $\langle F^1, v^* \rangle$ (see (10)) is bounded.*

3. Appendix

A.1. The Lax-Milgram theorem. Let V be a real Hilbert space and let $J(w,v)$ be a bilinear form defined on $V \times V$. Let this form be continuous, i.e., let there exist a constant $K > 0$ such that

$$|J(w,v)| \le K \|w\|_V \|v\|_V$$

holds $\forall w, v \in V$ and V-elliptic, i.e., let there exist a constant $\alpha > 0$ such that

$$J(w,w) \ge \alpha \|w\|^2_V$$

holds $\forall w \in V$. Further let F be a bounded linear functional from V^* dual of V. Then there exists one and only one element $z \in V$ such that

$$J(z,v) = \langle F, v \rangle \equiv Fv \quad \forall v \in V$$

and

$$\|z\|_V \le \alpha^{-1} \|F\|_{V^*}.$$

Let ω be as in Section 1 and let $\mathcal{D}(\omega)$ be a space of infinitely differentiable functions with compact support in ω.

A.2. Hardy's Inequality. For every $f \in \mathcal{D}(\omega)$ and $\nu \neq 1$ there holds the inequality

$$\int_\omega x_2^{\nu-2} f^2(x)\, d\omega \le C_\nu \int_\omega x_2^\nu\, |\nabla f(x)|^2\, d\omega, \tag{A.1}$$

where the positive constant C_ν is independent of f.

By completion of $\mathcal{D}(\omega)$ with the norm

$$||f||^2_{\overset{\circ}{W}{}^1_{2,\nu}(\omega)} := \int_\omega x_2^\nu\, |\nabla\, f(x)|^2\, d\omega,$$

we conclude that the inequality (A.1) holds for arbitrary $f \in \overset{\circ}{W}{}^1_{2,\nu}(\omega)$.

For proof see [2].

A.3. Korn's Weighted Inequality. Let $\varphi = (\varphi_1, \varphi_2) \in [\overset{\circ}{W}{}^1_{2,\nu}(\omega)]^2$ and $\nu \neq 1$. Then

$$\int_\omega x_2^\nu\, [\, |\nabla\, \varphi_1(x)|^2 + |\nabla\, \varphi_2(x)|^2\,]\, d\omega$$

$$\leq C_\nu \int_\omega x_2^\nu [\varphi_{1,1}^2(x) + \varphi_{2,2}^2(x) + (\varphi_{1,2}(x) + \varphi_{2,1}(x))^2]\, d\omega,$$

where the positive constant C_ν is independent of φ.

The proof can be found in [2], [11].

A.4. Trace Theorem. Let $0 < \nu < 1$ and $f \in \overset{\circ}{W}{}^1_{2,\nu}(\omega)$. Then the trace of the function f equals to zero on $\partial\omega$.

For proof see [2], [6], [7].

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