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ON A MODEL OF LAYERED PRISMATIC SHELLS

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Abstract. The present paper is devoted to a model for elastic layered prismatic shells. Using I. Vekua's dimension reduction method [1-3] hierarchical models for elastic layered prismatic shells are constructed. For each layer we construct hierarchical models assuming to be known stress vector components on the face surfaces of the layered body (structure) under consideration, while we calculate the values of X_{ij} and u_i on the interfaces from their Fourier-Legendre expansions there. So, we get coupled governing systems for the whole structure in the projection of the structure. Analogously, hierarchical models can be constructed, when on the face surfaces of the prismatic body displacements or mixed stress and displacement vector components are assumed to be known.

For the sake of simplicity we consider the case of two plies, in the zeroth approximation (hierarchical model).

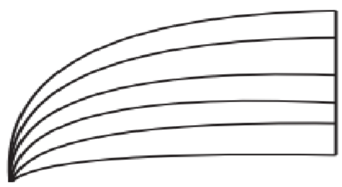

Figure 1. Cross-section of a cusped layered prismatic shell

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Let a layered prismatic shell consist of two prismatic shells as plies with the upper $\overset{(+)}{h}_{\gamma}(x_1, x_2)$, $\gamma = 1, 2$, and lower $\overset{(-)}{h}_{\gamma}(x_1, x_2)$, $\gamma = 1, 2$, face surfaces, herewith,

$$\overset{(+)}{h}_{2}(x_1, x_2) \equiv \overset{(-)}{h}_{1}(x_1, x_2), \qquad (x_1, x_2) \in \omega,$$

where ω is the common for the both prismatic shells projection on the plane $x_3 = 0$. S denotes the joint lateral cylindrical surface parallel to the x_3-axis according to the definition of prismatic shells (see [1-3]). Allowing cusped edges for each ply, evidently, the thickness of the plies

$$2h_{\gamma} := \overset{(+)}{h}_{\gamma}(x_1, x_2) - \overset{(-)}{h}_{\gamma}(x_1, x_2) \geq 0, \quad \gamma = 1, 2,$$

and the thickness of the layered prismatic shell

$$\begin{aligned} 2h &:= \overset{(+)}{h}_{1}(x_1, x_2) - \overset{(-)}{h}_{2}(x_1, x_2) \\ &= \overset{(+)}{h}_{1}(x_1, x_2) - \overset{(-)}{h}_{1}(x_1, x_2) + \overset{(-)}{h}_{1}(x_1, x_2) - \overset{(-)}{h}_{2}(x_1, x_2) = 2h_1 + 2h_2 \geq 0. \end{aligned}$$

Let

$$X^{\gamma}_{ijl}(x_1,x_2,t),\ \ e^{\gamma}_{ijl}(x_1,x_2,t),\ \ u^{\gamma}_{il}(x_1,x_2,t),\ \ \gamma=1,2,$$

be l-th order moments of the stress X^{γ}_{ij} and strain e^{γ}_{ij} tensor and displacement vector u^{γ}_{i}components of the plies.

Under well-known restrictions the following Fourier-Legendre expansions are convergent

$$\begin{aligned}\left(X^{\gamma}_{ij},e^{\gamma}_{ij},u^{\gamma}_{i}\right)\ (x_1,x_2,x_3,t)&=\sum_{l=0}^{\infty}a_{\underline{\gamma}}\left(l+\frac{1}{2}\right)\\ &\times\left(X^{\gamma}_{ijl},e^{\gamma}_{ijl},u^{\gamma}_{il}\right)\ (x_1,x_2,t)P_l(a_{\gamma}x_3-b_{\gamma})dx_3,\end{aligned}\tag{1}$$

where

$$\begin{aligned}&\left(X^{\gamma}_{ijl},e^{\gamma}_{ijl},u^{\gamma}_{il}\right)\ (x_1,x_2,t)\\ &=\int\limits_{\overset{(-)}{h}}^{\overset{(+)}{h}}\left(X^{\gamma}_{ij},e^{\gamma}_{ij},u^{\gamma}_{i}\right)\ (x_1,x_2,x_3,t)P_l(a_{\underline{\gamma}}x_3-b_{\underline{\gamma}})dx_3,\quad l=0,1,2,...,\end{aligned}\tag{2}$$

$$a_{\gamma}:=\frac{1}{h_{\gamma}},\quad b_{\gamma}:=\frac{\tilde{h}_{\gamma}}{h_{\underline{\gamma}}},\quad 2\tilde{h}_{\gamma}:=\overset{(+)}{h}_{\gamma}+\overset{(-)}{h}_{\gamma},\quad \gamma=1,2.$$

A bar under one of repeated indices means that in this case we do not use Einstein's summation convention.

Physical problem. Determine the stress-strain state of the elastic layered prismatic shell considered as a three-dimensional (3D) body (plies and the whole body may occupy non-Lipschitz domains), when on the face surfaces of the body stress-vectors

$$Q_{\overset{(+)}{\nu_1}i}(x_1,x_2,\overset{(+)}{h}_1(x_1,x_2),t)\ \text{and}\ Q_{\overset{(-)}{\nu_2}i}(x_1,x_2,\overset{(-)}{h}_2(x_1,x_2),t)\ \text{are known,}\tag{3}$$

where $\overset{(+)}{\nu_1}$ and $\overset{(-)}{\nu_2}$ are outward to the body normals to the upper (of the first ply) and lower (of the second ply) face surfaces ($\overset{(+)}{\nu_2}\equiv-\overset{(-)}{\nu_1}$), on the lateral surfaces arbitrary admissible boundary conditions (BC) are prescribed, and on the interface conditions of the type of the following conditions are fulfilled:

$$\begin{aligned}&\left(X^{1}_{ji}\overset{(+)}{\nu}_{2i},u^{1}_{j}\right)\left(x_1,x_2,x_3=\overset{(-)}{h}_1(x_1,x_2)\equiv\overset{(+)}{h}_2(x_1,x_2),t\right)\\ &=\left(X^{2}_{ji}\overset{(+)}{\nu}_{2i},u^{2}_{j}\right)\left(x_1,x_2,x_3=\overset{(+)}{h}_2(x_1,x_2)\equiv\overset{(-)}{h}_1(x_1,x_2),t\right),\quad j=1,2,3.\end{aligned}\tag{4}$$

For clearness, on the lateral surface $S=S_1\bigcup S_2$ we confine ourselves to boundary conditions (BCs) in displacements

$$u^{\gamma}_{i}\Big|_{S_{\underline{\gamma}}},\quad \gamma=1,2,\ \ \text{are prescribed.}\tag{5}$$

Integrating the governing equations of the 3D linear theory of elasticity for each ply, we get [4]

$$
\begin{aligned}
&X^{\gamma}_{j\beta0,\beta}(x_1,x_2,t) - X^{\gamma}_{j\beta}\left(x_1,x_2,\overset{(+)}{h}_{\gamma}(x_1,x_2),t\right)\overset{(+)}{h}_{\underline{\gamma},\beta} \\
&+X^{\gamma}_{j3}\left(x_1,x_2,\overset{(+)}{h}_{\gamma}(x_1,x_2),t\right) + X^{\gamma}_{j\beta}\left(x_1,x_2,\overset{(-)}{h}_{\gamma}(x_1,x_2),t\right)\overset{(-)}{h}_{\underline{\gamma},\beta} \\
&-X^{\gamma}_{j3}\left(x_1,x_2,\overset{(-)}{h}_{\gamma}(x_1,x_2),t\right) + \Phi^{\gamma}_{j0}(x_1,x_2,t) = \rho\ddot{u}^{\gamma}_{j0}(x_1,x_2,t), \\
&\qquad\qquad j=1,2,3, \quad \gamma=1,2,
\end{aligned}
\tag{6}
$$

where Φ^{γ}_{j0}, $\gamma=1,2$, are the zero moments of the volume force components for each ply;

$$
X^{\gamma}_{ij0}(x_1,x_2,t) = \lambda e^{\gamma}_{kk0}(x_1,x_2,t)\,\delta_{ij} + 2\mu e^{\gamma}_{ij0}(x_1,x_2,t), \quad i,j=1,2,3; \tag{7}
$$

$$
\begin{aligned}
&e^{\gamma}_{i\beta0}(x_1,x_2,t) \\
&= \frac{1}{2}\left[u^{\gamma}_{i0,\beta}(x_1,x_2,t) + u^{\gamma}_{i}\left(x_1,x_2,\overset{(-)}{h}_{\gamma},t\right)\overset{(-)}{h}_{\underline{\gamma},\beta} - u^{\gamma}_{i}\left(x_1,x_2,\overset{(+)}{h}_{\gamma},t\right)\overset{(+)}{h}_{\underline{\gamma},\beta}\right]
\end{aligned}
\tag{8}
$$

$$
+\frac{1}{2}\begin{cases}
u^{\gamma}_{\beta0,\alpha}(x_1,x_2,t) - u^{\gamma}_{\beta}\left(x_1,x_2,\overset{(+)}{h}_{\gamma},t\right)\overset{(+)}{h}_{\underline{\gamma},\alpha} + u^{\gamma}_{\beta}\left(x_1,x_2,\overset{(-)}{h}_{\gamma},t\right)\overset{(-)}{h}_{\underline{\gamma},\alpha}, & \alpha=1,2, \\
u^{\gamma}_{\beta}\left(x_1,x_2,\overset{(+)}{h}_{\underline{\gamma}},t\right) - u^{\gamma}_{\beta}\left(x_1,x_2,\overset{(-)}{h}_{\underline{\gamma}},t\right), & i=3,
\end{cases}
$$

$$
e^{\gamma}_{330}(x_1,x_2,t) = u^{\gamma}_{3}\left(x_1,x_2,\overset{(+)}{h}_{\underline{\gamma}},t\right) - u^{\gamma}_{3}\left(x_1,x_2,\overset{(-)}{h}_{\underline{\gamma}},t\right), \quad \gamma=1,2. \tag{9}
$$

In the zeroth approximation it is assumed that (see (1))

$$
u^{\gamma}_{i}(x_1,x_2,x_3,t) \cong \frac{u^{\gamma}_{i0}(x_1,x_2,t)}{2h_{\underline{\gamma}}} =: \frac{1}{2}v^{\gamma}_{i0}(x_1,x_2,t), \quad x_3 \in \left[\overset{(-)}{h}_{\gamma}, \overset{(+)}{h}_{\gamma}\right], \quad \gamma=1,2, \quad i=1,2,3. \tag{10}
$$

$$
\left(X^{\gamma}_{ij}, e^{\gamma}_{ij}\right)(x_1,x_2,x_3,t) \cong \frac{1}{2h_{\underline{\gamma}}}\left(X^{\gamma}_{ij0}, e^{\gamma}_{ij0}\right)(x_1,x_2,t), \quad x_3 \in \left[\overset{(-)}{h}_{\gamma}, \overset{(+)}{h}_{\gamma}\right], \quad \gamma=1,2, \quad i,j=1,2,3. \tag{11}
$$

From (9), (10) it follows that

$$
e^{\gamma}_{330} \cong 0, \quad \gamma=1,2. \tag{12}
$$

From (8), (10), (12) it follows that

$$
\begin{aligned}
&e^{\gamma}_{i\beta0}(x_1,x_2,t) \cong \frac{1}{2}[u^{\gamma}_{i0,\beta}(x_1,x_2,t) + \frac{1}{2}v^{\gamma}_{i0}(x_1,x_2,t)\overset{(-)}{h}_{\underline{\gamma},\beta} - \frac{1}{2}v^{\gamma}_{i0}(x_1,x_2,t)\overset{(+)}{h}_{\underline{\gamma},\beta}] \\
&+\frac{1}{2}\begin{cases}
u^{\gamma}_{\beta0,\alpha}(x_1,x_2,t) - \frac{1}{2}v^{\gamma}_{\beta0}(x_1,x_2,t)\overset{(+)}{h}_{\underline{\gamma},\alpha} + \frac{1}{2}v^{\gamma}_{\beta0}(x_1,x_2,t)\overset{(-)}{h}_{\underline{\gamma},\alpha}, & i=\alpha, \\
0, & i=3,
\end{cases} \\
&\qquad\qquad \gamma=1,2.
\end{aligned}
\tag{13}
$$

In what follows instead of "≅" we write "=".
For $i = \alpha$ from (13) we obtain

$$e^{\gamma}_{\alpha\beta 0} = \frac{1}{2}\left[\left(h_{\underline{\gamma}} v^{\gamma}_{\alpha 0}\right)_{,\beta} - v^{\gamma}_{\alpha 0} h_{\underline{\gamma},\beta} + \left(h_{\underline{\gamma}} v^{\gamma}_{\beta 0}\right)_{,\alpha} - v^{\gamma}_{\beta 0} h_{\underline{\gamma},\alpha}\right] = \frac{1}{2} h_{\underline{\gamma}} \left(v^{\gamma}_{\alpha 0,\beta} + v^{\gamma}_{\beta 0,\alpha}\right), \qquad \alpha,\beta,\gamma = 1,2. \tag{14}$$

For $i = 3$ from (13) we obtain

$$e^{\gamma}_{\beta 30} \equiv e^{\gamma}_{3\beta 0} = \frac{1}{2}\left[\left(h_{\underline{\gamma}} v^{\gamma}_{30}\right)_{,\beta} - h_{\underline{\gamma},\beta} v^{\gamma}_{30}\right] = \frac{1}{2} h_{\underline{\gamma}} v^{\gamma}_{30,\beta}, \quad \beta,\gamma = 1,2. \tag{15}$$

From (14) it follows that

$$e^{\gamma}_{\alpha\alpha 0} = h_{\underline{\gamma}} v^{\gamma}_{\alpha 0,\alpha}, \qquad \gamma = 1,2. \tag{16}$$

Taking into account (3), (10), and (11) from (6) we have

$$\begin{aligned} &X^{1}_{j\beta 0,\beta}(x_1,x_2,t) + Q_{\overset{(+)}{\nu_1} j}(x_1,x_2,t)\sqrt{\left(\overset{(+)}{h}_{1,1}\right)^2 + \left(\overset{(+)}{h}_{1,2}\right)^2 + 1} \\ &+\frac{1}{2h_1}\left[X^{1}_{j\beta 0}(x_1,x_2,t)\overset{(-)}{h}_{1,\beta} - X^{1}_{j30}(x_1,x_2,t)\right] + \Phi^{1}_{j0}(x_1,x_2,t) \\ &= \rho \ddot{u}^{1}_{j0}(x_1,x_2,t), \qquad j = 1,2,3; \end{aligned} \tag{17}$$

$$\begin{aligned} &X^{2}_{j\beta 0,\beta}(x_1,x_2,t) - \frac{1}{2h_2}\left[X^{2}_{j\beta 0}(x_1,x_2,t)\overset{(+)}{h}_{2,\beta} - X^{2}_{j30}(x_1,x_2,t)\right] \\ &+Q_{\overset{(-)}{\nu_2} j}(x_1,x_2,t)\sqrt{\left(\overset{(-)}{h}_{2,1}\right)^2 + \left(\overset{(-)}{h}_{2,2}\right)^2 + 1} \\ &+\Phi^{2}_{j0}(x_1,x_2,t) = \rho \ddot{u}^{2}_{j0}(x_1,x_2,t), \qquad j = 1,2,3. \end{aligned} \tag{18}$$

From (7), by virtue of (12), (14), (15), (16), it follows that

$$X^{\gamma}_{\alpha\beta 0} = \lambda h_{\underline{\gamma}} v^{\gamma}_{\delta 0,\delta}\delta_{\alpha\beta} + \mu h_{\underline{\gamma}}\left(v^{\gamma}_{\alpha 0,\beta} + v^{\gamma}_{\beta 0,\alpha}\right), \qquad \alpha,\beta,\gamma = 1,2; \tag{19}$$

$$X^{\gamma}_{3\beta 0} \equiv X^{\gamma}_{\beta 30} = 2\mu e^{\gamma}_{3\beta 0} = \mu h_{\underline{\gamma}} v^{\gamma}_{30,\beta}, \qquad \beta,\gamma = 1,2; \tag{20}$$

$$X^{\gamma}_{330} = \lambda e^{\gamma}_{\beta\beta} + 2\mu e^{\gamma}_{330} = \lambda h_{\underline{\gamma}} v^{\gamma}_{\beta 0,\beta}, \qquad \gamma = 1,2. \tag{21}$$

Thus, from (17), (19), (20) for $j = \alpha$ it follows that

$$\begin{aligned} &\mu\left[\left(h_1 v^{1}_{\alpha 0,\beta}\right)_{,\beta} + \left(h_1 v^{1}_{\beta 0,\alpha}\right)_{,\beta}\right] + \lambda\left(h_1 v^{1}_{\beta 0,\beta}\right)_{,\alpha} \\ &+\tfrac{1}{2}\left\{\overset{(-)}{h}_{1,\beta}\left[\lambda v^{1}_{\gamma,\gamma}\delta_{\alpha\beta} + \mu\left(v^{1}_{\alpha 0,\beta} + v^{1}_{\beta 0,\alpha}\right)\right] - \mu v^{1}_{30,\alpha}\right\} \\ &+Q_{\overset{(+)}{v_1}\alpha}\sqrt{\left(\overset{(+)}{h}_{1,1}\right)^2 + \left(\overset{(+)}{h}_{1,2}\right)^2 + 1} + \Phi^{1}_{\alpha 0} = \rho h_1 \ddot{v}^{1}_{\alpha 0}, \quad \alpha = 1,2; \end{aligned} \tag{22}$$

and from (17), (20), (21) for $j = 3$ it follows that

$$\begin{aligned}&\mu\left(h_1 v^1_{30,\beta}\right)_{,\beta} + \frac{1}{2}\left(\mu \overset{(-)}{h}_{1,\beta} v^1_{30,\beta} - \lambda v^1_{\beta 0,\beta}\right)\\&+Q_{\overset{(+)}{\nu_1} 3}\sqrt{\left(\overset{(+)}{h}_{1,1}\right)^2 + \left(\overset{(+)}{h}_{1,2}\right)^2 + 1} + \Phi^1_{30} = \rho h_1 \ddot{v}^1_{30}.\end{aligned} \tag{23}$$

On the one hand, because of the first condition on the interface (see (4)), according to (11), in the zeroth approximation

$$\frac{1}{2h_1} X^1_{ji0} \overset{(+)}{\nu_{2i}} = X^1_{ji} \overset{(+)}{\nu_{2i}} = X^2_{ji} \overset{(+)}{\nu_{2i}} = \frac{1}{2h_2} X^2_{ji0} \overset{(+)}{\nu_{2i}}, \qquad j = 1, 2, 3,$$

which are known after solving an initial boundary value problem (IBVP) for the first ply. On the other hand,

$$\begin{aligned}\overset{(+)}{\nu_{2\beta}} &= \frac{-\overset{(+)}{h}_{2,\beta}}{\sqrt{\left(\overset{(+)}{h}_{2,1}\right)^2 + \left(\overset{(+)}{h}_{2,2}\right)^2 + 1}}, \qquad \beta = 1, 2,\\ \overset{(+)}{\nu_{23}} &= \frac{1}{\sqrt{\left(\overset{(+)}{h}_{2,1}\right)^2 + \left(\overset{(+)}{h}_{2,2}\right)^2 + 1}}.\end{aligned}$$

That is why,

$$\begin{aligned}&-\frac{1}{2h_2}\left[X^2_{j\beta 0}(x_1, x_2, t)\overset{(+)}{h}_{2,\beta} - X^2_{j30}(x_1, x_2, t)\right]\\&= \frac{1}{2h_2} X^2_{ji0}(x_1, x_2, t) \overset{(+)}{\nu_{2\,i}} \sqrt{\left(\overset{(+)}{h}_{2,1}\right)^2 + \left(\overset{(+)}{h}_{2,2}\right)^2 + 1}\\&= X^2_{ji} \overset{(+)}{\nu_{2\,i}} \sqrt{\left(\overset{(+)}{h}_{2,1}\right)^2 + \left(\overset{(+)}{h}_{2,2}\right)^2 + 1}\\&= X^1_{ji} \overset{(+)}{\nu_{2\,i}} \sqrt{\left(\overset{(+)}{h}_{2,1}\right)^2 + \left(\overset{(+)}{h}_{2,2}\right)^2 + 1}\\&= \frac{1}{2h_1} X^1_{ji0}(x_1, x_2, t) \overset{(+)}{\nu_{2\,i}} \sqrt{\left(\overset{(+)}{h}_{2,1}\right)^2 + \left(\overset{(+)}{h}_{2,2}\right)^2 + 1}\\&=: Q^1_{\overset{(+)}{\nu_2}\, j} \frac{1}{2h_1}\sqrt{\left(\overset{(+)}{h}_{2,1}\right)^2 + \left(\overset{(+)}{h}_{2,2}\right)^2 + 1}, \quad j = 1, 2, 3,\end{aligned} \tag{24}$$

and the last quantities are known.

So, in view of (24), from (18) we get

$$
\begin{aligned}
&X^2_{j\beta0,\beta}(x_1,x_2,t)+\frac{1}{2h_1}Q^1_{\overset{(+)}{\nu_2}j}\sqrt{\left(\overset{(+)}{h}_{2,1}\right)^2+\left(\overset{(+)}{h}_{2,2}\right)^2+1}\\
&+Q_{\overset{(-)}{\nu_2}j}(x_1,x_2,t)\sqrt{\left(\overset{(-)}{h}_{2,1}\right)^2+\left(\overset{(-)}{h}_{2,2}\right)^2+1}\\
&+\Phi^2_{j0}(x_1,x_2,t)=\rho\ddot{u}^2_{j0}(x_1,x_2,t),\quad j=1,2,3.
\end{aligned}
\tag{25}
$$

Thus, from (25), taking into account (19), for $j=\alpha$ it follows that

$$
\begin{aligned}
&\mu\left[\left(h_2v^2_{\alpha0,\beta}\right)_{,\beta}+\left(h_2v^2_{\beta0,\alpha}\right)_{,\beta}\right]+\lambda\left(h_2v^2_{\beta0,\beta}\right)_{,\alpha}\\
&+\frac{1}{2h_1}Q^1_{\overset{(+)}{\nu_2}\alpha}\sqrt{\left(\overset{(+)}{h}_{2,1}\right)^2+\left(\overset{(+)}{h}_{2,2}\right)^2+1}\\
&+Q_{\overset{(-)}{\nu_2}\alpha}\sqrt{\left(\overset{(-)}{h}_{2,1}\right)^2+\left(\overset{(-)}{h}_{2,2}\right)^2+1}+\Phi^2_{\alpha0}=\rho h_2\ddot{v}^2_{\alpha0},\quad \alpha=1,2;
\end{aligned}
\tag{26}
$$

and from (25), (20) for $j=3$ it follows that

$$
\begin{aligned}
&\mu\left(h_2v^2_{30,\beta}\right)_{,\beta}+\frac{1}{2h_1}Q^1_{\overset{(+)}{\nu_2}3}\sqrt{\left(\overset{(+)}{h}_{2,1}\right)^2+\left(\overset{(+)}{h}_{2,2}\right)^2+1}\\
&+Q_{\overset{(-)}{\nu_2}3}\sqrt{\left(\overset{(-)}{h}_{2,1}\right)^2+\left(\overset{(-)}{h}_{2,2}\right)^2+1}+\Phi^2_{30}=\rho h_2\ddot{v}^2_{30}.
\end{aligned}
\tag{27}
$$

In the case of n plies the systems of the first $n-1$ plies are similar to system (22), (23) and the system for the last, i.e., n-th ply is similar to system (26), (27).

BCs for each ply prescribed on the boundary of the layered body projection should be added to the governing system. We get the corresponding boundary data from the boundary data on the lateral boundary of the layered body multiplying the last data on $P_l(a_\gamma x_3-b_\gamma)$ and integrating with respect to x_3 within the limits $\overset{(-)}{h}_\gamma(x_1,x_2)$ and $\overset{(+)}{h}_\gamma(x_1,x_2)$, $\gamma=\overline{1,n}$, on the non-cusped edge of the ply; while by setting BCs along the cusped edge of the ply peculiarities arising in the theory of cusped prismatic shells [3] should be taken in consideration. So, on the lateral boundary of the layered prismatic shell we can consider all the reasonable BCs in displacements, stresses, and mixed ones as well, herewith on the boundaries of the different plies BCs of different genus (kind) can be prescribed.

Using the energy potentials for each ply we follow [1] in proving uniqueness of solutions of BVPs in the zeroth approximation corresponding to the 3D Physical Problem 1.

Analogously, we handle the case of the N-th hierarchical model (approximation).

In the case when no of plies are cusped ones we use the well-known methods (see [1,2,5,6]) applied for non-cusped prismatic shells; if at least one of the plies is a cusped prismatic shell we use the above mentioned methods combined with the well-known methods developed for cusped prismatic shells (see [3,7,8] and references given there); if all the plies are cusped ones then we use the above methods [3].

Let us now consider a special case of deformation of the above layered prismatic shell in the static case when

$$\overset{(\pm)}{h}_\gamma = const, \ \ v^\gamma_{j0} = v^\gamma_{j0}(x_2), \quad j = 1,2,3, \quad \gamma = 1,2,$$

$$x_2 \in [0, L], \quad x_1 \in \,]-\infty, +\infty[\,.$$

For $\gamma = 1$ from (22), (23) we get

$$\mu h_1 v^1_{10,22} + Q_{\overset{(+)}{\nu_1}1}(x_2) + \Phi^1_{10}(x_2) = 0, \tag{28}$$

$$(\lambda + 2\mu) h_1 v^1_{20,22} - \frac{\mu}{2} v^1_{30,2} + Q_{\overset{(+)}{\nu_1}2}(x_2) + \Phi^1_{20}(x_2) = 0, \tag{29}$$

$$\mu h_1 v^1_{30,22} - \frac{\lambda}{2} v^1_{20,2} + Q_{\overset{(+)}{\nu_1}3}(x_2) + \Phi^1_{30}(x_2) = 0, \tag{30}$$

Let

$$V^1_j := v^1_{j0,2}, \quad j = 2,3.$$

Then (29), (30) we rewrite as

$$(\lambda + 2\mu) h_1 V^1_{2,2} - \frac{\mu}{2} V^1_3 + Q_{\overset{(+)}{\nu_1}2} + \Phi^1_{20} = 0, \tag{31}$$

$$\mu h_1 V^1_{3,2} - \frac{\lambda}{2} V^1_2 + Q_{\overset{(+)}{\nu_1}3} + \Phi^1_{30} = 0.$$

Hence,

$$V^1_2 = \frac{2\mu h_1}{\lambda} V^1_{3,2} + \frac{2}{\lambda}(Q_{\overset{(+)}{\nu_1}3} + \Phi^1_{30})$$

and substituting the latter into (31), we get

$$\frac{2\mu(\lambda+2\mu)h_1^2}{\lambda} V^1_{3,22} - \frac{\mu}{2} V^1_3 + \frac{2(\lambda+2\mu)h_1}{\lambda}(Q_{\overset{(+)}{\nu_1}3} + \Phi^1_{30})_{,2} + Q_{\overset{(+)}{\nu_1}2} + \Phi^1_{20} = 0. \tag{32}$$

Integrating (28) we have

$$v^1_{10} = \frac{1}{\mu h_1} \int_0^{x_2} (x-\tau)[Q_{\overset{(+)}{\nu_1}1}(\tau) + \Phi^1_{10}(\tau)] d\tau + c_1 x_2 + c_2, \ \ c_1,\, c_2 = const. \tag{33}$$

Let, for the sake of simplicity,

$$Q_{\overset{(+)}{\nu_1}} \equiv 0, \quad \Phi^1_{j0} = 0, \quad j = 1, 2, 3. \tag{34}$$

Then, from (32) we have

$$\frac{2(\lambda+2\mu)\mu h_1^2}{\lambda} V^1_{3,22} - \frac{\mu}{2} V^1_3 = 0.$$

Looking for the solution in the form

$$V^1_3 = e^{\eta x_2},$$

we obtain

$$\frac{2(\lambda+2\mu)h_1^2}{\lambda}\eta^2 e^{\eta x_2} - \frac{1}{2} e^{\eta x_2} = 0$$

and

$$\eta_{1;2} = \pm\frac{1}{2h_1}\sqrt{\frac{\lambda}{\lambda+2\mu}}.$$

Hence,

$$V^1_3 = c_3 e^{\eta_1 x_2} + c_4 e^{\eta_2 x_2}, V^1_2 = \frac{2\mu h_1}{\lambda}(c_3\eta_1 e^{\eta_1 x_2} + c_4 \eta_2 e^{\eta_2 x_2}), \qquad c_3,\, c_4 = const.$$

Finally, taking into account (33), we obtain

$$v^1_{10} = c_1 x_2 + c_2, \tag{35}$$

$$v^1_{20} = \frac{2\mu h_1}{\lambda}\left[c_3(e^{\eta_1 x_2} - 1) + c_4(e^{\eta_2 x_2} - 1)\right] + c_5, \qquad c_5 = const, \tag{36}$$

$$v^1_{30} = \frac{c_3}{\eta_1}(e^{\eta_1 x_2} - 1) + \frac{c_4}{\eta_2}(e^{\eta_2 x_2} - 1) + c_6, \qquad c_6 = const. \tag{37}$$

According to (20), (21), by virtue of (36), (37), evidently,

$$X^1_{130} = \mu h_1 v^1_{30,1} = 0, \tag{38}$$

$$X^1_{230} = \mu h_1 v^1_{30,2} = \mu h_1\left(c_3 e^{\eta_1 x_2} + c_4 e^{\eta_2 x_2}\right), \tag{39}$$

$$X^1_{330} = \lambda h_1 v^1_{20,2} = 2\mu h_1^2\left(c_3\eta_1 e^{\eta_1 x_2} + c_4\eta_2 e^{\eta_2 x_2}\right). \tag{40}$$

Since $Q^1_{\overset{(+)}{\nu_2}\,1} = X^1_{130} = 0$ because of (38), for $\gamma = 2$ from (26), (27) we get

$$\mu h_2 v^2_{10,22} + Q_{\overset{(-)}{\nu_2}\,1} + \Phi^2_{10} = 0, \tag{41}$$

$$(\lambda+2\mu)h_2 v^2_{20,22} + \frac{1}{2h_1}X^1_{230} + Q_{\overset{(-)}{\nu_2}\,2} + \Phi^2_{20} = 0, \tag{42}$$

$$\mu h_2 v^2_{30,22} + \frac{1}{2h_1}X^1_{330} + Q_{\overset{(-)}{\nu_2}\,3} + \Phi^2_{30} = 0. \tag{43}$$

Solving them, in view of (39), (40), we obtain

$$v_{10}^2 = -\frac{1}{\mu h_2}\int\limits_0^{x_2}(x-\tau)\left[Q_{\overset{(-)}{\nu_2}1}(\tau)+\Phi_{10}^2(\tau)\right]d\tau + c_7x_2+c_8,$$

$$v_{20}^2 = -\int\limits_0^{x_2}(x_2-\tau)\left[\frac{\mu}{2(\lambda+2\mu)h_2}\left(c_3e^{\eta_1\tau}+c_4e^{\eta_2\tau}\right)\right]$$

$$+\frac{1}{(\lambda+2\mu)h_2}\left(Q_{\overset{(-)}{\nu_2}2}+\Phi_{20}^2\right)\Bigg]d\tau + c_9x_2+c_{10},$$

$$v_{30}^2 = -\int\limits_0^{x_2}(x_2-\tau)\left[\frac{h_1}{h_2}\left(c_3\eta_1e^{\eta_1\tau}+c_4\eta_2e^{\eta_2\tau}\right)+\frac{1}{\mu h_2}\left(Q_{\overset{(-)}{\nu_2}3}(\tau)+\Phi_{30}^2(\tau)\right)\right]d\tau$$

$$+c_{11}x_2+c_{12}, \quad c_7,\ c_8,\ c_9,\ c_{10},\ c_{11},\ c_{12}=const.$$

Let, for the sake of simplicity, in what follows

$$Q_{\overset{(-)}{\nu_2}j}\equiv 0, \quad \Phi_{j0}^2\equiv 0, \quad j=1,2,3, \tag{44}$$

then

$$v_{10}^2 = c_7x_2+c_8, \tag{45}$$

$$v_{20}^2 = -\int_0^{x_2}(x_2-\tau)\left[\frac{\mu}{2(\lambda+2\mu)h_2}\left(c_3e^{\eta_1\tau}+c_4e^{\eta_2\tau}\right)\right]d\tau + c_9x_2+c_{10} \tag{46}$$

$$= \frac{\mu}{2(\lambda+2\mu)h_2}\left[\left(\frac{c_3}{\eta_1}+\frac{c_4}{\eta_2}\right)x_2-\frac{c_3}{\eta_1^2}\left(e^{\eta_1x_2}-1\right)-\frac{c_4}{\eta_2^2}\left(e^{\eta_2x_2}-1\right)\right]$$

$$+c_9x_2+c_{10},$$

$$v_{30}^2 = -\int_0^{x_2}(x_2-\tau)\left[\frac{h_1}{h_2}\left(c_3\eta_1e^{\eta_1\tau}+c_4\eta_2e^{\eta_2\tau}\right)\right]d\tau + c_{11}x_2+c_{12} \tag{47}$$

$$= \frac{h_1}{h_2}\left[\left(c_3+c_4\right)x_2-\frac{c_3}{\eta_1}\left(e^{\eta_1x_2}-1\right)-\frac{c_4}{\eta_2}\left(e^{\eta_2x_2}-1\right)\right]+c_{11}x_2+c_{12}.$$

According to (20), (21) for $\gamma=2$, by virtue of (46), (47), evidently,

$$X_{130}^2 = \mu h_2 v_{30,1}^2 = 0, \tag{48}$$

$$X_{230}^2 = \mu h_2 v_{30,2}^2 = -\mu h_1\left[c_3\left(e^{\eta_1x_2}-1\right)+c_4\left(e^{\eta_2x_2}-1\right)\right]+\mu h_2c_{11}, \tag{49}$$

$$X_{330}^2 = \lambda h_2 v_{20,2}^2 = -\frac{\lambda\mu}{2(\lambda+2\mu)}\left[\frac{c_3}{\eta_1}\left(e^{\eta_1x_2}-1\right)+\frac{c_4}{\eta_2}\left(e^{\eta_2x_2}-1\right)\right]+\lambda h_2c_9. \tag{50}$$

Let us consider the following BCs

$$v_{j0}^\gamma(0)=d_{j0}^\gamma, \quad j=1,2,3, \quad \gamma=1,2; \tag{51}$$

$$v^1_{j0}(L) = d_{j1}, \quad j = 1, 2, 3; \tag{52}$$

$$v^2_{j0}(L) = d_{j2}, \quad j = 1, 2, 3; \tag{53}$$

Then, taking into account (35), from

$$v^1_{10}(0) = d^1_{10} \ \text{ and } \ v^1_{10}(L) = d_{11}$$

it follows that

$$c_2 = d^1_{10} \ \text{ and } \ c_1 = \frac{d_{11} - d^1_{10}}{L}, \tag{54}$$

whence,

$$v^1_{10} = \frac{d_{11} - d^1_{10}}{L} x_2 + d^1_{10}; \tag{55}$$

taking into account (45), from

$$v^2_{10}(0) = d^2_{10} \ \text{ and } \ v^2_{10}(L) = d_{12}$$

it follows that

$$c_8 = d^2_{10} \ \text{ and } \ c_7 = \frac{d_{12} - d^2_{10}}{L}, \tag{56}$$

whence,

$$v^2_{10} = \frac{d_{12} - d^2_{10}}{L} x_2 + d^2_{10}; \tag{57}$$

taking into account (36), (37), from

$$v^1_{20}(0) = d^1_{20} \ \text{ and } \ v^1_{30}(0) = d^1_{30}$$

it follows that

$$c_5 = d^1_{20} \ \text{ and } \ c_6 = d^1_{30}, \tag{58}$$

respectively;
from

$$v^1_{20}(L) = d_{21} \ \text{ and } \ v^1_{30}(L) = d_{31}$$

it follows that

$$\frac{2\mu h_1}{\lambda} \left[c_3(e^{\eta_1 L} - 1) + c_4(e^{\eta_2 L} - 1) \right] = d_{21} - d^1_{20}$$

and

$$\frac{e^{\eta_1 L} - 1}{\eta_1} c_3 + \frac{e^{\eta_2 L} - 1}{\eta_2} c_4 = d_{31} - d^1_{30},$$

respectively, whence,

$$\begin{aligned} c_3 &= \frac{\begin{vmatrix} d_{21} - d^1_{20} & \frac{2\mu h_1}{\lambda} \\ d_{31} - d^1_{30} & \eta_2^{-1} \end{vmatrix} \left(e^{\eta_2 L} - 1 \right)}{\frac{2\mu h_1 \left(e^{\eta_1 L} - 1 \right)\left(e^{\eta_2 L} - 1 \right)}{\lambda} \begin{vmatrix} 1 & 1 \\ \eta_1^{-1} & \eta_2^{-1} \end{vmatrix}} \\ &= \frac{\lambda (d_{21} - d^1_{20}) \eta_2^{-1} - 2(d_{31} - d^1_{30}) \mu h_1}{2\mu h_1 \left(e^{\eta_1 L} - 1 \right) \left(\eta_2^{-1} - \eta_1^{-1} \right)}, \end{aligned} \tag{59}$$

$$
\begin{aligned}
c_4 &= \frac{\begin{vmatrix} \frac{2\mu h_1}{\lambda} & d_{21} - d^1_{20} \\ \eta_1^{-1} & d_{31} - d^1_{30} \end{vmatrix} \left(e^{\eta_1 L} - 1\right)}{\frac{2\mu h_1 \left(e^{\eta_1 L}-1\right)\left(e^{\eta_2 L}-1\right)\left(\eta_2^{-1}-\eta_1^{-1}\right)}{\lambda}} \\
&= \frac{2\mu h_1 (d_{31} - d^1_{30}) - \lambda (d_{21} - d^1_{20}) \eta_1^{-1}}{2\mu h_1 \left(e^{\eta_2 L} - 1\right) \left(\eta_2^{-1} - \eta_1^{-1}\right)}.
\end{aligned} \tag{60}
$$

Taking into account (46), (47), from

$$v^2_{20}(0) = d^2_{20} \ \text{ and } \ v^2_{30}(0) = d^2_{30}$$

it follows that

$$c_{10} = d^2_{20} \ \text{ and } \ c_{12} = d^2_{30}, \tag{61}$$

respectively; from

$$v^2_{20}(L) = d_{22} \ \text{ and } \ v^2_{30}(L) = d_{32}$$

it follows that

$$
\begin{aligned}
&\frac{\mu}{2(\lambda + 2\mu) h_2} \left[\left(\frac{c_3}{\eta_1} + \frac{c_4}{\eta_2} \right) L - \frac{c_3}{\eta_1^2} \left(e^{\eta_1 L} - 1\right) - \frac{c_4}{\eta_2^2} \left(e^{\eta_2 L} - 1\right) \right] + c_9 L \\
&= d_{22} - d^2_{20}, \\
&\frac{h_1}{h_2} \left[(c_3 + c_4) L - \frac{c_3}{\eta_1} \left(e^{\eta_1 L} - 1\right) - \frac{c_4}{\eta_2} \left(e^{\eta_2 L} - 1\right) \right] + c_{11} L = d_{32} - d^2_{30},
\end{aligned}
$$

respectively, whence,

$$
\begin{aligned}
&L c_9 = d_{22} - d^2_{20} \\
&- \frac{\mu}{2(\lambda + 2\mu) h_2} \left[\left(\frac{c_3}{\eta_1} + \frac{c_4}{\eta_2} \right) L - \frac{c_3}{\eta_1^2} \left(e^{\eta_1 L} - 1\right) - \frac{c_4}{\eta_2^2} \left(e^{\eta_2 L} - 1\right) \right];
\end{aligned} \tag{62}
$$

$$L c_{11} = d_{32} - d^2_{30} - \frac{h_1}{h_2} \left[(c_3 + c_4) L - \frac{c_3}{\eta_1} \left(e^{\eta_1 L} - 1\right) - \frac{c_4}{\eta_2} \left(e^{\eta_2 L} - 1\right) \right]. \tag{63}$$

Thus, a unique solution of the BVP for system (28)-(30), (41)-(43) with BCs (51)-(53) has the form (55), (57), and (36), (37), (46), (47) with (59), (60), (62), (63).

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