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ON THE OPTIMALITY OF INITIAL ELEMENT FOR DELAY FUNCTIONAL DIFFERENTIAL EQUATIONS WITH THE MIXED INITIAL CONDITION

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Abstract. Necessary optimality conditions are obtained for the initial element of a nonlinear functional differential equation with constant delays in phase coordinates and with the mixed initial condition. Here the initial element implies the collection of initial and finally moments, delay parameters, initial vector and functions, control function. The mixed initial condition means that at the initial moment, some coordinates of the trajectory do not coincide with the corresponding coordinates of the initial function (a discontinuous part of the initial condition), whereas the others coincide (a continuous part of the initial condition). In this paper, the essential novelty is necessary condition of optimality for delay parameters, which contains the effect of mixed initial condition.

Keywords and phrases: Necessary conditions of optimality, delay functional differential equation, optimal element, optimization, mixed initial condition.

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1. Problem statement. Necessary conditions of optimality

Let R_x^n be the n-dimensional vector space of points

$$x=(x^1,...,x^n)^T, |x|^2=\sum_{i=1}^{n}(x^i)^2$$

where T means transpose; let $P\subset R_p^k, Z\subset R_z^m$ and $V\subset R_u^r$ be open sets and

$$O=\{x=(p,z)^T\in R_x^n: p\in P, z\in Z\},$$

where $k+m=n$; the n-dimensional function $f(t,x,p,z,u)$ is continuous on the set $I\times O\times P\times Z\times V$ and continuously differentiable with respect to x,p,z, where $I=[a,b]$; let $P_0\subset P$ be a convex and compact set of initial vectors p_0 and $0<\tau_1<\tau_2, 0<\sigma_1<\sigma_2$ be given numbers. Further, let Φ and G be sets of continuous initial functions $\varphi(t)\in P_1, t\in I_1$ and $g(t)\in Z_1, t\in I_1$, respectively, where $I_1=[\hat{\tau},b], \hat{\tau}=a-\max\{\tau_2,\sigma_2\}$ and $P_1\subset P, Z_1\subset Z$ are convex and compact sets. Let Ω be a set of piecewise control functions $u(t)\in U, t\in I$ with discontinuity points of the first kind, where $U\subset V$ is an arbitrary set. Suppose that, the scalar functions

$$q^i(t_0,t_1,\tau,\sigma,p,z,x),\ i=\overline{0,l}$$

are continuously differentiable with respect to all arguments

$$t_0, t_1 \in I, \tau \in [\tau_1, \tau_2], \sigma \in [\sigma_1, \sigma_2], p \in P, z \in Z, x \in O.$$

To each element

$$w = (t_0, t_1, \tau, \sigma, p_0, \varphi(\cdot), g(\cdot), u(\cdot)) \in W$$

$$= (a, b) \times (a, b) \times (\tau_1, \tau_2) \times (\sigma_1, \sigma_2) \times P_0 \times \Phi \times G \times \Omega,$$

where $t_0 < t_1$, we assign delay functional differential equation

$$\dot{x}(t) = f(t, x(t), p(t-\tau), z(t-\sigma), u(t)) \tag{1}$$

with the initial condition

$$\begin{cases} x(t) = (p(t), z(t))^T = (\varphi(t), g(t))^T, t \in [\hat{\tau}, t_0), \\ x(t_0) = (p_0, g(t_0))^T. \end{cases} \tag{2}$$

The condition (2) is said to be a mixed condition. It consists of two parts: the first part is $p(t) = \varphi(t), t \in [\hat{\tau}, t_0), p(t_0) = p_0$, it is the so-called discontinuous part, because, in general, $p(t_0) \neq \varphi(t_0)$; the second part is $z(t) = g(t), t \in [\hat{\tau}, t_0]$, it is the so-called continuous part because always $z(t_0) = g(t_0)$.

Definition 1.1. Let $w = (t_0, t_1, \tau, \sigma, p_0, \varphi(\cdot), g(\cdot), u(\cdot)) \in W$. A function

$$x(t) = x(t; w) = (p(t; w), z(t; w))^T \in O, t \in [\hat{\tau}, t_1]$$

is called a solution of equation (1) with the mixed initial condition (2) or a solution corresponding to the element w and defined on the interval $[\hat{\tau}, t_1]$ if it satisfies condition (2) and is absolutely continuous on the interval $[t_0, t_1]$ and satisfies equation (1) almost everywhere on $[t_0, t_1]$.

Definition 1.2. An element $w \in W$ is said to be admissible if the corresponding solution $x(t)$ is defined on the interval $[\hat{\tau}, t_1]$ and satisfies the conditions

$$q^i(t_0, t_1, \tau, \sigma, p_0, g(t_0), x(t_1)) = 0, \quad i = \overline{1, l}. \tag{3}$$

We denote the set of admissible elements by W_0.

Definition 1.3. An element

$$w_0 = (t_{00}, t_{10}, \tau_0, \sigma_0, p_{00}, \varphi_0(\cdot), g_0(\cdot), u_0(\cdot)) \in W_0$$

is said to be optimal if for any $w = (t_0, t_1, \tau, \sigma, p_0, \varphi(\cdot), g(\cdot), u(\cdot)) \in W_0$

$$q^0(t_{00}, t_{10}, \tau_0, \sigma_0, p_{00}, g_0(t_{00}), x_0(t_{10})) \leq q^0(t_0, t_1, \tau, \sigma, p_0, g(t_0), x(t_1)). \tag{4}$$

Here $x_0(t) = x(t; w_0)$, $x(t) = x(t; w)$.

Problem (1)-(4) is called an optimization problem with respect to initial element. It consists in finding an optimal initial element w_0.

Theorem 1.1. *Let* $w_0 = (t_{00}, t_{10}, \tau_0, \sigma_0, p_{00}, \varphi_0(\cdot), g_0(\cdot), u_0(\cdot))$ *be an optimal element and let* $x_0(t) = (p_0(t), z_0(t))^T$ *be corresponding solution and the following conditions hold*

1.1. $t_{00} + \tau_0 < t_{10}$;

1.2. the functions $\varphi_0(t), g_0(t), t \in J_1$ *are absolutely continuous and* $\dot{\varphi}_0(t), \dot{g}_0(t)$ *are bounded*;

1.3. the function $\dot{g}_0(t)$ *is continuous at the point* t_{00}*;*

1.4. the function $u_0(t)$ *is continuous at the point* $t_{00} + \tau_0$.

Then there exists a non-zero vector $\pi = (\pi_0, \ldots, \pi_l)$*, where* $\pi_0 \le 0$ *and a solution* $\Psi(t)$*, of the equation*

$$\begin{cases} \dot{\Psi}(t) = -\Psi(t) f_{0x}[t] - (\Psi(t+\tau_0) f_{0p}[t+\tau_0], \Psi(t+\sigma_0) f_{0z}[t+\sigma_0]), t \in [t_{00}, t_{10}], \\ \Psi(t) = 0, t > t_{10} \end{cases}$$

such that the conditions listed below hold:

1.5. the conditions for the function

$$\Psi(t) = (\psi(t), \chi(t)) = (\psi_1(t), \ldots, \psi_k(t), \chi_1(t), \ldots, \chi_m(t))$$

and vectors $p_{00}, g_0(t_{00})$

$$\Psi(t_{10}) = \pi Q_{0x},$$

$$(\pi Q_{0p_0} + \psi(t_{00})) p_{00} = \max_{p_0 \in P_0} (\pi Q_{0p_0} + \psi(t_{00})) p_0,$$

$$(\pi Q_{0z} + \chi(t_{00})) g_0(t_{00}) = \max_{g \in Z_1} (\pi Q_{0z} + \chi(t_{00})) g;$$

1.6. the integral maximum principle for the optimal initial functions $\varphi_0(t)$ *and* $g_0(t)$

$$\int_{t_{00}-\tau_0}^{t_{00}} \Psi(t+\tau_0) f_{0p}[t+\tau_0] \varphi_0(t) dt = \max_{\varphi(\cdot) \in \Phi} \int_{t_{00}-\tau_0}^{t_{00}} \Psi(t+\tau_0) f_{0p}[t+\tau_0] \varphi(t) dt,$$

$$\int_{t_{00}-\sigma_0}^{t_{00}} \Psi(t+\sigma_0) f_{0z}[t+\sigma_0] g_0(t) dt = \max_{g(\cdot) \in G} \int_{t_{00}-\sigma_0}^{t_{00}} \Psi(t+\sigma_0) f_{0z}[t+\sigma_0] g(t) dt;$$

1.7. the integral maximum principle for the optimal control $u_0(t)$

$$\int_{t_{00}}^{t_{10}} \Psi(t) f_0[t] dt = \max_{u(\cdot) \in \Omega} \int_{t_{00}}^{t_{10}} \Psi(t) f(t, x_0(t), p_0(t-\tau_0), z_0(t-\sigma_0), u(t)) dt;$$

1.8. the condition for the optimal final moment t_{10}

$$\pi Q_{0t_1} = -\Psi(t_{10}) f_0[t_{10}];$$

1.9. the condition for the optimal initial moment t_{00}

$$\pi Q_{0t_0} + (\pi Q_{0z} + \chi(t_{00})) \dot{g}(t_{00}) = \Psi(t_{00}) f_0[t_{00}]$$

$$+ \Psi(t_{00}+\tau_0) \{ f[t_0+\tau_0; p_{00}] - f[t_0+\tau_0; \varphi_0(t_{00})] \};$$

1.10. the conditions for the optimal delays τ_0, σ_0

$$\pi Q_{0\tau} = \Psi(t_{00}+\tau_0)\{f[t_0+\tau_0;p_{00}] - f[t_0+\tau_0;\varphi_0(t_{00})]\}$$

$$+\int_{t_{00}}^{t_{10}} \Psi(t) f_{0p}[t]\dot{p}_0(t-\tau_0)dt,$$

$$\pi Q_{0\sigma} = \int_{t_{00}}^{t_{10}} \Psi(t) f_{0z}[t]\dot{z}_0(t-\sigma_0)dt.$$

Here

$$f_0[t] = f(t, x_0(t), p_0(t-\tau_0), z_0(t-\sigma_0), u_0(t)),$$

$$f_{0x}[t] = f_x(t, x_0(t), p_0(t-\tau_0), z_0(t-\sigma_0), u_0(t)),$$

$$f[t;p_0] = f(t, x_0(t), p_0, z_0(t-\sigma_0), u_0(t)),$$

$$Q_{0x} = Q_x(t_{00}, t_{10}, \tau_0, \sigma_0, p_{00}, g_0(t_{00}), x_0(t_{10})),$$

$$Q(t_0, t_1, \tau, \sigma, p_0, z, x) = (q^0(t_0, t_1, \tau, \sigma, p_0, z, x), \dots, q^l(t_0, t_1, \tau, \sigma, p_0, z, x)^T.$$

Some comments: the expression

$$\Psi(t_{00}+\tau_0)\{f[t_0+\tau_0;p_{00}] - f[t_0+\tau_0;\varphi_0(t_{00})]\}$$

is the effect of the discontinuous part of the mixed initial condition; the term

$$(\pi Q_{0z} + \chi(t_{00}))\dot{g}(t_{00})$$

is the effect of the continuous part of the initial condition.

Theorem 1.1 is proved by a method given in [1]. Finally we note that the optimization problems for various classes functional differential equations with fixed delays and with the mixed initial condition are investigated in [2-5].

2. Problem with the integral functional

Let $p_0 \in P$ and $x_1 \in O$ be fixed points.Consider the following problem

$$\dot{x}(t) = f(t, x(t), p(t-\tau), z(t-\sigma), u(t)), t \in [t_0, t_1],$$

$$\begin{cases} x(t) = (\varphi(t), g(t))^T, t \in [\hat{\tau}, t_0), \\ x(t_0) = (p_0, g(t_0))^T, \\ x(t_1) = x_1, \end{cases}$$

$$\int_{t_0}^{t_1} f^0(t, x(t), p(t-\tau), z(t-\sigma), u(t))dt \to min,$$

where the function $f^0(t, x, p, z, u)$ is continuous on the set $I \times O \times P \times Z \times V$ and continuously differentiable with respect to x, p, z.

Theorem 2.1. *Let $w_0 = (t_{00}, t_{10}, \tau_0, \sigma_0, \varphi_0(\cdot), g_0(\cdot), u_0(\cdot))$ be an optimal element and let $x_0(t) = (p_0(t), z_0(t))^T$ be corresponding solution and conditions 1.1-1.4 of Theorem 1.1 hold. Then there exists a non-trivial solution*

$$H(t) = (\psi_0(t), \psi(t), \chi(t)) = (\psi_0(t), \Psi(t)), \psi_0(t) = const \leq 0,$$

of the equation

$$\begin{cases} \dot{\Psi}(t) = -H(t)F_{0x}[t] - (H(t+\tau_0)F_{0p}[t+\tau_0], H(t+\sigma_0)F_{0z}[t+\sigma_0]), t \in [t_{00}, t_{10}], \\ H(t) = 0, t > t_{10}. \end{cases}$$

where $F = (f^0, f)^T$, such that the conditions listed below hold

2.1. the condition for the vector $g_0(t_{00})$

$$\chi(t_{00})g_0(t_{00}) = \max_{g \in Z_1} \chi(t_{00})g;$$

2.2. the integral maximum principle for the optimal initial functions $\varphi_0(t)$ and $g_0(t)$

$$\int_{t_{00}-\tau_0}^{t_{00}} H(t+\tau_0)F_{0p}[t+\tau_0]\varphi_0(t)dt = \max_{\varphi(\cdot)\in\Phi} \int_{t_{00}-\tau_0}^{t_{00}} H(t+\tau_0)F_{0p}[t+\tau_0]\varphi(t)dt,$$

$$\int_{t_{00}-\sigma_0}^{t_{00}} H(t+\sigma_0)F_{0z}[t+\sigma_0]g_0(t)dt = \max_{g(\cdot)\in G} \int_{t_{00}-\sigma_0}^{t_{00}} H(t+\sigma_0)F_{0z}[t+\sigma_0]g(t)dt;$$

2.3. the integral maximum principle for the optimal control $u_0(t)$

$$\int_{t_{00}}^{t_{10}} H(t)F_0[t]dt = \max_{u(\cdot)\in\Omega} \int_{t_{00}}^{t_{10}} H(t)F(t, x_0(t), p_0(t-\tau), z_0(t-\sigma_0), u_0(t))dt;$$

2.4. the condition for the optimal final moment t_{10}

$$\pi Q_{0t_1} = -H(t_{10})F_0[t_{10}];$$

2.5. the condition for the optimal initial moment t_{00}

$$\chi(t_{00})\dot{g}(t_{00}) = H(t_{00})F_0[t_{00}] + H(t_{00}+\tau_0)\{F[t_0+\tau_0; p_{00}] - F[t_0+\tau_0; \varphi_0(t_{00})]\};$$

2.6. the conditions for the optimal delays τ_0, σ_0

$$H(t_{00}+\tau_0)\{F[t_0+\tau_0; p_0] - F[t_0+\tau_0; \varphi_0(t_{00})]\} + \int_{t_{00}}^{t_{10}} H(t)F_{0p}[t]\dot{p}_0(t-\tau_0)dt = 0,$$

$$\int_{t_{00}}^{t_{10}} H(t)F_{0z}[t]\dot{z}_0(t-\sigma_0)dt = 0.$$

Here

$$F_0[t] = F(t, x_0(t), p_0(t-\tau_0), z_0(t-\sigma_0), u_0(t)), F_{0x}[t] = F_x(t, x_0(t), p_0(t-\tau_0),$$

$$z_0(t-\sigma_0),u_0(t)),F[t;p_0]=F(t,x_0(t),p_0,z_0(t-\sigma_0),u_0(t)).$$

3. Problem for the linear equation

Let $t_{01}<t_{02}<t_1$ be fixed numbers, with $t_1-t_{02}>\max\{\tau_2,\sigma_2\}$. To each element

$$\mu=(t_0,\tau,\sigma,p_0,\varphi(\cdot),g(\cdot),u(\cdot))\in\Pi=(t_{01},t_{02})\times(\tau_1,\tau_2)\times(\sigma_1,\sigma_2)\times P_0\times\Phi\times G$$

we assign the linear delay differential equation

$$\dot{x}(t)=Ax(t)+Bp(t-\tau)+Cz(t-\sigma)+Du(t)$$

with the mixed initial condition

$$\begin{cases} x(t)=(\varphi(t),g(t))^T, t\in[\hat{\tau},t_0),\\ x(t_0)=(p_0,g(t_0))^T, \end{cases}$$

where A,B,C and D are constant matrices with appropriate dimensions.

Let $y\in O$ be a given vector.

Definition 3.1. An element $\mu_0=(t_{00},\tau_0,\sigma_0,p_0,\varphi_0(\cdot),g_0(\cdot),u_0(\cdot))\in\Pi$ is said to be optimal if for any $\mu=(t_0,\tau,\sigma,\varphi(\cdot),g(\cdot),u(\cdot))\in\Pi$

$$|x(t_1;\mu_0)-y|^2\le|x(t_1;\mu)-y|^2$$

Theorem 3.1. *Let $\mu_0=(t_{00},\tau_0,\sigma_0,p_0,\varphi_0(\cdot),g_0(\cdot),u_0(\cdot))$ be an optimal element and let $x_0(t)=(p_0(t),z_0(t))^T$ be corresponding solution and conditions 1.2-1.3 of Theorem 1.1 hold. Then there exists a solution $\Psi(t)$ of the equation*

$$\begin{cases} \dot{\Psi}(t)=-\Psi(t)A-(\Psi(t+\tau_0)B,\Psi(t+\sigma_0)C), t\in[t_{00},t_{10}],\\ \Psi(t)=0, t>t_{10} \end{cases}$$

such that the conditions listed below hold

3.1. the condition for the function

$$\Psi(t)=(\psi(t),\chi(t))=(\psi_1(t),\dots,\psi_k(t),\chi_1(t),\dots,\chi_m(t))$$

and vectors $p_{00},g_0(t_{00})$

$$\Psi(t_{10})=-2(x_0(t_1)-y)^T,\psi(t_{00})p_{00}=\max_{p_0\in P_1}\psi(t_{00})p_0,$$

$$\chi(t_{00})g_0(t_{00})=\max_{g\in Z_1}\chi(t_{00})g;$$

3.2. the condition for the optimal initial moment t_{00}

$$\chi(t_{00})\dot{g}(t_{00})=\Psi(t_{00})\Big[Ax_0(t_{00})+Bp_0(t_{00}-\tau_0)+Cz(t_{00}-\sigma_0)+Du_0(t_{00})\Big]$$

$$+\Psi(t_{00}+\tau_0)B(p_{00}-\varphi_0(t_{00});$$

3.3. the conditions for the optimal delays τ_0, σ_0

$$\Psi(t_{00}+\tau_0)B(x_{00}-\varphi_0(t_{00}))+\int_{t_{00}}^{t_{10}}\Psi(t)B\dot{p}_0(t-\tau_0)dt=0,$$

$$\int_{t_{00}}^{t_{10}}\Psi(t)C\dot{z}_0(t-\sigma_0)dt=0;$$

3.4. the integral maximum principle for the optimal initial functions $\varphi_0(t)$ *and* $g_0(t)$

$$\int_{t_{00}-\tau_0}^{t_{00}}\Psi(t+\tau_0)B\varphi_0(t)dt=\max_{\varphi(\cdot)\in\Phi}\int_{t_{00}-\tau_0}^{t_{00}}\Psi(t+\tau_0)B\varphi(t)dt,$$

$$\int_{t_{00}-\sigma_0}^{t_{00}}\Psi(t+\sigma_0)Cg_0(t)dt=\max_{g(\cdot)\in G}\int_{t_{00}-\sigma_0}^{t_{00}}\Psi(t+\sigma_0)Cg(t)dt.$$

3.5. the integral maximum principle for the optimal control function $u_0(t)$

$$\int_{t_{00}}^{t_{10}}\Psi(t)Du_0(t)dt=\max_{u(\cdot)\in\Omega}\int_{t_{00}}^{t_{10}}\Psi(t)Du(t)dt.$$

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