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FUNDAMENTAL SOLUTION IN THE THEORY OF POROELASTICITY OF STEADY VIBRATIONS FOR SOLIDS WITH DOUBLE POROSITY

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Abstract. In this paper the 2D full coupled theory of steady vibrations of poroelasticity for materials with double porosity is considered. There the fundamental and singular matrixes of solutions are constructed in terms of elementary functions. Using the fundamental matrix we will construct the simple and double layer potentials and study their properties near the boundary.

Keywords and phrases: Double porosity, fundamental solution, steady vibrations.

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1. Introduction

Porous materials play an important role in many branches of engineering,e.g., the petroleum industry,chemical engineering , geomechanics and biomechanics. The general 3D theory of consolidation for elastic materials with single porosity was formulated by Biot [1]. A theory of consolidation for elastic materials with double porosity was presented in [2-4], where the physical and mathematical foundations of this theory were considered. In this papers the theory of Aifantis unifies a model proposed by Biot for the consolidation of deformable single porosity media with a model proposed by Barenblatt for seepage in undeformable media with two degrees of porosity. However, Aifantis' quasi-static theory ignored the cross-coupling effect between the volume change of the pores and fissures in the system. The cross-coupled terms were included in the equations of conservation of mass for the pore and fissure fluid and in Darcy's law for solids with double porosity by several authors [5,7].

The poroelasticity is an affective and useful model for deformation-driven bone fluid movement in bone tissue. The suggested double porosity model would consider the bone fluid pressure in the vascular porosity and the bone fluid pressure in the lacunar-canalicular porosity. The extensive review of the results in the theory of bone poroelasticity can be found in [8].

In [9,10] the full coupled linear theory of elasticity for solids with double porosity is considered. Four spatial cases of the dynamical equations are considered. The fundamental solutions are constructed by means of elementary functions and the basic properties of the fundamental solutions are established. The fundamental solution of quasi-static equations of the linear theory elasticity for double porosity solids is constructed and basic properties are established in [11].

In [12-16] the Aifantis' quasi-static theory of elasticity for solids with double porosity is considered. Explicit solutions of the BVPs of the theory of consolidation with double porosity for half-plane and for half-space are constructed, Uniqueness and existence theorems of solutions of two and three-dimensional boundary value problems of the theory of consolidation with double porosity are proved.

In this paper the 2D full coupled theory of steady vibrations of poroelasticity for materials with double porosity is considered. There the fundamental and singular matrixes of solutions are constructed in terms of elementary functions. Using the fundamental matrix we will construct the simple and double layer potentials and study their properties near the boundary.

2. Basic equations

Let $\mathbf{x} = (x_1, x_2)$ be a point of the Euclidean 2D space R^2. Let D^+ be a bounded 2D domain (surrounded by the curve S) and let D^- be the complement of $D^+ \cup S$. $\partial_{\mathbf{x}} = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\right)$. Let us assume that the domain D^+ is filled with an isotropic material with double porosity.

The system of homogeneous equations of motion in the 2D linear theory of elasticity for solids with double porosity can be written as follows [11]

$$
\begin{aligned}
&\mu\Delta\mathbf{u}' + (\lambda+\mu) graddiv\mathbf{u}' - grad(\beta_1 p_1' + \beta_2 p_2') - \rho\ddot{\mathbf{u}}' = \mathbf{0},\\
&-\beta_1 div\dot{\mathbf{u}}' + k_1\Delta p_1' - \alpha_1 \dot{p}'_1 - \gamma(p_1' - p_2') = 0,\\
&-\beta_2 div\dot{\mathbf{u}}' + k_2\Delta p_2' - \alpha_2 \dot{p}'_2 + \gamma(p_1' - p_2') = 0.
\end{aligned}
$$

Let us suppose that

$$
(\mathbf{u}', p_1', p_2') = Re[(\mathbf{u}, p_1, p_2)(\mathbf{x})e^{-i\omega t}],
$$

then we obtain the following system of homogeneous equations of steady vibrations in the 2D linear theory of elasticity for solids with double porosity

$$
\begin{aligned}
&\mu\Delta\mathbf{u} + (\lambda+\mu) graddiv\mathbf{u} - grad(\beta_1 p_1 + \beta_2 p_2) + \rho\omega^2\mathbf{u} = 0,\\
&i\omega\beta_1 div\mathbf{u} + (k_1\Delta + a_1)p_1 + \gamma p_2 = 0,\\
&i\omega\beta_2 div\mathbf{u} + \gamma p_1 + (k_2\Delta + a_2)p_2 = 0,
\end{aligned}
\tag{1}
$$

where $\mathbf{u} = (u_1, u_2)^T$ is the displacement vector in a solid, p_1 and p_2 are the pore and fissure fluid pressures respectively. $a_j = i\omega\alpha_j - \gamma$, $\omega > 0$ is the oscillation frequency, β_1 and β_2 are the effective stress parameters, $\gamma > 0$ is the internal transport coefficient and corresponds to a fluid transfer rate with respect to the intensity of flow between the pore and fissures, α_1 and α_2 measure the compressibilities of the pore and fissure system, respectively

λ, μ, are constitutive coefficients, $k_j = \frac{\kappa_j}{\mu'}$, μ' is the fluid viscosity, κ_1 and κ_2 are the macroscopic intrinsic permeabilities associated with matrix and fissure porosity, respectively, Δ is the 2D Laplace operator. The superscript "T" denotes transposition.

Note that, neglecting inertial effect ($\rho = 0$) in (1) we obtain the system of homogeneous equations of steady vibrations in the Aifantis' quasi-static theory of elasticity for solids with double porosity.

We introduce the matrix differential operator

$$\mathbf{A}(\partial_{\mathbf{x}}, \omega) = \| A_{lj}(\partial_{\mathbf{x}}, \omega) \|_{4x4}, \quad l, j = 1, 2, 3, 4,$$

where

$$A_{lj} := \delta_{lj}(\mu\Delta + \rho\omega^2) + (\lambda + \mu)\frac{\partial^2}{\partial x_l \partial x_j}, \quad l, j = 1, 2,$$

$$A_{j3} := -\beta_1 \frac{\partial}{\partial x_j}, \quad A_{j4} := -\beta_2 \frac{\partial}{\partial x_j}, \quad j = 1, 2,$$

$$A_{3j} := i\omega\beta_1 \frac{\partial}{\partial x_j}, \quad A_{4j} := i\omega\beta_2 \frac{\partial}{\partial x_j}, \quad j = 1, 2, \quad A_{33} := k_1\Delta + a_1,$$

$$A_{34} := \gamma, \quad A_{43} := \gamma, \quad A_{44} := k_2\Delta + a_2,$$

$\delta_{\alpha\gamma}$ is the Kronecker delta. Then the system (1) can be rewritten as

$$\mathbf{A}(\partial_{\mathbf{x}}, \omega)\mathbf{U} = 0, \tag{2}$$

where $\mathbf{U} = (u_1, u_2, p_1, p_2)^T$.

We consider the system of the equations

$$\begin{aligned} &\mu\Delta\mathbf{u} + (\lambda + \mu) graddiv\mathbf{u} + i\omega grad(\beta_1 p_1 + \beta_2 p_2) = 0, \\ &-\beta_1 div\mathbf{u} + (k_1\Delta + a_1)p_1 + \gamma p_2 = 0, \\ &-\beta_2 div\mathbf{u} + \gamma p_1 + (k_2\Delta + a_2)p_2 = 0. \end{aligned} \tag{3}$$

The latter system (3) may be written in the form

$$\mathbf{A}^T(\partial_{\mathbf{x}}, \omega)\mathbf{U} = 0, \tag{4}$$

where $A^T(\partial_{\mathbf{x}}, \omega)$ is the transpose of matrix $\mathbf{A}(\partial_{\mathbf{x}}, \omega)$.

We assume that $\mu\mu_0 k_1 k_2 \neq 0$, where $\mu_0 := \lambda + 2\mu$. Obviously, if the last condition is satisfied, then $\mathbf{A}(\partial_{\mathbf{x}}, \omega)$ is the elliptic differential operator.

4. The basic fundamental matrix

In this section we will construct the fundamental matrix of solutions for the system (2).

By the direct calculations we get

$$detA = \mu\mu_0 k_1 k_2 (\Delta + \lambda_1^2)(\Delta + \lambda_2^2)(\Delta + \lambda_3^2)(\Delta + \lambda_4^2),$$

where $\lambda_4^2=\frac{\rho\omega^2}{\mu}$, and λ_j^2, $j=1,2,3,$ are roots of cubic algebraic equation (with respect to ξ)

$$(\rho\omega^2-\mu_0\xi)(k_1k_2\xi^2-k_0\xi+a_1a_2-\gamma^2)-i\omega\xi(\alpha_{12}-\alpha_{11}\xi)=0, \tag{5}$$

where

$$\alpha_{11}=k_2\beta_1^2+k_1\beta_2^2,\quad \alpha_{12}=a_2\beta_1^2+a_1\beta_2^2-2\gamma\beta_1\beta_2,$$
$$k_0=a_1k_2+a_2k_1.$$

We introduce the matrix differential operator $\mathbf{B}(\partial_{\mathbf{x}},\omega)$ consisting of cofactors of elements of the matrix $\mathbf{A}^T$ divided on $\mu\mu_0k_1k_2$:

$$B(\partial_{\mathbf{x}},\omega)=\frac{1}{\mu\mu_0k_1k_2}\parallel B_{lj}\parallel_{4x4},\ \ l,j=1,2,3,4,$$

where

$$B_{lj}=\delta_{lj}\mu_0k_1k_2(\Delta+\lambda_1^2)(\Delta+\lambda_2^2)(\Delta+\lambda_3^2)$$
$$-\xi_j\xi_l\{(\lambda+\mu)[k_1k_2\Delta\Delta+k_0\Delta+a_1a_2-\gamma^2]+i\omega(\alpha_{11}\Delta+\alpha_{12})\},$$
$$B_{j3}=\mu(\Delta+\lambda_4^2)[\beta_1(k_2\Delta+a_2)-\beta_2\gamma]\xi_j,\quad l,j=1,2,$$
$$B_{j4}=\mu(\Delta+\lambda_4^2)[\beta_2(k_1\Delta+a_1)-\beta_1\gamma]\xi_j,\ \ \xi_j=\frac{\partial}{\partial x_j},\quad j=1,2,$$
$$B_{3j}=-i\omega\mu(\Delta+\lambda_4^2)[\beta_1(k_2\Delta+a_2)-\beta_2\gamma]\xi_j,\quad j=1,2,$$
$$B_{4j}=-i\omega\mu(\Delta+\lambda_4^2)[\beta_2(k_1\Delta+a_1)-\beta_1\gamma]\xi_j,\quad j=1,2,$$
$$B_{33}=\mu(\Delta+\lambda_4^2)[(\mu_0\Delta+\rho\omega^2)(k_2\Delta+a_2)+i\omega\beta_2^2\Delta],$$
$$B_{44}=\mu(\Delta+\lambda_4^2)[(\mu_0\Delta+\rho\omega^2)(k_1\Delta+a_1)+i\omega\beta_1^2\Delta],$$
$$B_{34}=-\mu(\Delta+\lambda_4^2)[(\mu_0\Delta+\rho\omega^2)\gamma+i\omega\beta_1\beta_2\Delta],$$
$$B_{43}=-\mu(\Delta+\lambda_4^2)[(\mu_0\Delta+\rho\omega^2)\gamma+i\omega\beta_1\beta_2\Delta].$$

Substituting the vector $\mathbf{U}(\mathbf{x})=\mathbf{B}(\partial_{\mathbf{x}},\omega)\mathbf{\Psi}$ into (2), where $\mathbf{\Psi}$ is a four-component vector function, we get

$$(\Delta+\lambda_1^2)(\Delta+\lambda_2^2)(\Delta+\lambda_3^2)(\Delta+\lambda_4^2)\mathbf{\Psi}=0.$$

From here, after some calculations, the vector Ψ can be represented as

$$\Psi=-\sum_{m=1}^{4}d_m\varphi_m, \tag{6}$$

where

$$\varphi_m=\frac{\pi}{2i}H_0^{(1)}(\lambda_m r),$$

$H_0^{(1)}(\lambda_m r)$ is Hankel's function of the first kind with the index 0

$$
\begin{aligned}
&H_0^{(1)}(\lambda_m r) = \frac{2i}{\pi} J_0(\lambda_m r)\ln r + \frac{2i}{\pi}\left(\ln\frac{\lambda_m}{2} + C - \frac{i\pi}{2}\right) J_0(\lambda_m r)\\
&-\frac{2i}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^k}{(k!)^2}\left(\frac{\lambda_m r}{2}\right)^{2k}\left(\frac{1}{k} + \frac{1}{k-1} + ... + 1\right), \quad m = 1,2,3,4,
\end{aligned}
\tag{7}
$$

$$
J_0(\lambda_m r) = \sum_{k=0}^{\infty}\frac{(-1)^k}{(k!)^2}\left(\frac{\lambda_m r}{2}\right)^{2k}, \quad r^2 = (x_1 - y_1)^2 + (x_2 - y_2)^2,
$$

$$
\sum_{j=1}^{4} d_j = 0, \quad \sum_{j=1}^{4} d_j\lambda_j^2 = 0, \quad \sum_{j=1}^{4} d_j\lambda_j^4 = 0, \quad \sum_{j=1}^{3} d_j(\lambda_4^2 - \lambda_j^2) = 0,
$$

$$
\sum_{j=1}^{4} d_j\lambda_j^6 = 1, \quad d_j = \prod_{\substack{m=1\\ j\neq m}}^{4}\frac{1}{\lambda_j^2 - \lambda_m^2}.
$$

Substituting (6) into $\mathbf{U} = \mathbf{B\Psi}$, we obtain the matrix of fundamental solutions for the equation (2) which we denote by $\mathbf{\Gamma}(\mathbf{x}\text{-}\mathbf{y},\omega)$

$$
\mathbf{\Gamma}(\mathbf{x}\text{-}\mathbf{y},\omega) = \| \Gamma_{kj}(\mathbf{x}\text{-}\mathbf{y},\omega) \|_{4\times 4} \tag{8}
$$

where

$$
\begin{aligned}
&\Gamma_{kj} = \frac{\delta_{kj}}{\mu}\varphi_4 + i\omega\sum_{l=1}^{3} N_l\frac{\partial^2(\varphi_l - \varphi_4)}{\partial x_k\partial x_j}, \quad k,j = 1,2,\\
&\Gamma_{j3} = -\sum_{l=1}^{3} N_{l3}\frac{\partial\varphi_l}{\partial x_j}, \quad j = 1,2, \quad \Gamma_{j4} = -\sum_{l=1}^{3} N_{l4}\frac{\partial\varphi_l}{\partial x_j},\\
&\Gamma_{3j} = i\omega\sum_{l=1}^{3} N_{3l}\frac{\partial\varphi_l}{\partial x_j}, \quad \Gamma_{4j} = i\omega\sum_{l=1}^{3} N_{4l}\frac{\partial\varphi_l}{\partial x_j}, \quad j = 1,2,\\
&N_l = \frac{\delta_l(\alpha_{12} - \alpha_{11}\lambda_l^2)}{\rho\omega^2 - \mu_0\lambda_l^2}, \quad \delta_l = \frac{d_l(\lambda_4^2 - \lambda_l^2)}{\mu_0 k_1 k_2},\\
&N_{l3} = \delta_l\left[\beta_1(a_2 - k_2\lambda_l^2) - \beta_2\gamma\right], \quad N_{l4} = \delta_l\left[\beta_2(a_1 - k_1\lambda_l^2) - \beta_1\gamma\right],\\
&N_{3l} = \delta_l\left[\beta_1(a_2 - k_2\lambda_l^2) - \beta_2\gamma\right], \quad N_{4l} = \delta_l\left[\beta_2(a_1 - k_1\lambda_l^2) - \beta_1\gamma\right],\\
&\Gamma_{33} = -\sum_{l=1}^{3}\delta_l[(\rho\omega^2 - \mu_0\lambda_l^2)(a_2 - k_2\lambda_l^2) - i\omega\beta_2^2\lambda_l^2]\varphi_l,\\
&\Gamma_{44} = -\sum_{l=1}^{3}\delta_l[(\rho\omega^2 - \mu_0\lambda_l^2)(a_1 - k_1\lambda_l^2) - i\omega\beta_1^2\lambda_l^2]\varphi_l,
\end{aligned}
$$

$$
\begin{aligned}
\Gamma_{34} &= \sum_{l=1}^{3}\delta_l[(\rho\omega^2 - \mu_0\lambda_l^2)\gamma - i\omega\beta_1\beta_2\lambda_l^2]\varphi_l,\\
\Gamma_{43} &= \sum_{l=1}^{3}\delta_l[(\rho\omega^2 - \mu_0\lambda_l^2)\gamma - i\omega\beta_1\beta_2\lambda_l^2]\varphi_l.
\end{aligned}
$$

Clearly

$$
\frac{\pi}{2i}H_0^{(1)}(\lambda r) = \ln|\mathbf{x} - \mathbf{y}| - \frac{\lambda^2}{4}|\mathbf{x} - \mathbf{y}|^2\ln|\mathbf{x} - \mathbf{y}| + const + O(|\mathbf{x} - \mathbf{y}|^2).
$$

It is evident that all elements of $\mathbf{\Gamma}(\mathbf{x}\text{-}\mathbf{y},\omega)$ are single-valued functions on the whole plane and they have a logarithmic singularity at most. It can be shown that columns of the matrix $\mathbf{\Gamma}(\mathbf{x}\text{-}\mathbf{y},\omega)$ are solutions to the system (1) with respect to $\mathbf{x}$ for any $\mathbf{x}\neq\mathbf{y}$. By applying the methods, as in the classical theory of elasticity, we can similarly prove the following;

Theorem 1. *The elements of the matrix* $\mathbf{\Gamma}(\mathbf{x}\text{-}\mathbf{y},\omega)$ *have a logarithmic singularity as* $\mathbf{x}\to\mathbf{y}$ *and each column of the matrix* $\mathbf{\Gamma}(\mathbf{x}\text{-}\mathbf{y},\omega)$, *considered as a vector, is a solution of system* (1) *at every point* $\mathbf{x}$, *if* $\mathbf{x}\neq\mathbf{y}$.

Remark. The operator $\mathbf{A}(\partial_{\mathbf{x}},\omega)\mathbf{U}$ is not self adjoint. Obviously, it is possible to construct the fundamental solution of the adjoined operator in quite a similar manner. Let's consider the matrixes $\widetilde{\mathbf{\Gamma}}(\mathbf{x},\omega):=\mathbf{\Gamma}^T(-\mathbf{x},\omega)$ and $\widetilde{\boldsymbol{A}}(\partial_{\mathbf{x}},\omega):=\boldsymbol{A}^T(-\partial_{\mathbf{x}},\omega)$. The following basic properties of $\widetilde{\mathbf{\Gamma}}(\mathbf{x},\omega)$ may be easily verified:

Theorem 2. *Each column of the matrix* $\widetilde{\mathbf{\Gamma}}(\mathbf{x}\text{-}\mathbf{y},\omega)$, *considered as a vector, satisfies the associated system* $\widetilde{\boldsymbol{A}}(\partial_{\mathbf{x}},\omega)\widetilde{\mathbf{\Gamma}}(\mathbf{x}\text{-}\mathbf{y},\omega)=0$, *at every point* $\mathbf{x}$, if $\mathbf{x}\neq\mathbf{y}$ *and the elements of the matrix* $\widetilde{\mathbf{\Gamma}}(\mathbf{x}\text{-}\mathbf{y},\omega)$ *have a logarithmic singularity as* $\mathbf{x}\to\mathbf{y}$.

5. Singular matrix of solutions

Using the basic fundamental matrix, we will construct the so-called singular matrix of solutions and study their properties.

Write now the expressions for the components of the stress vector, which acts on an elements of the arc with the normal $\mathbf{n}=(n_1,n_2)$. Denoting the stress vector by $\boldsymbol{P}(\partial_{\mathbf{x}},\mathbf{n})\mathbf{U}$, we have

$$\boldsymbol{P}(\partial_{\mathbf{x}},\mathbf{n})\mathbf{U}=\boldsymbol{T}(\partial_{\mathbf{x}},\mathbf{n})\mathbf{u}-\mathbf{n}(\beta_1p_1+\beta_2p_2), \tag{9}$$

where

$$\boldsymbol{T}(\partial_{\mathbf{x}},\mathbf{n})\mathbf{u}=\begin{pmatrix}\mu\dfrac{\partial}{\partial n}+(\lambda+\mu)n_1\dfrac{\partial}{\partial x_1} & (\lambda+\mu)n_1\dfrac{\partial}{\partial x_2}+\mu\dfrac{\partial}{\partial s}\\ (\lambda+\mu)n_2\dfrac{\partial}{\partial x_1}-\mu\dfrac{\partial}{\partial s} & \mu\dfrac{\partial}{\partial n}+(\lambda+\mu)n_2\dfrac{\partial}{\partial x_2}\end{pmatrix}\mathbf{u},$$

$$\frac{\partial}{\partial n}=n_1\frac{\partial}{\partial x_1}+n_2\frac{\partial}{\partial x_2},\quad \frac{\partial}{\partial s}=n_2\frac{\partial}{\partial x_1}-n_1\frac{\partial}{\partial x_2}.$$

We introduce the following notations $\boldsymbol{R}(\partial_{\mathbf{x}},\mathbf{n})$ and $\widetilde{\boldsymbol{R}}(\partial_{\mathbf{x}},\mathbf{n})$, where

$$\boldsymbol{R}(\partial_{\mathbf{x}},\mathbf{n})=\begin{pmatrix}T_{11}(\partial x,n) & T_{12}(\partial x,n) & -\beta_1n_1 & -\beta_2n_1\\ T_{21}(\partial x,n) & T_{22}(\partial x,n) & -\beta_1n_2 & -\beta_2n_2\\ 0 & 0 & k_1\dfrac{\partial}{\partial n} & 0\\ 0 & 0 & 0 & k_2\dfrac{\partial}{\partial n}\end{pmatrix},$$

$$\widetilde{\boldsymbol{R}}(\partial_{\mathbf{x}},\mathbf{n})=\begin{pmatrix} T_{11}(\partial x,n) & T_{12}(\partial x,n) & -i\omega n_1\beta_1 & -i\omega n_1\beta_2\\ T_{21}(\partial x,n) & T_{22}(\partial x,n) & -i\omega n_2\beta_1 & -i\omega n_2\beta_2\\ 0 & 0 & k_1\dfrac{\partial}{\partial n} & 0\\ 0 & 0 & 0 & k_2\dfrac{\partial}{\partial n}\end{pmatrix},$$

By Applying the operator $\boldsymbol{R}(\partial_{\mathbf{x}},\mathbf{n})$ to the matrix $\mathbf{\Gamma}(\mathbf{x\text{-}y}$ and the operator $\widetilde{\boldsymbol{R}}(\partial_{\mathbf{x}},\mathbf{n})$ to the matrix $\widetilde{\mathbf{\Gamma}}(\mathbf{x\text{-}y})$, we shall construct the so-called singular matrixes of solutions respectively

$$\boldsymbol{R}(\partial_{\mathbf{x}},\mathbf{n})\mathbf{\Gamma}(\mathbf{x}-\mathbf{y})=\|R_{pq}\|_{4\times4},\quad \widetilde{\boldsymbol{R}}(\partial_{\mathbf{x}},\mathbf{n})\widetilde{\mathbf{\Gamma}}(\mathbf{y}-\mathbf{x})=\|\widetilde{R}_{pq}\|_{4\times4},$$

The elements R_{pq} are following:

$$R_{11}=\frac{\partial\varphi_4}{\partial n}+i\omega\left[2\mu\frac{\partial}{\partial s}\frac{\partial^2}{\partial x_1x_2}-\rho\omega^2n_1\frac{\partial}{\partial x_1}\right]\sum_{l=1}^{3}N_l(\varphi_l-\varphi_4),$$
$$R_{12}=\frac{\partial\varphi_4}{\partial s}+i\omega\left[2\mu\frac{\partial}{\partial s}\frac{\partial^2}{\partial x_2^2}-\rho\omega^2n_1\frac{\partial}{\partial x_2}\right]\sum_{l=1}^{3}N_l(\varphi_l-\varphi_4),$$
$$R_{21}=-\frac{\partial\varphi_4}{\partial s}-i\omega\left[2\mu\frac{\partial}{\partial s}\frac{\partial^2}{\partial x_1^2}+\rho\omega^2n_2\frac{\partial}{\partial x_1}\right]\sum_{l=1}^{3}N_l(\varphi_l-\varphi_4),$$

$$R_{22}=\frac{\partial\varphi_4}{\partial n}-i\omega\left[2\mu\frac{\partial}{\partial s}\frac{\partial^2}{\partial x_1x_2}+\rho\omega^2n_2\frac{\partial}{\partial x_2}\right]\sum_{l=1}^{3}N_l(\varphi_l-\varphi_4),$$
$$R_{13}=\left(-2\mu\frac{\partial}{\partial s}\frac{\partial}{\partial x_2}+n_1\rho\omega^2\right)\sum_{l=1}^{3}N_{l3}\varphi_l,\ R_{23}=\left(2\mu\frac{\partial}{\partial s}\frac{\partial}{\partial x_1}+n_2\rho\omega^2\right)\sum_{l=1}^{3}N_{l3}\varphi_l,$$
$$R_{14}=\left(2\mu\frac{\partial}{\partial s}\frac{\partial}{\partial x_2}-n_1\rho\omega^2\right)\sum_{l=1}^{3}N_{l4}\varphi_l,\ R_{24}=-\left(2\mu\frac{\partial}{\partial s}\frac{\partial}{\partial x_1}+n_2\rho\omega^2\right)\sum_{l=1}^{3}N_{l4}\varphi_l,$$
$$R_{3j}=k_1\frac{\partial}{\partial n}\Gamma_{3j},\quad R_{33}=k_1\frac{\partial}{\partial n}\Gamma_{33},\quad R_{34}=k_1\frac{\partial}{\partial n}\Gamma_{34},$$
$$R_{4j}=k_2\frac{\partial}{\partial n}\Gamma_{4j},\quad R_{43}=k_2\frac{\partial}{\partial n}\Gamma_{43},\quad R_{44}=k_2\frac{\partial}{\partial n}\Gamma_{44},\quad j=1,2,$$

The elements $\tilde{R}_{pq}$ are following:

$$\tilde{R}_{kl}=R_{kl},\quad k,l=1,2,\quad \tilde{R}_{13}=i\omega\left[\rho\omega^2n_1-2\mu\frac{\partial}{\partial s}\frac{\partial}{\partial x_2}\right]\sum_{l=1}^{3}N_{3l}\varphi_l,$$
$$\tilde{R}_{23}=i\omega\left[\rho\omega^2n_2+2\mu\frac{\partial}{\partial s}\frac{\partial}{\partial x_1}\right]\sum_{l=1}^{3}N_{3l}\varphi_l,\ \tilde{R}_{14}=i\omega\left[-\rho\omega^2n_1+2\mu\frac{\partial}{\partial s}\frac{\partial}{\partial x_2}\right]\sum_{l=1}^{3}N_{4l}\varphi_l,$$
$$\tilde{R}_{24}=-i\omega\left[\rho\omega^2n_2+2\mu\frac{\partial}{\partial s}\frac{\partial}{\partial x_1}\right]\sum_{l=1}^{3}N_{4l}\varphi_l,\quad \tilde{R}_{3j}=-k_1\frac{\partial}{\partial n}\Gamma_{j3},$$
$$\tilde{R}_{33}=k_1\frac{\partial}{\partial n}\Gamma_{33},\quad \tilde{R}_{34}=k_1\frac{\partial}{\partial n}\Gamma_{43},\quad \tilde{R}_{4j}=-k_2\frac{\partial}{\partial n}\Gamma_{j4},$$
$$\tilde{R}_{43}=k_2\frac{\partial}{\partial n}\Gamma_{34},\quad \tilde{R}_{44}=k_2\frac{\partial}{\partial n}\Gamma_{44},\quad j=1,2.$$

It is well-known that in the case of a Lyapunov curve $S \in C^{1,\alpha}$ the function $\dfrac{\partial \ln r}{\partial n}$, for $\mathbf{x}, \mathbf{y} \in S$ has a week singularity and $\dfrac{\partial \ln r}{\partial n}$ is integrable in the sense of the principal Cauchy value. Consequently, $\dfrac{\partial \ln r}{\partial n}$ is a singular kernel on S. It is obvious that, $\tilde{\boldsymbol{R}}^{\tau}(\partial_{\mathbf{y}}, \mathbf{n})\boldsymbol{\Gamma}^T(\mathbf{x}\text{-}\mathbf{y}, \omega)$ and $\boldsymbol{R}(\partial_{\mathbf{x}}, \mathbf{n})\boldsymbol{\Gamma}(\mathbf{x}-\mathbf{y})$ are singular kernels (in the sense of Cauchy).

Theorem 3. *Every column of the matrix* $[\boldsymbol{R}(\partial_{\mathbf{y}}, \mathbf{n})\boldsymbol{\Gamma}(\mathbf{y}\text{-}\mathbf{x}, \omega)]^T$, *considered as a vector, is a solution of the system* $\widetilde{\mathbf{A}}(\partial_{\mathbf{x}}, \omega) = 0$ *at any point* $\mathbf{x}$ if $\mathbf{x} \neq \mathbf{y}$ *and the elements of the matrix* $[\boldsymbol{R}(\partial_{\mathbf{y}}, \mathbf{n})\boldsymbol{\Gamma}(\mathbf{y}\text{-}\mathbf{x}, \omega)]^T$ *contain a singular part, which is integrable in the sense of the Cauchy principal value.*

Theorem 4. *Every column of the matrix* $\left[\tilde{\boldsymbol{R}}(\partial_{\mathbf{y}}, \mathbf{n})\tilde{\boldsymbol{\Gamma}}(\mathbf{y}\text{-}\mathbf{x}, \omega)\right]^T$, *considered as a vector, is a solution of the system* $\mathbf{A}(\partial_{\mathbf{x}}, \omega)\mathbf{U} = 0$ at *any point* $\mathbf{x}$ *if* $\mathbf{x} \neq \mathbf{y}$ *and the elements of the matrix* $\left[\tilde{\boldsymbol{R}}(\partial_{\mathbf{y}}, \mathbf{n})\tilde{\boldsymbol{\Gamma}}(\mathbf{y}\text{-}\mathbf{x}, \omega)\right]^T$, *contain a singular part, which is integrable in the sense of the Cauchy principal value.*

We introduce the potential of a single-layer

$$\mathbf{Z}^{(1)}(\mathbf{x}, \mathbf{g}) = \frac{1}{4i}\int_S \boldsymbol{\Gamma}(\mathbf{x}-\mathbf{y}, \omega)\mathbf{g}(\mathbf{y})dS$$

and the potential of a double-layer

$$\mathbf{Z}^{(2)}(\mathbf{x}, \mathbf{h}) = \frac{1}{4i}\int_S [\tilde{\boldsymbol{R}}^{\tau}(\partial_{\mathbf{y}}, \mathbf{n})\boldsymbol{\Gamma}^T(\mathbf{x}\text{-}\mathbf{y}, \omega)]^T\mathbf{h}(\mathbf{y})dS, \tag{10}$$

where $\boldsymbol{\Gamma}$ is given by (8), $\mathbf{g}$ and $\boldsymbol{h}$ are four-component continuous (or Holder continuous) vectors. S is a closed Lyapunov curve.

Now let us consider the operation $\boldsymbol{R}(\partial\mathbf{x}, \mathbf{n})$ acting on a single-layer potential. We obtain

$$\boldsymbol{R}(\partial_{\mathbf{x}}, \mathbf{n})\mathbf{Z}^{(1)}(\mathbf{x}, \boldsymbol{g}) = \frac{1}{4i}\int_S \boldsymbol{R}(\partial_{\mathbf{x}}, \mathbf{n})\boldsymbol{\Gamma}(\mathbf{x}-\mathbf{y}, \omega)\boldsymbol{g}(\mathbf{y})d\mathbf{y}. \tag{11}$$

The following theorem is valid:

Theorem 5. *The vectors* $\boldsymbol{Z}^{(j)}$, $j = 1, 2$, *are solutions of the system (2) in both the domains* D^+ *and* D^-. *When* $\boldsymbol{x} \longrightarrow \boldsymbol{z} \in S$, *from (10) and (11) we obtain*

$$[\boldsymbol{Z}^{(2)}(\boldsymbol{z}, \boldsymbol{h})]^{\pm} = \pm\boldsymbol{h}(\boldsymbol{z}) + \frac{1}{4i}\int_S [\tilde{\boldsymbol{R}}^{\tau}(\partial_{\boldsymbol{y}}, \boldsymbol{n})\boldsymbol{\Gamma}^T(\boldsymbol{z}\text{-}\boldsymbol{y}, \omega)]^T\boldsymbol{h}(\boldsymbol{y})dS,$$

$$[\boldsymbol{R}(\partial_z, \boldsymbol{n})\boldsymbol{Z}^{(1)}(\boldsymbol{z}, \boldsymbol{g})]^{\pm} = \mp\boldsymbol{g}(\boldsymbol{z}) + \frac{1}{4i}\int_S \boldsymbol{R}(\partial_z, \boldsymbol{n})\boldsymbol{\Gamma}(\boldsymbol{z}-\boldsymbol{y}, \omega)\boldsymbol{g}(\boldsymbol{y})d\boldsymbol{y},$$

For the regularity of potentials (10)-(11) in the domains D^+ and D^- it is sufficient to assume that $S \in C^{2,\beta}$, $(0 < \beta < 1)$ $\mathbf{g}, \mathbf{h} \in C^{1,\alpha}(S)$, $(0 < \alpha < \beta)$.

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