Proceedings of I. Vekua Institute of Applied Mathematics
Vol. 63, 2013

NEGATIVE BINOMIAL DISTRIBUTION AND RIEMANN ZETA-FUNCTION

Makhaldiani N.

Abstract. Description of the Negative binomial distribution (NBD) and Riemann zeta-function is given. New equation connecting NBD and zeta-function analyzed.

Keywords and phrases: Riemann zeta-function.

AMS subject classification (2000): 37C30.

1. Negative binomial distribution

With the advent of any new hadron accelerator the quantities first studied are charged particle multiplicities. The multiparticle production can be described by the probability distribution P_n which is a superposition of some unknown distribution of sources, and the Poisson distribution describing particle emission from one source. This is a typical situation in many microscopic models of multiparticle production.

Negative binomial distribution (NBD) is defined as

$$P_n = \frac{\Gamma(n+r)}{n!\Gamma(r)} p^n (1-p)^r, \ \sum_{n\geq 0} P_n = 1, \tag{1}$$

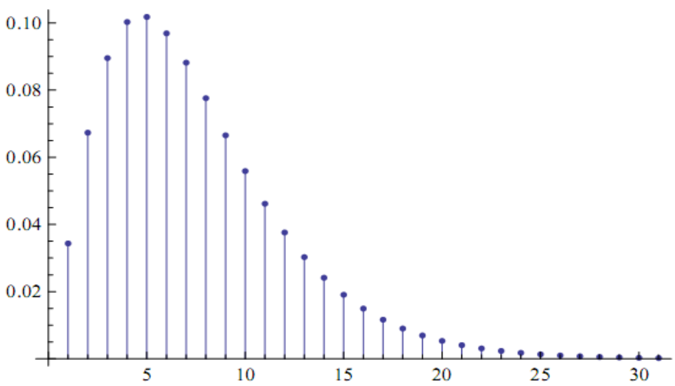

Figure 1: $P_n, (1), -r = 2.8, p = 0.3, < n >= 6$

Hadronic collisions at high energies (LHC) lead to charged multiplicity distributions whose shapes are well fitted by a single NBD in fixed intervals of central (pseudo)rapidity η [1]. NBD provides a very good parametrization for multiplicity distributions in e^+e^- annihilation; in deep inelastic lepton scattering; in proton-proton collisions; in proton-nucleus scattering.

It is interesting to understand how NBD fits such a different reactions?

1.1. Generating function for NBD

Let us consider NBD for normed topological cross sections

$$\begin{aligned}
\frac{\sigma_n}{\sigma} &= P_n = \frac{\Gamma(n+k)}{\Gamma(n+1)\Gamma(k)}\Big(\frac{k}{<n>}\Big)^k\Big((1+\frac{k}{<n>}\Big)^{-(n+k)} \\
&= \frac{\Gamma(n+k)}{\Gamma(n+1)\Gamma(k)}\Big(1+\frac{k}{<n>}\Big)^{-n}\Big(1+\frac{<n>}{k}\Big)^{-k} \\
&= \frac{\Gamma(n+k)}{\Gamma(n+1)\Gamma(k)}\Big(\frac{<n>}{<n>+k}\Big)^n\Big(\frac{k}{k+<n>}\Big)^k, \\
&= \frac{\Gamma(k+n)}{\Gamma(k)n!}\frac{(\frac{k}{<n>})^k}{(1+\frac{k}{<n>})^{k+n}}, \\
r &= k > 0,\ p = \frac{<n>}{<n>+k}.
\end{aligned} \tag{2}$$

The generating function for NBD is

$$F(h) = \Big(1+\frac{<n>}{k}(1-h)\Big)^{-k} = \Big(1+\frac{<n>}{k}\Big)^{-k}(1-ph))^{-k}, \tag{3}$$

1.2. Multiplicative properties of NBD and corresponding motion equations

An useful property of NBD with parameters $<n>,\ k$ is that it is (also) the distribution of a sum of k independent random variables, with mean $<n>/k$, drawn from a Bose-Einstein distribution[1]

$$\begin{aligned}
P_n &= \frac{1}{<n>+1}\Big(\frac{<n>}{<n>+1}\Big)^n \\
&= (e^{\beta\hbar\omega/2} - e^{-\beta\hbar\omega/2})e^{-\beta\hbar\omega(n+1/2)},\ T = \beta^{-1} = \frac{\hbar\omega}{\ln\frac{<n>+1}{<n>}} \\
&\sum_{n\geq 0} P_n = 1,\ \sum nP_n = <n> = \frac{1}{e^{\beta\hbar\omega-1}},\ T \simeq \hbar\omega <n>,\ <n> \gg 1, \\
P(x) &= \sum_n x^n P_n = (1+<n>(1-x))^{-1}.
\end{aligned} \tag{4}$$

This is easily seen from the generating function in (3), remembering that the generating function of a sum of independent random variables is the product of their generating functions.

Indeed, for

$$n = n_1 + n_2 + ... + n_k, \tag{5}$$

[1] A Bose-Einstein, or geometrical, distribution is a thermal distribution for single state systems.

with n_i independent of each other, the probability distribution of n is

$$P_n = \sum_{n_1,...,n_k} \delta(n - \sum n_i) p_{n_1}...p_{n_k},$$
$$P(x) = \sum_n x^n P_n = p(x)^k \tag{6}$$

This has a consequence that an incoherent superposition of N emitters that have a negative binomial distribution with parameters $k, <n>$ produces a negative binomial distribution with parameters $Nk, N<n>$.

So, for the GF of NBD we have (N=2)

$$F(k,<n>)F(k,<n>) = F(2k, 2<n>) \tag{7}$$

And more general formula (N=m) is

$$F(k,<n>)^m = F(mk, m<n>) \tag{8}$$

We can put this equation in the closed nonlocal form

$$Q_q F = F^q, \tag{9}$$

where

$$Q_q = q^D, \quad D = \frac{kd}{dk} + \frac{<n>d}{d<n>} = \frac{x_1 d}{dx_1} + \frac{x_2 d}{dx_2} \tag{10}$$

Note that temperature defined in (4) gives an estimation of the Glukvar temperature when it radiates hadrons. If we take $\hbar\omega = 100MeV$, to $T \simeq T_c \simeq 200MeV$ corresponds $<n> \simeq 1.5$ If we take $\hbar\omega = 10MeV$, to $T \simeq T_c \simeq 200MeV$ corresponds $<n> \simeq 20$. A singular behavior of $<n>$ may indicate corresponding phase transition and temperature. At that point we estimate characteristic quantum $\hbar\omega$. We see that universality of NBD in hadron-production is similar to the universality of black body radiation.

1.3. NBD motivated equations

For NBD distribution we have corresponding multiplication (convolution) formulas

$$(P \star P)_n \equiv \sum_{m=0}^{n} P_m(k,<n>)P_{n-m}(k,<n>)$$
$$= P_n(2k, 2<n>) = Q_2 P_n(k,<n>), ... \tag{11}$$

So, we can say, that star-product on the distributions of NBD corresponds ordinary product for GF.

It will be nice to have similar things for string field theory (SFT) [4].

SFT motion equation is

$$Q\Phi = \Phi \star \Phi \tag{12}$$

For stringfield GF F we may have

$$QF = F^2. \tag{13}$$

By construction we know the solution of the nice equation (9) as GF of NBD, F. We obtain corresponding differential equations, if we consider $q = 1 + \varepsilon$, for small ε,

$$\begin{aligned}
&(D(D-1)...(D-m+1) - (lnF)^m)\Psi = 0,\\
&\Big(\frac{\Gamma(D+1)}{\Gamma(D+1-m)} - (\ln F)^m\Big)\Psi = 0,\\
&(D_m - \Phi^m)\Psi = 0, m = 1, 2, 3, ...\\
&D_m = \frac{\Gamma(D+1)}{\Gamma(D+1-m)}, \Phi = \ln F,
\end{aligned} \tag{14}$$

with the solution $\Psi = F = \exp(\Phi)$. In the case of the NBD and p-adic string, we have correspondingly

$$\begin{aligned}
&D = \frac{x_1 d}{dx_1} + \frac{x_2 d}{dx_2};\\
&D = -\frac{1}{2}\triangle,\ \triangle = -\partial^2_{x_0} + \partial^2_{x_1} + ... + \partial^2_{x_{n-1}}.
\end{aligned} \tag{15}$$

These equations have meaning not only for integer m.

2. Riemann hypothesis

If I were to awaken after having slept for a thousand years,
my first question would be:
Has the Riemann hypothesis been proven?
David Hilbert

The Riemann hypothesis (RH), the most important open question in number theory and, possibly, in the whole of mathematics, was first formulated by Bernhard Riemann in 1859, was included in David Hilbert's list of challenging problems for 20th-century mathematicians, and is widely believed to be true. Yet a proof remains tantalizingly out of reach.

What the RH says is that the non-trivial zeros of the Riemann zeta-function all have real part equal to 1/2. Hilbert and Polya put forward the idea that the zeros of the Riemann zeta-function may have a spectral origin : the values of t_n such that $1/2 + it_n$ is a non trivial zero of ζ might be the eigenvalues of a self-adjoint operator. This would imply the RH.

2.1. Zeros of the Riemann zeta-function

The Riemann zeta-function $\zeta(s)$ is defined for complex $s = \sigma + it$ and $\sigma > 1$ by the expansion

$$\zeta(s) = \sum_{n\geq 1} n^{-s},\ Re\ s > 1,$$

$$= \delta_x^{-s} \frac{x}{1-x}|_{x\to 1} = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} e^{-\delta_x t} \frac{x}{1-x}|_{x\to 1}$$
$$= \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} e^{t\partial_\tau} \frac{1}{e^\tau - 1}|_{\tau\to 0}$$
$$= \frac{1}{\Gamma(s)} \int_0^\infty \frac{t^{s-1}dt}{e^t - 1},\ x = e^{-\tau},\ \delta_x = x\partial_x = \frac{xd}{dx} \tag{16}$$

All complex zeros, $s = \alpha + i\beta$, of $\zeta(\sigma + it)$ function lie in the critical stripe $0 < \sigma < 1$, symmetrically with respect to the real axe and critical line $\sigma = 1/2$. So, it is enough to investigate zeros with $\alpha \leq 1/2$ and $\beta > 0$. These zeros are of three type, with small, intermediate and big ordinates.

2.2. Riemann hypothesis

The Riemann hypothesis (RH) [6] states that the (non-trivial) complex zeros of $\zeta(s)$ lie on the critical line $\sigma = 1/2$.

At the beginning of the XX century Polya and Hilbert made a conjecture that the imaginary part of the Riemann zeros could be the oscillation frequencies of a physical system (ζ - (mem)brane). After the advent of Quantum Mechanics, the Polya-Hilbert conjecture was formulated as the existence of a self-adjoint operator whose spectrum contains the imaginary part of the Riemann zeros.

The Riemann hypothesis (RH) is a central problem in Pure Mathematics due to its connection with Number theory and other branches of Mathematics and Physics.

2.3. The functional equation for zeta-function

The functional equation is (see e.g. [6])

$$\zeta(1-s) = \frac{2\Gamma(s)}{(2\pi)^s} \cos\Big(\frac{\pi s}{2}\Big)\zeta(s) \tag{17}$$

From this equation we see the real (trivial) zeros of zeta-function:

$$\zeta(-2n) = 0,\ n = 1, 2, ... \tag{18}$$

Also, at s=1, zeta has pole with reside 1.

From Field theory and statistical physics point of view, the functional equation (17) is duality relation, with self dual (or critical) line in the complex plane, at $s = 1/2 + i\beta$,

$$\zeta\Big(\frac{1}{2} - i\beta\Big) = \frac{2\Gamma(s)}{(2\pi)^s} \cos\Big(\frac{\pi s}{2}\Big)\zeta\Big(\frac{1}{2} + i\beta\Big), \tag{19}$$

we see that complex zeros lie symmetrically with respect to the real axe.

On the critical line, (nontrivial) zeros of zeta corresponds to the infinite value of the free energy,

$$F = -T \ln \zeta. \tag{20}$$

At the point with $\beta = 14.134725...$ is located the first zero. In the interval $10 < \beta < 100$, zeta has 29 zeros. The first few million zeros have been computed and all lie on the critical line. It has been proved that uncountably many zeros lie on critical line.

The first relation of zeta-function with prime numbers is given by the following formula,

$$\zeta(s) = \prod_p (1 - p^{-s})^{-1}, \ Re\ s > 1. \tag{21}$$

Another formula, which can be used on critical line, is

$$\begin{aligned}
&\zeta(s) = (1 - 2^{1-s})^{-1} \sum_{n \geq 1} (-1)^{n+1} n^{-s}, \ Re\ s > 0 \\
&= \frac{e^{i\pi(\delta_x + 1)}}{(1 - 2^{1-s})\delta_x^s} \frac{x}{1-x}|_{x \to 1} \\
&= \frac{1}{1 - 2^{1-s}} \frac{1}{\Gamma(s)} \int_0^\infty dt t^{s-1} e^{i\pi} e^{(i\pi - t)\delta_x} \frac{1}{x^{-1} - 1}|_{x \to 1} \\
&= \frac{1}{1 - 2^{1-s}} \frac{1}{\Gamma(s)} \int_0^\infty dt t^{s-1} e^{(t - i\pi)\partial_\tau} \frac{e^{i\pi}}{e^\tau - 1}|_{\tau \to 0} \\
&= \frac{1}{1 - 2^{1-s}} \frac{1}{\Gamma(s)} \int_0^\infty \frac{t^{s-1} dt}{e^t + 1}, \\
&\int_0^\infty \frac{t^{s-1} dt}{e^t + 1} = \int_0^\infty dt t^{s-1} e^{-t} \sum_{n \geq 0} (-1)^n e^{-nt} = \Gamma(s) \sum_{n \geq 1} (-1)^{n+1} n^{-s}
\end{aligned} \tag{22}$$

2.4. From Qlike to zeta equations

Let us consider the values $q = n, n = 1, 2, 3, ...$ and take sum of the corresponding equations (9), we find [5]

$$\zeta(-D)F = \frac{F}{1 - F} \tag{23}$$

In the case of the NBD we know the solutions of this equation.

Now we invent a Hamiltonian H with spectrum corresponding to the set of nontrivial zeros of the zeta-function, in correspondence with Riemann hypothesis,

$$\begin{aligned}
&-D_n = \frac{n}{2} + iH_n, \ H_n = i\left(\frac{n}{2} + D_n\right), \\
&D_n = x_1\partial_1 + x_2\partial_2 + ... + x_n\partial_n, \ H_n^+ = H_n = \sum_{m=1}^n H_1(x_m), \\
&H_1(x) = i\left(\frac{1}{2} + x\partial_x\right) = -\frac{1}{2}(x\hat{p} + \hat{p}x), \ \hat{p} = -i\partial_x
\end{aligned} \tag{24}$$

The Hamiltonian $H = H_n$ is hermitian, its spectrum is real. The case $n = 1$ corresponds to the Riemann hypothesis.

The case $n = 2$, corresponds to NBD,

$$\zeta(1+iH_2)F = \frac{F}{1-F},\ \zeta(1+iH_2)|_F = \frac{1}{1-F},$$
$$F(x_1,x_2;h) = (1+\frac{x_1}{x_2}(1-h))^{-x_2} \tag{25}$$

Let us scale $x_2 \to \lambda x_2$ and take $\lambda \to \infty$ in (25), we obtain

$$\zeta(\frac{1}{2}+iH(x))e^{-\varepsilon x} = \frac{1}{e^{\varepsilon x}-1},$$
$$H(x) = i(\frac{1}{2}+x\partial_x) = -\frac{1}{2}(x\hat{p}+\hat{p}x),\ H^+ = H, \varepsilon = 1-h. \tag{26}$$

Now we scale $x \to xy$, multiply the equation by y^{s-1} and integrate

$$\begin{aligned}\zeta(\frac{1}{2}+iH(x))\int_0^\infty dye^{-\varepsilon xy}y^{s-1} &= \int_0^\infty dy\frac{y^{s-1}}{e^{\varepsilon xy}-1} = \frac{1}{(\varepsilon x)^s}\Gamma(s)\zeta(s),\\ &= \zeta(\frac{1}{2}+iH(x))\frac{1}{(\varepsilon x)^s}\Gamma(s)\end{aligned} \tag{27}$$

so

$$\zeta(\frac{1}{2}+iH(x))x^{-s} = \zeta(s)x^{-s} \Rightarrow H(x)\psi_E = E\psi_E,$$
$$\psi_E = cx^{-s},\ s = \frac{1}{2}+iE, \tag{28}$$

where the complex constant c is arbitrary, since the solutions are not square-integrable. To the normalization

$$\int_0^\infty dx\psi_E^*(x)\psi_{E'}(x) = \delta(E-E'), \tag{29}$$

corresponds $c = 1/\sqrt{2\pi}$.

We have seen that

$$\zeta(\frac{1}{2}+iH)e^{-\varepsilon x} = \frac{1}{e^{\varepsilon x}-1},$$
$$H = i(\frac{1}{2}+x\partial_x) = -\frac{1}{2}(xp+px), p = -i\partial_x, \tag{30}$$

than

$$e^{-\varepsilon x} = \int dEa(E,\varepsilon)\psi_E(x) = \int_{-\infty}^\infty dEx^{-1/2-iE}a(E,\varepsilon),$$
$$a(E,\varepsilon) = \frac{1}{2\pi}\int_0^\infty dxx^{-1/2+iE}e^{-\varepsilon x} = \frac{\varepsilon^{-1/2-iE}}{2\pi}\Gamma(1/2+iE);$$
$$\frac{1}{e^{\varepsilon x}-1} = \int dEb(E,\varepsilon)\psi_E(x),$$
$$b(E,\varepsilon) = \frac{1}{2\pi}\int_0^\infty dxx^{-1/2+iE}\frac{1}{e^{\varepsilon x}-1} = \frac{\varepsilon^{-1/2-iE}}{2\pi}\Gamma(1/2+iE)\zeta(\frac{1}{2}+iE),$$

$$\zeta(\frac{1}{2}+iH)\psi_E=\zeta(\frac{1}{2}+iE)\psi_E,\ \zeta(\frac{1}{2}+iE)a(E,\varepsilon)=b(E,\varepsilon) \tag{31}$$

There have been a number of approaches to understanding the Riemann hypothesis based on physics (for a comprehensive list see [7]). According to the idea of Berry and Keating, [2] the real solutions E_n of

$$\zeta(\frac{1}{2}+iE_n)=0, \tag{32}$$

are energy levels, eigenvalues of a quantum Hermitian operator (the Riemann operator) associated with the one-dimensional classical hyperbolic Hamiltonian

$$H_c=xp, \tag{33}$$

where x and p are the conjugate coordinate and momentum.

2.5. Some calculations with zeta-function values

From the equation (26) we have

$$\begin{aligned}&\zeta(\frac{1}{2}+iH(x))e^{-\varepsilon x}=\frac{1}{e^{\varepsilon x}-1},\ H=i(\frac{1}{2}+x\partial_x),\\&\zeta(-x\partial_x)(1-\varepsilon x+\frac{(\varepsilon x)^2}{2}+...)=\frac{1}{\varepsilon x}(1-(\frac{\varepsilon x}{2}+\frac{(\varepsilon x)^2}{6}+...)+\\&+(\frac{\varepsilon x}{2}+...)^2+...),\end{aligned} \tag{34}$$

so

$$\zeta(0)=-\frac{1}{2},\ \zeta(-1)=-\frac{1}{12},... \tag{35}$$

A little calculation shows that, the $(\varepsilon x)^2$ terms cancels on the r.h.s, in accordance with $\zeta(-2)=0$.

More curious question concerns with the term $1/\varepsilon x$ on the r.h.s. To it corresponds the term with actual infinitesimal coefficient on the l.h.s.

$$\frac{1}{\zeta(1)}\frac{1}{\varepsilon x}, \tag{36}$$

in the spirit of the nonstandard analysis (see, e.g. [3]), we can imagine that such a terms always present but on the l.h.s we may not note them.

For other values of zeta-function we will use the following expansion

$$\begin{aligned}&\frac{1}{e^x-1}=\frac{1}{x+\frac{x^2}{2}+\frac{x^3}{3!}+...}=\frac{1}{x}-\frac{1}{2}+\sum_{k\geq 1}\frac{B_{2k}x^{2k-1}}{(2k)!},\\&B_2=\frac{1}{6},\ B_4=-\frac{1}{30},\ B_6=\frac{1}{42},...\end{aligned} \tag{37}$$

and obtain

$$\zeta(1-2n)=-\frac{B_{2n}}{2n},\ n\geq 1. \tag{38}$$

REFERENCES

1. Aamodt K. et al. [ALICE collaboration] *Eur. Phys. J.* **C65** (2010) 111 [`arXiv:1004.3034`], [`arXiv:1004.3514`].

2. Berry M. Speculations on the Riemann operator, in proceedings of Symposium on Supersymmetry and Trace formulae, *Cambridge*, 1997.

3. Davis M. Applied nonstandard analysis, *New York*, 1977.

4. Kaku M. Strings, Conformal Fields, and M-Theory, *Springer, New York*, 2000.

5. Makhaldiani M. Renormdynamics, coupling constant unification and universal properties of the multiparticle production, *XXI International Baldin Seminar on High Energy Physics Problems* September 10-15, 2012, PoS(Baldin ISHEPP XXI)068.

6. Titchmarsh E.C. The Theory of the Riemann zeta-function, *Clarendon Press, Oxford*, 1986.

7. Watkins M. at http://secamlocal.ex.ac.uk/$\sim$ mwatkins/zeta/physics.htm.

Received 08.07.2013; revised 09.10.2013; accepted 29.11.2013.

Author's addresses:

N. Makhaldiani
Laboratory of Information Technologies
Joint Institute for Nuchlear Research
Dubna
E-mail: mnv@jinr.ru