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THE METHOD OF THE SMALL PARAMETER FOR NONLINEAR NON-SHALLOW SPHERICAL SHELLS

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Abstract. In the present paper we consider the geometrically nonlinear and non-shallow spherical shells. Using the method of the small parameter approximate solutions of I. Vekua's equations for approximations $N = 0$ are constructed.

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I. Vekua constructed several versions of the refined linear theory of thin and shallow shells, containing the regular processes, by means of the method of reduction of 3-D problems of elasticity to 2-D [1], [2]. This method for non-shallow shells in case of geometrical and physical non-linear theory was generalized by T. Meunargia [3], [4].

The equations of equilibrium of an elastic shell-type body in a vector form which is convenient for the reduction to the tow-dimensional equations:

$$\hat{\nabla}_i \boldsymbol{\sigma}^i + \boldsymbol{\Phi} = 0, \tag{1}$$

where g is the discriminant of the metric quadratic form of the space, $\hat{\nabla}_i$ are covariant derivatives with respect to the space coordinates x^i, $\boldsymbol{\Phi}$ is an external force, $\boldsymbol{\sigma}^i$ are the contravariant constituents of the stress vectors, x^1, x^2, x^3 are curvilinear coordinates.

Hooke's law has the following form:

$$\boldsymbol{\sigma}^i = E^{ijpq} e_{pq} \left(\boldsymbol{R}_j + \partial_j \boldsymbol{U} \right), \tag{2}$$

where e_{pq} are covariant components of the strain tensor

$$e_{pq} = \frac{1}{2}\Big(\boldsymbol{R}_p \partial_q \boldsymbol{U} + \boldsymbol{R}_q \partial_p \boldsymbol{U} + \partial_p \boldsymbol{U} \partial_q \boldsymbol{U} \Big),$$

coefficients of elasticity of the first order for isotropic elastic bodies are expressed by the Lame coefficients

$$E^{ijpq} = \lambda g^{ij} g^{pq} + \mu \left(g^{ip} g^{jq} + g^{iq} g^{ip} \right) \quad \left(g^{ij} = \boldsymbol{R}^i \boldsymbol{R}^j \right),$$

$\boldsymbol{R}_i$ and $\boldsymbol{R}^i$ are covariant and contravariant base vectors of the space and $\boldsymbol{U}$ is the displacement vector.

we consider the system of equilibrium equations of the two-dimensional geometrically nonlinear and non-shallow spherical shells which is obtained from the three-dimensional problems of the theory of elasticity for isotropic and homogeneous shell by the method of I. Vekua.

The system of equilibrium equations of the two-dimensional geometrically nonlinear and non-shallow spherical shells may be written in the following form (approximation $N=0$) [3]:

$$\begin{aligned} &\nabla_\alpha \overset{(0)}{\sigma}{}^{\alpha\beta} + \varepsilon \overset{(0)}{\sigma}{}^{\beta 3} + \overset{(0)}{F}{}^{\beta} = 0, \\ &\nabla_\alpha \overset{(0)}{\sigma}{}^{\alpha 3} - \varepsilon \left(\overset{(0)}{\sigma}{}_1^1 + \overset{(0)}{\sigma}{}_2^2 \right) + \overset{(0)}{F}{}^{3} = 0, \end{aligned} \tag{3}$$

where

$$\overset{(0)}{\boldsymbol{F}} = \overset{(0)}{\boldsymbol{\Phi}} + \frac{1}{2h}\left[(1+\varepsilon)^2 \overset{(+)}{\boldsymbol{\sigma}}{}^3 - (1-\varepsilon)^2 \overset{(-)}{\boldsymbol{\sigma}}{}^3 \right],$$

$$\left(\overset{(0)}{\sigma}{}^{ij},\ \overset{(0)}{\boldsymbol{\Phi}} \right) = \frac{1}{2h}\int_{-h}^{h} \left(1+\frac{x_3}{R_0}\right)^2 (\sigma^{ij},\ \boldsymbol{\Phi}) dx_3, \quad \boldsymbol{u} = \frac{1}{2h}\int_{-h}^{h} \boldsymbol{U} dx_3,$$

$$\boldsymbol{\sigma}^3(x_1,x_2,\pm h) = \overset{(\pm)}{\boldsymbol{\sigma}}{}^3.$$

Here σ^{ij} are contravariant components of the stress tensor, P_m are Legendre polynomials of order m, $\varepsilon = \dfrac{h}{R_0}$ is a small parameter, R_0 is the radius of the midsurface of the sphere.

Hooke's law has the form:

$$\begin{aligned} &\overset{(0)}{\boldsymbol{\sigma}}{}^{\alpha} = \lambda(\boldsymbol{r}^\gamma \partial_\gamma \boldsymbol{u})\boldsymbol{r}^\alpha + \mu\Big[2(\boldsymbol{r}^\alpha \partial^\alpha \boldsymbol{u})\boldsymbol{r}_\alpha + (\boldsymbol{r}^\alpha \partial^\beta \boldsymbol{u})\boldsymbol{r}_\beta + (\boldsymbol{r}^\beta \partial^\alpha \boldsymbol{u})\boldsymbol{r}_\beta \\ &+(\boldsymbol{n}\partial^\alpha \boldsymbol{u})\boldsymbol{n}\Big] + \left(1+\frac{\varepsilon^2}{3}+\frac{\varepsilon^4}{5}+\cdots\right)\Big\{\lambda\Big[(\boldsymbol{r}^\gamma\partial_\gamma \boldsymbol{u})\partial^\alpha \boldsymbol{u} \\ &+\frac{1}{2}(\partial^\gamma \boldsymbol{u}\partial_\gamma \boldsymbol{u})\boldsymbol{r}^\alpha\Big] + \mu[2(\boldsymbol{r}^\alpha\partial^\alpha \boldsymbol{u})\partial_\alpha \boldsymbol{u} + (\partial^\alpha \boldsymbol{u}\partial_\alpha \boldsymbol{u})\boldsymbol{r}^\alpha + (\boldsymbol{r}^\alpha \partial^\beta \boldsymbol{u})\partial_\beta \boldsymbol{u} \\ &+(\boldsymbol{r}^\beta\partial^\alpha \boldsymbol{u})\partial_\beta \boldsymbol{u} + (\boldsymbol{r}^\beta \partial^\alpha \boldsymbol{u})\partial_\beta \boldsymbol{u} + (\partial^\beta \boldsymbol{u}\partial^\alpha \boldsymbol{u})\boldsymbol{r}_\beta]\Big\} + \\ &(1+\varepsilon^2+\varepsilon^4+\cdots) \\ &\times\left\{\frac{\lambda}{2}(\partial^\gamma \boldsymbol{u}\partial_\gamma \boldsymbol{u})\partial^\alpha \boldsymbol{u} + \mu(\partial^\alpha \boldsymbol{u}\partial^\alpha \boldsymbol{u})\partial_\alpha \boldsymbol{u} + \mu(\partial^\beta \boldsymbol{u}\partial^\alpha \boldsymbol{u})\partial_\beta \boldsymbol{u}\right\}, \\ &\overset{(0)}{\boldsymbol{\sigma}}{}_3 = \lambda\left[\boldsymbol{r}^\gamma\partial_\gamma \boldsymbol{u} + \frac{1}{2}\partial^\gamma \boldsymbol{u}\partial_\gamma \boldsymbol{u}\right]\boldsymbol{n} + \mu(\boldsymbol{n}\partial^\gamma \boldsymbol{u})(\boldsymbol{r}_\gamma + \partial_\gamma \boldsymbol{u}), \end{aligned} \tag{4}$$

$$(\alpha \neq \beta, \quad \alpha,\beta = 1,2, \quad \gamma = 1,2).$$

Introduce the notations

$$\overset{(0)}{\boldsymbol{\sigma}}{}_i = \boldsymbol{T}_i, \quad \overset{(0)}{\boldsymbol{F}} = \boldsymbol{X}'.$$

To find components of the displacements vector and stress tensor, we take the following series of expansions with respect to the small parameter ε [5], [6]:

$$(u_i,\ \boldsymbol{T}_i,\ X_i) = \sum_{k=1}^{\infty} (\overset{(k)}{u}{}_i, \overset{(k)}{\boldsymbol{T}}{}_i, \overset{(k)}{X}{}'_i)\varepsilon^k, \tag{5}$$

Substituting the above expansions into relations (3), (4) and then equalizing the coefficients of expansions for ε^n, we obtain the following system of equations:

$$\begin{aligned}&4\mu\partial_{\bar z}\left(\frac{1}{\Lambda}\partial_z \overset{(k)}{u}_+\right)+2(\lambda+\mu)\partial_{\bar z}\overset{(k)}{\theta}=\overset{(k)}{X}_+\left(\overset{(1)}{u}_i,\ldots,\overset{(k-1)}{u}_i\right),\\ &\mu\nabla^2\overset{(k)}{u}_3=\overset{(k)}{X}_3\left(\overset{(1)}{u}_i,\ldots,\overset{(k-1)}{u}_i\right),\end{aligned}\tag{6}$$

where

$$x^1=\tan\frac{\theta}{2}\cos\varphi,\quad x^2=\tan\frac{\theta}{2}\sin\varphi,$$

are the isometric coordinates on the shell midsurface of the spherical shell, θ and φ are the geographical coordinates:

$$z=x^1+ix^2,\quad \Lambda=\frac{4R_0^2}{(1+z\bar z)^2},\quad \nabla^2=\frac{4}{\Lambda}\partial^2_{z\bar z}$$

$$\overset{(k)}{u}_+=\overset{(k)}{u}_1+i\overset{(k)}{u}_2,$$

$$\overset{(k)}{\theta}=\frac{1}{\Lambda}\left(\partial_z\overset{(k)}{u}_++\partial_{\bar z}\overset{(k)}{\overline{u}}_+\right).$$

Introducing the well-known differential operators

$$\partial_z=\frac{1}{2}\left(\partial x^1-i\partial x^2\right),\quad \partial_{\bar z}=\frac{1}{2}\left(\partial x^1+i\partial x^2\right).$$

$\overset{(k)}{X}_+$ and $\overset{(k)}{X}_3$ are expressed by $\overset{(1)}{u}_+, \overset{(1)}{u}_3, \ldots, \overset{(k-1)}{u}_+, \overset{(k-1)}{u}_3$ and it is assumed that they are already found.

When deducing the system (6) we used the formula [1]

$$\frac{1}{\Lambda}\partial_z\Lambda\partial_{\bar z}U^+=\partial_{\bar z}\left(\frac{1}{\Lambda}\partial_z U_+\right)+2K\varepsilon^2U_+,$$

where K is the Gaussian curvature of the midsurface of the shell.

Simple calculations show that general solutions of the system (6) can be represented by means of three analytic functions of z in the form

$$\begin{aligned}\overset{(k)}{u}_+=&-\frac{\varkappa}{\pi}\iint_D\frac{\Lambda(\zeta,\bar\zeta)\varphi'(\zeta)d\xi d\eta}{\bar\zeta-\bar z}+\left(\frac{1}{\pi}\iint_D\frac{\Lambda(\zeta,\bar\zeta)d\xi d\eta}{\bar\zeta-\bar z}\right)\overline{\varphi'(z)}\\ &-\overline{\psi(z)}+\frac{1}{8\mu h^2}\frac{\lambda+\mu}{\lambda+2\mu}\frac{1}{\pi}\iint_D\frac{\overset{(k)}{F}_+(\zeta,\bar\zeta)d\xi d\eta}{\bar\zeta-\bar z}\end{aligned}\tag{7}$$

$$\overset{(k)}{u}_3=f(z)+\overline{f(z)}-\frac{2}{\pi}\iint_D\overset{(k)}{X}_3\ln|\zeta-z|d\xi d\eta.\tag{8}$$

where $\varphi'(z), f(z)$ and $\psi(z)$ are analytic functions of $z = x_1 + ix_2 \in D$, and $\zeta = \xi + i\eta$.

Further,

$$\overset{(k)}{F}_+(z,\bar{z}) = -\frac{1}{\pi}\iint\limits_D \left(\frac{\overline{\overset{(k)}{X}_+}}{\bar{\zeta}-\bar{z}} - \frac{\varkappa \overset{(k)}{X}_+}{\zeta - z} \right) d\xi d\eta, \quad \left(\varkappa = \frac{\lambda+3\mu}{\lambda+\mu}\right).$$

D is the domain of the plane Ox_1x_2 onto which the midsurface S of the shell Ω is mapped topologically.

Here we present a general scheme of solution of the boundary value problem when the domain D is a circle of radius r_0.

For the first boundary problem (in stresses):

$$\overset{(k)}{T}_{ll} + i\overset{(k)}{T}_{ls} = (\lambda+\mu)\overset{(k)}{\theta} + 2\mu\partial_{\bar{z}}\Big(\frac{1}{\Lambda}\overset{(k)}{u}_+\Big)\frac{d\bar{z}}{dz} = \overset{(k)}{P}_+, \quad |z| = r_0, \tag{9}$$

$$\overset{(k)}{T}_{ln} = \frac{2\mu}{\Lambda}\left(\partial_z \overset{(k)}{u}_3 e^{i\varphi} + \partial_{\bar{z}}\overset{(k)}{u}_3 e^{-i\varphi}\right) = \overset{(k)}{P}_3, \quad |z| = r_0, \tag{10}$$

where $\overset{(k)}{P}_+$ and $\overset{(k)}{P}_3$ are the known values expressed in terms of the solutions $\overset{(k)}{u}_i$ $(k = 1, 2, ..., n-1)$, of the previous approximations.

Let the expansions

$$\varphi'(z) = \sum_{n=0}^{\infty} a_n z^n, \quad \psi(z) = \sum_{n=0}^{\infty} b_n z^n, \quad f(z) = \sum_{n=0}^{\infty} c_n z^n$$

$$\hat{P}_+ = \sum_{n=-\infty}^{\infty} A_k e^{ik\theta}, \quad \hat{P}_3 = \sum_{n=-\infty}^{\infty} B_k e^{ik\theta}$$

hold. Here $\hat{P}_+$ and $\hat{P}_3$ are expressed by the particular solutions of equation (6) and by means of $\overset{(k)}{P}_+$ and $\overset{(k)}{P}_3$.

From the boundary condition and comparing the coefficients for $e^{in\varphi}$ and taking into account that the resultant vector and principal moment are equal to zero, we obtain

$$a_n = \frac{A_n}{2\mu}\frac{1}{[1+2\varkappa(1+r_0^2)r_0^2]r_0^n} \quad (n = 0, 1, ...),$$

$$b_n = \frac{1}{2\mu}\frac{1}{(1+r_0^2)(n+2r_0^2)r_0^{n-1}}\Big[A_{n-1}\frac{1-(k+1+(k+2)r_0^2)}{1+2\varkappa(1+r_0^2)r_0^2\beta_{n+1}(r_0)} - \bar{A}_{-n-1}\Big],$$

where

$$\beta_n(r) = \frac{1}{z^{n+2}}\int\limits_0^z \frac{(z-t)t^n dt}{(1+z\bar{z})^3}$$

and for the coefficients c_n, we have

$$c_n = \frac{1}{2\mu}\frac{R_0}{1-r_0^2}\frac{B_n}{nr_0^{n-1}}.$$

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