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HARMONIC VIBRATION OF A CUSPED PLATES IN THE FIRST APPROXIMATION OF VEKUA'S HIERARCHICAL MODELS

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Abstract. In the case of harmonic vibration we study the well-posedness of boundary value problems for elastic cusped prismatic shells in the first approximation of I.Vekua's hierarchical models.

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1. Introduction

In the present paper in the case of harmonic vibration we study well-posedness of boundary value problems for elastic cusped prismatic shells in the first approximation of I.Vekua's hierarchical models in case of harmonic vibration. One can find survey of results concerning cusped prismatic shells in [6]. To the investigation of cusped plates within the framework of classical Kirchhoff-Love model the works of E. Makhover [8], G. Jaiani [7], N. Chinchaladze [1] are devoted. Vibration under action of fluids is considered by N. Chinchaladze [2], N. Chinchaladze and R. Gilbert [4].

We consider symmetric cusped prismatic shells, i.e., plates of variable thickness with cusped edges. We assume that the cusped plate projection ω has a Lipschitz boundary $\partial\omega = \overline{\gamma}_0 \cup \overline{\gamma}_1$, where $\overline{\gamma}_0$ is a segment of the x_1-axis and γ_1 lies in the upper half-plane $x_2 > 0$; moreover, in some neighborhood of an edge of the plate which may be cusped, the plate thickness has the following form

$$2h(x_1, x_2) = \overset{(+)}{h}(x_1, x_2) - \overset{(-)}{h}(x_1, x_2) = h_0 x_2^{\kappa},$$

$$h_0 = \text{const} > 0, \quad \kappa = \text{const} \geq 0, \quad x_2 \geq 0.$$

Then γ_0 will be a cusped edge for $\kappa > 0$.

In what follows X_{ij} and e_{ij} are the stress and strain tensors, respectively, u_i are the displacements, Φ_i are the volume force components, ρ is the density, λ and μ are the Lamé constants, δ_{ij} is the Kronecker delta. Moreover, repeated indices imply summation, bar under one of the repeated indices means that we do not sum.

By u_{ir}, X_{ijr}, e_{ijr}, Φ_{jr} we denote the r-th order moments of the corresponding quantities u_i, X_{ij}, e_{ij}, Φ_j as defined below:

$$\Big(u_{ir},\, X_{ijr},\, e_{ijr},\, \Phi_{jr}\Big)(x_1,x_2,t)$$
$$:= \int\limits_{\overset{(-)}{h}(x_1,x_2)}^{\overset{(+)}{h}(x_1,x_2)} \Big(u_i,\, X_{ij},\, e_{ij},\, \Phi_j\Big)(x_1,x_2,x_3,t)P_r(ax_3-b)\,dx_3, \quad j=\overline{1,3}.$$

I.Vekua's hierarchical models for elastic prismatic shells are the mathematical models (see, e.g., [11], [12], and [6]). Their constructing is based on the multiplication of the basic equations of linear elasticity by Legendre polynomials $P_r(ax_3-b)$, where

$$a(x_1,x_2) := \frac{1}{h(x_1,x_2)}, \quad b(x_1,x_2) := \frac{\overset{(+)}{h}(x_1,x_2) + \overset{(-)}{h}(x_1,x_2)}{\overset{(+)}{h}(x_1,x_2) - \overset{(-)}{h}(x_1,x_2)},$$

and then integration with respect to x_3 within the limits $\overset{(-)}{h}(x_1,x_2)$ and $\overset{(+)}{h}(x_1,x_2)$. By constructing Vekua's hierarchical models in Vekua's first version on upper and lower face surfaces stress vectors are assumed to be known.

The mathematical model of elastic cusped plates with variable thickness, in the $N=1$ approximation of Vekua's hierarchical method, is described by the following degenerating hyperbolic system ([6])

$$\begin{aligned}
&\rho h v_{\beta0,tt} - \mu\big[(hv_{\alpha0,\beta})_{,\alpha} + (hv_{\beta0,\alpha})_{,\alpha}\big] - \lambda(hv_{\gamma0,\gamma})_{,\beta} - 3\lambda(hv_{31})_{,\beta} = \Phi^{(0)}_{\beta},\\
&\rho h v_{30,tt} - \mu(hv_{30,\alpha})_{,\alpha} - 3\mu(hv_{\alpha1})_{,\alpha} = \Phi^{(0)}_{3}, \qquad (1)\\
&3\rho h^3 v_{\beta1,tt} - 3\mu\Big[(h^3v_{\alpha1,\beta})_{,\alpha} + (h^3v_{\beta1,\alpha})_{,\alpha}\Big] - 3\lambda\big(h^3v_{\gamma1,\gamma}\big)_{,\beta}\\
&+3\Big[\mu h(v_{30,\beta} + 3v_{\beta1})\Big] = 3h\Phi^{(1)}_{\beta}, \qquad \beta=1,2,\\
&3\rho h^3 v_{31,tt} - 3\mu(h^3v_{31,\alpha})_{,\alpha} + 3\Big[\lambda h v_{\gamma0,\gamma} + 3(\lambda+2\mu)hv_{31}\Big] = 3h\Phi^{(1)}_{3}.
\end{aligned}$$

where

$$\overset{r}{\Phi}_j := Q_{\overset{(+)}{n}j}\sqrt{1+\Big(\overset{(+)}{h}_{,1}\Big)^2 + \Big(\overset{(+)}{h}_{,2}\Big)^2}\,+$$
$$+(-1)^r Q_{\overset{(-)}{n}j}\sqrt{1+\Big(\overset{(-)}{h}_{,1}\Big)^2 + \Big(\overset{(-)}{h}_{,2}\Big)^2} + \Phi_{jr},\, j=\overline{1,3}, \quad r=0,1;$$

$Q_{\overset{(+)}{n}j}$ and $Q_{\overset{(-)}{n}j}$ are components of the stress vectors acting on the upper and lower face surfaces with normals $\overset{(+)}{n}$ and $\overset{(-)}{n}$, respectively. Φ_{j0} and Φ_{j1}

are the zero and first moments of the volume forces Φ_j; v_{j0} and v_{j1} are the components of the zero and first weighted moment of the displacement vector $v_{jr} := h^{-r-1}u_{jr}$. The ranges of Latin and Greek indices are $\{1,2,3\}$ and $\{1,2\}$ correspondingly.

2. Harmonic vibration of the cusped plate in case of first approximation of Vekua's hierarchical models

We will consider the case of harmonic vibration, i.e.,

$$v_{ir}(x,t) := e^{-\mathbf{1}\nu t}\overset{0}{v}_{ir}(x), \quad \Phi_i^{(r)}(x,t) := e^{-\mathbf{1}\nu t}\overset{0}{\Phi}{}_i^{(r)}(x),$$

$$\nu = \text{const} > 0, \quad i = \overline{1,3}, \quad r = 0,1.$$

For $\overset{0}{v}_{i0}(x)$ and $\overset{0}{v}_{i1}(x)$ taking into account (1) we get the following system (in what follows we omit the overscript index 0 if it will not lead to a misunderstanding)

$$\begin{aligned}
&-\rho\nu^2 h v_{\beta 0} - \mu\big[(hv_{\alpha 0,\beta})_{,\alpha} + (hv_{\beta 0,\alpha})_{,\alpha}\big] - \lambda(hv_{\gamma 0,\gamma})_{,\beta} - 3\lambda(hv_{31})_{,\beta} = \Phi_\beta^{(0)},\\
&-\rho\nu^2 h v_{30} - \mu(hv_{30,\alpha})_{,\alpha} - 3\mu(hv_{\alpha 1})_{,\alpha} = \Phi_3^{(0)},\\
&-3\rho\nu^2 h^3 v_{\beta 1} - 3\mu\Big[(h^3 v_{\alpha 1,\beta})_{,\alpha} + (h^3 v_{\beta 1,\alpha})_{,\alpha}\Big] - 3\lambda\big(h^3 v_{\gamma 1,\gamma}\big)_{,\beta}\\
&+3\Big[\mu h(v_{30,\beta} + 3v_{\beta 1})\Big] = 3h\Phi_\beta^{(1)}, \qquad \beta = 1,2,\\
&-3\rho\nu^2 h^3 v_{31} - 3\mu(h^3 v_{31,\alpha})_{,\alpha} + 3\Big[\lambda h v_{\gamma 0,\gamma} + 3(\lambda + 2\mu)hv_{31}\Big] = 3h\Phi_3^{(1)}.
\end{aligned} \tag{2}$$

Denoting by $L^{(1)}(x,\partial)$ the 6×6 matrix differential operator, generated by the left-hand side expressions of system (2). We can rewrite (2) in the following vector form

$$L^{(1)}(x,\partial)v(x) = F(x), \quad x \in \omega, \tag{3}$$

where

$$L^{(1)}(x,\partial) := \left\| \begin{matrix} L_{11} & L_{12} & L_{13} & L_{14} & L_{15} & L_{16}\\ L_{21} & L_{22} & L_{23} & L_{24} & L_{25} & L_{26}\\ L_{31} & L_{32} & L_{33} & L_{34} & L_{35} & L_{36}\\ L_{41} & L_{42} & L_{43} & L_{44} & L_{45} & L_{46}\\ L_{51} & L_{52} & L_{53} & L_{54} & L_{55} & L_{56}\\ L_{61} & L_{62} & L_{63} & L_{64} & L_{65} & L_{66} \end{matrix} \right\|,$$

$$L_{11} := -\rho\nu^2 h - h(2\mu+\lambda)\frac{\partial^2}{\partial x_1^2} - h\mu\frac{\partial^2}{\partial x_2^2} - h_{,2}\,\mu\frac{\partial}{\partial x_2},$$

$$L_{12} := -h(\mu+\lambda)\frac{\partial^2}{\partial x_1 \partial x_2} - h_{,2}\,\mu\frac{\partial}{\partial x_1}, \quad L_{13} = L_{14} = L_{15} = 0,$$

$$L_{16} := -3\lambda h\frac{\partial}{\partial x_1}, \quad L_{21} := -h(\mu+\lambda)\frac{\partial^2}{\partial x_1 \partial x_2} - h_{,2}\,\mu\frac{\partial}{\partial x_1},$$

$$L_{22} := -\rho\nu^2 h - h(2\mu+\lambda)\frac{\partial^2}{\partial x_2^2} - h\mu\frac{\partial^2}{\partial x_1^2} - h_{,2}(\mu+\lambda)\frac{\partial}{\partial x_2},$$

$$L_{23} = L_{24} = L_{25} = 0, \quad L_{26} := -3\lambda(h_{,2} + h\frac{\partial}{\partial x_2}),$$

$$L_{31} = L_{32} = L_{36} = 0, \quad L_{33} := -\rho\nu^2 h - h\mu\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right) + h_{,2}\mu\frac{\partial}{\partial x_2},$$

$$L_{34} := -3\mu h\frac{\partial}{\partial x_1}, \quad L_{35} := -3\mu(h_{,2} + h\frac{\partial}{\partial x_2}),$$

$$L_{41} = L_{42} = L_{46} = 0, \quad L_{43} := 3\mu h\frac{\partial}{\partial x_1}, \quad L_{45} := -3(\mu+\lambda)h^3\frac{\partial^2}{\partial x_1 x_2},$$

$$L_{44} := -3\rho\nu^2 h^3 - 3(2\mu+\lambda)h^3\frac{\partial^2}{\partial x_1^2} - 3\mu h^3\frac{\partial^2}{\partial x_2^2} + 3h^2 h_{,2}\frac{\partial}{\partial x_2} + 9\mu h,$$

$$L_{51} = L_{52} = L_{56} = 0, \quad L_{53} := 3\mu h\frac{\partial}{\partial x_2},$$

$$L_{54} := -3(\mu+\lambda)h^3\frac{\partial^2}{\partial x_1 x_2} - 3\lambda h^2 h_{,2}\frac{\partial}{\partial x_1},$$

$$L_{55} := -3\rho\nu^2 h^3 v_{21} - 3\mu h^3\frac{\partial^2}{\partial x_1^2} - 3(2\mu+\lambda)\frac{\partial^2}{\partial x_2^2} - 9(2\mu+\lambda)h^2 h_{,2}\frac{\partial}{\partial x_2} + 9\mu h,$$

$$L_{61} := 3\lambda h\frac{\partial}{\partial x_1}, \quad L_{62} := -3\lambda h\frac{\partial}{\partial x_2}, \quad L_{63} = L_{64} = L_{65} = 0,$$

$$L_{66} := -3\rho\nu^2 h^3 - 3\mu\left(3h^2 h_{,\alpha}\frac{\partial}{\partial x_\alpha} + h^3\frac{\partial^2}{\partial x_\alpha^2}\right) + 9(\lambda+2\mu)h,$$

$$v := (v_{10}, v_{20}, v_{30}, v_{11}, v_{21}, v_{31})^\top,$$

$$F := (\Phi_1^{(0)}, \Phi_2^{(0)}, \Phi_3^{(0)}, 3h\Phi_1^{(1)}, 3h\Phi_2^{(1)}, 3h\Phi_3^{(1)}),$$

the symbol $(\cdot)^\top$ means transposition.

Let

$$v, v^* \in c^2(\omega) \cap c^1(\overline{\omega}), \qquad v^* := (v_{10}^*, v_{20}^*, v_{30}^*, v_{11}^*, v_{21}^*, v_{31}^*)^\top,$$

where v and v^* are arbitrary vectors of the above class. After multiplication (3) by v^* and integration by parts we obtain the following Green‘s formula

$$\int_\omega L^{(1)} v \cdot v^* d\omega = B^{(1)}(v, v^*) - \int_{\partial\omega} T_n v \cdot v^* d\partial\omega = \int_\omega F \cdot v^* d\omega. \tag{4}$$

Here and in what follows the $\cdot$ denotes the scalar product of two vectors, $n := (n_1, n_2)$ is the inward normal to $\partial\omega$,

$$\begin{aligned} B^{(1)}(v, v^*) := \int_\omega &\{h[\rho\nu^2 v_{j0} v_{j0}^* + \mu(v_{\alpha 0,\beta} v_{\beta 0,\alpha}^* + v_{j0,\alpha} v_{j0,\alpha}^*) + \lambda v_{\alpha 0,\alpha} v_{\beta 0,\beta}^*] \\ &+ 3h^3\rho\nu^2 v_{j1} v_{j1}^* - 3\lambda(h v_{31})_{,\alpha} v_{\alpha 0}^* - 3\mu(h v_{\alpha 1})_{,\alpha} v_{30}^* \\ &+ 3h^3(\mu v_{\alpha 1,\beta} v_{\beta 1,\alpha}^* + \mu v_{j1,\alpha} v_{j1,\alpha}^* + \lambda v_{\alpha 1,\alpha} v_{\beta 1,\beta}^*) \\ &+ 3\lambda h v_{\alpha 0,\alpha} v_{31}^* + 3\mu h v_{30,\alpha} v_{\alpha 1}^* + 9\mu h v_{\alpha 1} v_{\alpha 1}^* + 9(\lambda+2\mu) h v_{31} v_{31}^*\} d\omega, \end{aligned} \tag{5}$$

$$T_n := \{\sigma_{n10}, \sigma_{n20}, \sigma_{n30}, 3h\sigma_{n11}, 3h\sigma_{n21}, 3h\sigma_{n31}\},$$

with

$$\sigma_{nir} = \sigma_{ijr} n_j = \{\lambda\delta_{ij}(\sum_{s=r}^{1} h^{s+1} \overset{r}{b}_{ks} v_{ks} + h^{r+1} v_{kr,k})\} n_j$$

$$+\{\mu[\sum_{s=r}^{1} h^{s+1}(\overset{r}{b}_{ir} v_{js} + \overset{r}{b}_{js} v_{ir}) + h^{r+1}(v_{ir,j} + v_{jr,i})]\} n_j, \quad i = 1,2,3, \quad r = 0,1,$$

where (see [6])

$$\overset{0}{b}_{\alpha 0} := -\frac{\overset{(+)}{h}_{,\alpha} - \overset{(-)}{h}_{,\alpha}}{2h}, \;\; \overset{1}{b}_{\alpha 1} := -\frac{\overset{(+)}{h}_{,\alpha} - \overset{(-)}{h}_{,\alpha}}{h}, \;\; \alpha = 1,2; \;\; \overset{0}{b}_{30} = \overset{1}{b}_{31} = 0,$$

$$\overset{1}{b}_{j0} = 0, \;\; \overset{0}{b}_{j1} := -3\frac{\overset{(+)}{h}_{,\alpha} + \overset{(-)}{h}_{,\alpha}}{2h} = 0, \;\; j = 1,2,3;$$

σ_{nir}, $i = 1,2,3$, $r = 0,1$, denote the zero and first moments of the corresponding components of the 3D stresses σ_{ni}, $i = 1,2,3$. From now on, throughout the paper we assume that the plate is symmetric, i.e.

$$\overset{(-)}{h} = -\overset{(+)}{h}, \;\; 2h = h_0 x_2^{\kappa}, \;\; h_0 = const > 0, \;\; \kappa = const \geq 0, \;\; x_2 \geq 0.$$

If we consider BVPs for system (3) with homogeneous boundary conditions for which the curvilinear integral along $\partial\omega$ in (4) disappears, we arrive at the equation

$$B^{(1)}(v, v^*) = \int_{\omega} F \cdot v^* d\omega.$$

Let us consider the following Dirichlet problem in the classical setting: Fing a 6-dimensional vector

$$v = (v_{10}, v_{20}, v_{30}, v_{11}, v_{21}, v_{31})^{\top}$$

in ω satisfying the system of differential equations (3) in ω and the homogeneous Dirichlet boundary condition on

$$[v(x)]^{+} = 0, \quad x \in \partial\omega. \tag{6}$$

Note that throughout the paper, for smooth classical solutions, equation (3) and boundary condition (6) are understood in the classical point-wise sense, while for generalized weak solutions of equation (3) is understood in the distributional sense and boundary condition (6) understood in the usual trace sense. To derive the weak setting of the above problem, we have to apply Green‘s formulas (4). We arrive at the variational equation:

$$B^{(1)}(v, v^*) = \langle F, v^* \rangle, \tag{7}$$

where the bilinear form $B^{(1)}(v,v^*)$ is defined by (5) and

$$\langle F, v^*\rangle = \int_\omega (\Phi_j^{(0)} v_{j0}^* + 3h\Phi_j^{(1)} v_{j1}^*)d\omega. \tag{8}$$

Note that the bilinear form (5) can be represented as follows

$$B^{(1)}(v,v^*) := \int_\omega \Big\{(h\rho\nu^2 v_{i0}v_{i0}^* + 3h^3\rho\nu^2 v_{i1}v_{i1}^*) + \sum_{r=0}^{1}\Big(r+\frac{1}{2}\Big) \times a[\lambda e_{kkr}(v)e_{iir}(v^*) + 2\mu e_{ijr}(v)e_{ijr}(v^*)]\Big\}d\omega,$$

where e_{ijr} $(r=0,1)$ is given by the following expression

$$e_{ij0} = \frac{1}{2}h(v_{i0,j}+v_{j0,i}), \quad e_{ij1} = \frac{1}{2}h^2(v_{i1,j}+v_{j1,i}).$$

Further, we construct the vectors in $\Omega := \{(x;x_3) : x\in\omega, -h(x) < x_3 < h(x)\}$:

$$w_i(x,x_3) = \frac{1}{2}v_{i0}(x) + \frac{3}{2}x_3 v_{i1}(x), \quad i=1,2,3, \tag{9}$$

$$w_i^*(x,x_3) = \frac{1}{2}v_{i0}^*(x) + \frac{3}{2}x_3 v_{i1}^*(x), \quad i=1,2,3. \tag{10}$$

It can be shown that

$$B(w,w^*) := \int_\Omega \Big[2\rho\nu^2 w_i w_i^* + \sigma_{ij}(w)e_{ij}(w^*)\Big]d\Omega = B^{(1)}(v,v^*), \tag{11}$$

where $w(x,x_3) := (w_1,w_2,w_3)$ and $w^*(x,x_3) := (w_1^*,w_2^*,w_3^*)$ are vectors and $B(w,w^*)$ is the bilinear form corresponding to the 3D potential energy for the displacement vector w. Owing to positive definiteness of the potential energy for $2\lambda+3\mu>0$ and $\mu>0$.

$$\begin{aligned}
B(w,w) &\ge 2\rho\nu^2\sum_{i=1}^{3}\int_\Omega w_i^2 d\Omega + c_2\sum_{i,j=1}^{3}\int_\Omega [e_{ij}(w)]^2 d\Omega = \sum_{i=1}^{3}\int_\Omega 2\rho\nu^2 w_i^2 d\Omega \\
&+c_2\int_\omega d\omega\int_{-h}^{h}\Big(\frac{1}{2}ae_{ij0}(v)+\frac{3}{2}a^2x_3e_{ij1}(v)\Big)\cdot\Big(\frac{1}{2}ae_{ij0}(v)+\frac{3}{2}a^2x_3e_{ij1}(v)\Big)dx_3 \\
&= \sum_{i=1}^{3}\int_\Omega 2\rho\nu^2 w_i^2 d\Omega + c_2\int_\omega\sum_{i,j=1}^{3}\Big(\frac{1}{2}e_{ij0}^2(v)+\frac{3}{2}e_{ij1}^2(v)\Big)\frac{d\omega}{h} \qquad (12) \\
&= \int_\omega d\omega\int_{-h}^{h}2\rho\nu^2\cdot\frac{1}{4}(v_{i0}v_{i0}+9x_3^2v_{i1}v_{i1}+6x_3v_{i0}v_{i1})dx_3 + c_2\sum_{i,j=1}^{3}\int_\omega\frac{1}{2}e_{ij0}^2(v)\frac{d\omega}{h} \\
&= \sum_{i=1}^{3}\int_\omega h\rho\nu^2(v_{i0}^2+3h^2v_{i1}^2)d\omega + c_2\sum_{i,j=1}^{3}\int_\omega\frac{1}{2}e_{ij0}^2(v)\frac{d\omega}{h}.
\end{aligned}$$

After denoting by $c_0 := min\{1, c_2\}$ we obtain

$$B^{(1)}(v, v^*) \geq c_0 \int_\omega \Big(h\rho\nu^2 \sum_{i=1}^{3} (v_{i0}^2 + 3h^2 v_{i1}^2) + \frac{1}{2} \sum_{i,j=1}^{3} e_{ij}^2(v) \cdot \frac{1}{h} \Big) d\omega.$$

Remark 1. In view of (11) and (12) we conclude that $B^{(1)}(v, v) = 0$ yields $v = 0$. Indeed, if $B^{(1)}(v, v) = 0$, then $B(w, w) = 0$ by (12). In turn, the latter equality for the strain tensor e_{ij} corresponding to the displacement vector w implies that $e_{ij}(w) = 0$, $i, j = 1, 2, 3$, i.e., w is a rigid displacement. Since w vanishes on the part of the lateral boundary Γ_1 of Ω (which contains at least three points not belonging to a straight line) it follows that $w = 0$ in Ω. Therefore v_{ir}, $r = 0, 1$, due to formulas (9) and (10).

Denote by $D(\omega)$ a space of infinitely differentiable functions with compact support in ω and introduce the linear form $[D(\omega)]^6$ by the formula:

$$\begin{aligned}
(v, v^*)_{X^\kappa_{1,\nu}} &= \int_\omega \Big[h\rho\nu^2 (v_{i0} v_{i0}^* + 3h^2 v_{i1} v_{i1}^*) \\
&\quad + \Big(\frac{1}{2} e_{ij0}(v) e_{ij0}(v^*) + \frac{3}{2} e_{ij1}(v) e_{ij1}(v^*) \Big) \frac{1}{h} \Big] d\omega \\
&= \int_\omega \Big[h\rho\nu^2 (v_{i0} v_{i0}^* + 3h^2 v_{i1} v_{i1}^*) \Big] d\omega \\
&\quad + \frac{1}{8} \sum_{i,j=1}^{3} \int_\omega \Big\{ \Big[h(v_{i0,j} + v_{j0,i}) \Big] \Big[h(v_{i0,j}^* + v_{j0,i}^*) \Big] \\
&\quad + 3 \Big[h^2 (v_{i1,j} + v_{j1,i}) \Big] \Big[h^2 (v_{i1,j}^* + v_{j1,i}^*) \Big] \Big\} \frac{d\omega}{h}.
\end{aligned}$$

Denote by $X^\kappa_{1,\nu} := X^\kappa_{1,\nu}(\omega)$ the completion of the space $[D(\omega)]^6$ with the help of the norm:

$$\begin{aligned}
\|v\|^2_{X^\kappa_{1,\nu}} &= \int_\omega \Big[h\rho\nu^2 (v_{10}^2 + v_{20}^2 + v_{30}^2 + 3h^2 v_{11}^2 + 3h^2 v_{21}^2 + 3h^2 v_{31}^2) \\
&+ \frac{h}{8} \Big(4v_{10,1}^2 + 4v_{20,2}^2 + 2(v_{10,2} + v_{20,1}^2)^2 + 2v_{30,1}^2 + 2v_{30,2}^2 \Big) \\
&+ \frac{3h^3}{8} \Big(4v_{11,1}^2 + 4v_{21,2}^2 + 2(v_{11,2} + v_{21,1})^2 + 2v_{31,1}^2 + 2v_{31,2}^2 \Big) \Big] d\omega. \qquad (13)
\end{aligned}$$

$X^\kappa_{1,\nu}$ is a Hilbert space.

Now we can formulate the weak setting of the homogeneous Dirichlet problem (6), (7):

Find a vector $v = (v_{10}, v_{20}, v_{30}, v_{11}, v_{21}, v_{31})^\top \in X^\kappa_{1,\nu}$, satisfying the equality

$$B^{(1)}(v, v^*) = \langle F, v^* \rangle \quad \text{for all} \quad v^* \in X^\kappa_{1,\nu}. \qquad (14)$$

Here, the vector F belongs to the adjoint space $[X^\kappa_{1,\nu}]^*$, in general, and $\langle \cdot, \cdot \rangle$ denotes duality brackets between the spaces $[X^\kappa_{1,\nu}]^*$ and $X^\kappa_{1,\nu}$.

Lemma 2. *The bilinear form $B^{(1)}(\cdot,\cdot)$ is bounded and strictly coercive in the space $X^\kappa_{1,\nu}(\omega)$, i.e., there are positive constant C_0 and C_1 such that*

$$|B^{(1)}(v,v^*)| \le C_1\|v\|_{X^\kappa_{1,\nu}}\|v^*\|_{X^\kappa_{1,\nu}}, \tag{15}$$

$$B^{(1)}(v,v) \geqslant C_0\|v\|^2_{X^\kappa_{1,\nu}} \tag{16}$$

for all $v,v^*\in X^\kappa_{1,\nu}$.

Proof. Since $[D(\omega)]^6$ is dense in $X^\kappa_{1,\nu}$ it suffices to show inequalities (15) and (16) for $v,v^*\in[D(\omega)]^6$. By the 6-dimensional vectors v and v^* defined in ω, we construct $3D$ vectors (9) and (10) defined in Ω. Owing to equalities (11), (12), and Hooke's law we have

$$\begin{aligned}
&|B^{(1)}(v,v^*)|^2=|B^{(1)}(w,w^*)|^2\\
&=\Big[\int_\Omega 2\rho\nu^2 w_iw_i^*+(2\mu e_{ij}(w)+\lambda\delta_{ij}e_{kk}(w))e_{ij}(w^*)d\Omega\Big]^2\\
&\le\Big|\int_\Omega 2\rho\nu^2 w_iw_i^*d\Omega\Big|^2+\Big|\int_\Omega(2\mu e_{ij}(w)+\lambda\delta_{ij}e_{kk}(w))e_{ij}(w^*)d\Omega\Big|^2\\
&\le\int_\Omega 2\rho\nu^2 w_i^2d\Omega\int_\Omega 2\rho\nu^2 w_i^{*2}d\Omega+C_2\sum_{i,j=1}^3\int_\omega e_{ij}^2(w)d\omega\sum_{i,j=1}^3\int_\omega e_{ij}^2(w^*)d\omega\\
&=\int_\omega 2\rho\nu^2h(v_{i0}^2+3h^2v_{i1}^2)d\omega\int_\omega 2\rho\nu^2h(v_{i0}^{*2}+3h^2v_{i1}^{*2})d\omega\\
&+C_2\int_\omega\sum_{i,j=1}^3\Big(\frac12e_{ij0}^2(v)+\frac32e_{ij1}^2(v)\Big)\frac{d\omega}{h}\int_\omega\sum_{i,j=1}^3\Big(\frac12e_{ij0}^2(v^*)+\frac32e_{ij1}^2(v^*)\Big)\frac{d\omega}{h}\\
&\le C_1\|v\|^2_{X^\kappa_{1,\nu}}\|v^*\|^2_{X^\kappa_{1,\nu}},
\end{aligned}$$

where

$$C_1:=\max\{2,C_2\}.$$

Whence (15) follows. Inequality (16) immediately follows from (11) and (12).

Theorem 3. *Let $F\in[X^\kappa_{1,\nu}]^*$. Then the variational problem (14) has a unique solution $v\in X^\kappa_{1,\nu}$ for an arbitrary value of the parameter κ and*

$$\|v\|_{X^\kappa_{1,\nu}}\le\frac{1}{C_0}\|F\|_{[X^\kappa_{1,\nu}]^*}.$$

Proof. The proof directly follows from the Lax-Milgram theorem (see Appendix A, Theorem A.1).

It can be easily shown that if $F\in[L(\omega)]^6$ and $\operatorname{supp}F\cap\overline{\gamma}_0=\emptyset$, then $F\in[X^\kappa_1]^*$ and

$$\langle F\,,\,v^*\rangle=\int_\omega F(x)\,v^*(x)\,d\omega,$$

since $v^* \in [H^1(\omega_\varepsilon)]^6$, where ε is a sufficiently small positive number such that $\operatorname{supp} F \subset \omega_\varepsilon = \omega \cap \{x_2 > \varepsilon\}$. Therefore,

$$|\langle F\,,\, v^*\rangle| = \Big| \int\limits_\omega F(x)\, v^*(x)\, d\omega \Big| \le ||F||_{[L_2(\omega)]^6}\, ||v^*||_{[L_2(\omega_\varepsilon)]^6}$$

$$\le ||F||_{[L_2(\omega)]^6}\, ||v^*||_{[H^1(\omega_\varepsilon)]^6} \le C_\varepsilon\, ||F||_{[L_2(\omega)]^6}\, ||v^*||_{X_1^\kappa}\,.$$

In this case we obtain the estimate

$$||v||_{X_1^\kappa} \le \frac{C_\varepsilon}{C_0}\, ||F||_{[L_2(\omega)]^6}\,.$$

Now we establish a representation of the space $X^\kappa_{1,\nu}$ as a weighted Sobolev space. To this end, we introduce the following space:

$$Y_1^\kappa := \Big[\overset{0}{W}{}^1_{2\,,\kappa}(\omega)\Big]^3 \times \Big[\overset{0}{W}{}^1_{2\,,3\kappa}(\omega)\Big]^3,$$

where $\overset{0}{W}{}^1_{2\,,\kappa_1}(\omega)$ is a completion $\mathcal{D}(\omega)$ by means of the norm

$$\|f\|^2_{\overset{0}{W}{}^1_{2\,,\kappa_1}}(\omega) := \int_\omega x_2^{\kappa_1}\Big(|\nabla f|^2\Big) d\omega, \quad \nabla f = (f_{,1}\,, f_{,2}\,).$$

The norm in the space Y_1^κ for a vector $(v_{10}, v_{20}, v_{30}, v_{11}, v_{21}, v_{31})$ reads as

$$\|v\|^2_{Y_1^\kappa} := \int_\omega \Big[x_2^\kappa\Big(\sum_{j=1}^3 |\nabla v_{j0}|^2\Big) + x_2^{3\kappa}\Big(\sum_{j=1}^3 |\nabla v_{j0}|^2\Big)\Big] d\omega.$$

Theorem 4. *Let $\kappa < 1$ and $\kappa \neq \frac{1}{3}$. Then the linear spaces $X^\kappa_{1,\nu}$ and Y_1^κ as sets of vector functions coincide and the norms $\|\cdot\|_{X^\kappa_{1,\nu}}$, $\|\cdot\|_{Y_1^\kappa}$ are equivalent.*

Proof. Rewrite formula (13) in the form

$$\|v\|^2_{X^\kappa_{1,\nu}} = \int_\omega \Big[h_1 x_2^\kappa \rho\nu^2(v_{10}^2 + v_{20}^2 + v_{30}^2) + 3h_1^3 x_2^{3\kappa}\rho\nu^2(v_{11}^2 + v_{21}^2 + v_{31}^2)$$
$$+ \frac{h_1 x_2^\kappa}{8}\Big(4v^2_{10,1} + 4v^2_{20,2} + 2(v_{10,2} + v^2_{20,1})^2 + 2v^2_{30,1} + 2v^2_{30,2}\Big)$$
$$+ \frac{3h_1^3 x_2^{3\kappa}}{8}\Big(4v^2_{11,1} + 4v^2_{21,2} + 2(v_{11,2} + v_{21,1})^2 + 2v^2_{31,1} + 2v^2_{31,2}\Big)\Big] d\omega,$$
$$h_1 := \frac{h_0}{2}.$$

Let us at first prove the following inequality,

$$\|v\|^2_{X^\kappa_{1,\nu}} \le C_3 \|v\|^2_{Y_1^\kappa}. \tag{17}$$

Let us denote by

$$I_1 := \int_\omega \Big[h_1 x_2^\kappa \rho\nu^2 (v_{10}^2 + v_{20}^2 + v_{30}^2) + 3h_1^3 x_2^{3\kappa}\rho\nu^2 (v_{11}^2 + v_{21}^2 + v_{31}^2)\Big] d\omega,$$

and by

$$I_2 := \int_\omega \frac{h_1 x_2^\kappa}{8}\Big(4v_{10,1}^2 + 4v_{20,2}^2 + 2(v_{10,2} + v_{20,1}^2)^2 + 2v_{30,1}^2 + 2v_{30,2}^2\Big)$$

$$+\frac{3h_1^3 x_2^{3\kappa}}{8}\Big(4v_{11,1}^2 + 4v_{21,2}^2 + 2(v_{11,2} + v_{21,1})^2 + 2v_{31,1}^2 + 2v_{31,2}^2\Big)\Big] d\omega.$$

The inequality,

$$I_2 \le C_4 \|v\|^2_{Y_1^\kappa} \tag{18}$$

is a consequence of Hardy's inequality (see [3]).

Let us now consider

$$|I_1| \le h_1\rho\nu^2 \Big|\int_\omega x_2^2 x_2^{\kappa-2}(v_{10}^2 + v_{20}^2 + v_{30}^2) d\omega\Big|$$

$$+3h_1^3\rho\nu^2 \Big|\int_\omega x_2^2 x_2^{3\kappa-2}(v_{11}^2 + v_{21}^2 + v_{31}^2) d\omega\Big|$$

$$\le h_1\rho\nu^2 l^2 \int_\omega x_2^{\kappa-2}(v_{10}^2 + v_{20}^2 + v_{30}^2) d\omega$$

$$+3h_1^3\rho\nu^2 l^2 \int_\omega x_2^{3\kappa-2}(v_{11}^2 + v_{21}^2 + v_{31}^2) d\omega$$

$$\le C_5 h_1\rho\nu^2 l^2 \int_\omega x_2^\kappa |\nabla v_{j0}|^2 d\omega + C_6 3h_1^3\rho\nu^2 l^2 \int_\omega x_2^{3\kappa}|\nabla v_{j1}|^2 d\omega$$

$$\le C_7 \int_\omega \Big(x_2^\kappa |\nabla v_{j0}|^2 + x_2^{3\kappa}|\nabla v_{j1}|^2\Big) d\omega$$

$$\text{if } \ \nu^2 \le \frac{1}{h_2\rho l^2}, \ \ h_2 := \max\{h_1; 3h_1^3\}, \ \ C_7 := \max\{C_5; C_6\},$$

i.e.,

$$I_1 \le C_7 \|v\|^2_{Y_1^\kappa} \ \text{ if } \ \nu^2 \le \frac{1}{h_2\rho l^2}, \ \ h_2 := \max\{\frac{h_0}{2}; \frac{3h_0^3}{8}\}. \tag{19}$$

From (18) and (19) in case of $\nu^2 \le 1/(h_2\rho l^2)$ we get (17).

Let $v \in X_{1,\nu}^\kappa$ and show that $v \in Y_1^\kappa$. We have to prove that

$$\|v\|^2_{Y_1^\kappa} \le C_0 \|v\|^2_{X_{1,\nu}^\kappa} \tag{20}$$

C_0 does not depend on v.

Denote by $\mathcal{D}(\omega)$ a space of infinitely differentiable functions with compact support in ω and introduce the linear form $[\mathcal{D}(\omega)]^{\prime}$by the formula

$$(v,v^*)_{X_1^\kappa}=\int_\omega\Big(\frac{1}{2}e_{ij0}(v)e_{ij0}(v^*)+\frac{3}{2}e_{ij1}(v)e_{ij1}(v^*)\Big)\frac{1}{h}d\omega$$
$$=\frac{1}{8}\sum_{i,j=1}^{3}\int_\omega\Big\{\Big[h(v_{i0,j}+v_{j0,i})\Big]\Big[h(v^*_{i0,j}+v^*_{j0,i})\Big]$$
$$+3\Big[h^2(v_{i1,j}+v_{j1,i})\Big]\Big[h^2(v^*_{i1,j}+v^*_{j1,i})\Big]\Big\}\frac{d\omega}{h}.$$

Denote by $X_1^\kappa:=X_1^\kappa(\omega)$ the completion of the space $[\mathcal{D}(\omega)]^6$ with the help of the norm

$$\|v\|^2_{X_1^\kappa}=\int_\omega\frac{h}{8}\Big(4v^2_{10,1}+4v^2_{20,2}+2(v_{10,2}+v^2_{20,1})^2+2v^2_{30,1}+2v^2_{30,2}\Big)$$
$$+\frac{3h^3}{8}\Big(4v^2_{11,1}+4v^2_{21,2}+2(v_{11,2}+v_{21,1})^2+2v^2_{31,1}+2v^2_{31,2}\Big)d\omega.$$

X_1^κ is a Hilbert space (see [3]).

It is evidently,

$$\|v\|^2_{X_1^\kappa}\le C_9\|v\|^2_{X^\kappa_{1,\nu}}.$$

On the other hand, the subset of

$$\|v\|^2_{Y_1^\kappa}\le C_{10}\|v\|^2_{X_1^\kappa}$$

is shown in [3]. The last two inequality leads to (20) if $\nu^2\le 1/(h_2\rho l^2)$, $h_2:=\max\{\frac{h_0}{2},\frac{3h_0^3}{8}\}$.

Remark 5. From the trace theorem (see Appendix A, Theorem A.4) it follows that
(i) if $\kappa<1/3$, then the components of the unique solution v to the problem 14 possesses the zero traces on $\partial\omega$;
(ii)if $1/3<\kappa<1$, then the components v_{i0} have the zero traces on the whole of the boundary $\partial\omega$, while the components v_{i1} have no traces on the part $\gamma_0\subset\partial\omega$ due to the order degeneration of equations (2).

Remark 6. From the Theorem 1.4 by Hardy's inequality it follows that for $\kappa<1$ and $\kappa\neq 1/3$ the linear functional defined by (8) is bounded if

$$x_2\Phi_j^{(0)},\quad x_2^{1/2}\Phi_j^{(1)}\in L_2(\omega),\quad j=1,2,3.$$

A. Appendix

A.1. The Lax-Milgram theorem. Let V be a real Hilbert space and let $J(w,v)$ be a bilinear form defined on $V\times V$. Let this form be continuous, i.e., let there exist a constant $K>0$ such that

$$|J(w,v)|\le K\|w\|_{V}\|v\|_{V}$$

holds $\forall w, v \in V$ and V-elliptic, i.e., let there exist a constant $\alpha > 0$ such that

$$J(w, w) \geqslant \alpha \|w\|_V^2$$

holds $\forall w \in V$. Further let F be a bounded linear functional from V^* dual of V. Then there exists one and only one element $z \in V$ such that

$$J(z, v) = \langle F, v \rangle \equiv Fv \quad \forall v \in V$$

and

$$\|z\|_V \leq \alpha^{-1} \|F\|_{V^*}.$$

Let ω be as in Section 1 and let $\mathcal{D}(\omega)$ be a space of infinitely differentiable functions with compact support in ω.

A.2. Hardy's Inequality. For every $f \in \mathcal{D}(\omega)$ and $\nu \neq 1$ there holds the inequality

$$\int\limits_{\omega} x_2^{\nu-2} f^2(x)\, d\omega \leq C_\nu \int\limits_{\omega} x_2^{\nu} |\nabla f(x)|^2\, d\omega, \tag{21}$$

where the positive constant C_ν is independent of f.

By completion of $\mathcal{D}(\omega)$ with the norm

$$||f||^2_{\overset{\circ}{W}{}^1_{2,\nu}(\omega)} := \int\limits_{\omega} x_2^{\nu} |\nabla f(x)|^2\, d\omega,$$

we conclude that the inequality (21) holds for arbitrary $f \in \overset{\circ}{W}{}^1_{2,\nu}(\omega)$.

For proof see [5].

A.3. Korn's Weighted Inequality. Let $\varphi = (\varphi_1, \varphi_2) \in [\overset{\circ}{W}{}^1_{2,\nu}(\omega)]^2$ and $\nu \neq 1$. Then

$$\int\limits_{\omega} x_2^{\nu} [\, |\nabla \varphi_1(x)|^2 + |\nabla \varphi_2(x)|^2 \,]\, d\omega$$

$$\leq C_\nu \int\limits_{\omega} x_2^{\nu} [\varphi_{1,1}^2(x) + \varphi_{2,2}^2(x) + (\varphi_{1,2}(x) + \varphi_{2,1}(x))^2 \,]\, d\omega,$$

where the positive constant C_ν is independent of φ.

The proof can be found in [5], [13].

A.4. Trace Theorem. Let $0 < \nu < 1$ and $f \in \overset{\circ}{W}{}^1_{2,\nu}(\omega)$. Then the trace of the function f equals to zero on $\partial\omega$.

For proof see [5], [9], [10].

R E F E R E N C E S

1. Chinchaladze N. Vibration of the plate with two cusped edges. *Proceedings of I.Vekua Institute of Applied Mathematics of Tbilisi State University*, **52** (2002), 30-48.

2. Chinchaladze N. On some nonclassical problems for differential equations and their applications to the theory of cusped prismatic shells. *Lecture Notes of TICMI*, **9** (2008), 1-92.

3. Chinchaladze N., Jaiani G., Gilbert R., Kharibegashvili S., Natroshvili D. Existence and uniqueness theorems for cusped prismatic shells in the N-th hierarchical model. *Mathematical Methods in Applied Sciences*, **31**, 11 (2008), 1345-1367.

4. Chinchaladze N., Gilbert R. Cylindrical vibration of an elastic cusped plate under action of an incompressible fluid in case of N=0 approximation of I.Vekuas hierarchical models. *Complex Variables*, **50**, 7-11 (2005), 479-496.

5. Devdariani G., Jaiani G.V., Kharibegashvili S.S., Natroshvili D. The first boundary value problem for the system of cusped prismatic shells in the first approximation. *Appl. Math. Inform.*, **5**, 2 (2000), 26-46.

6. Jaiani G. Cusped shell-like structures. *SpringerBriefs in Applied Science and Technology, Springer-Heidelberg-Dordrecht-London-New York*, 2011.

7. Jaiani G. Theory of cusped Euler-Bernoulli beams and Kirchhoff-Love plates. *Lecture Notes of TICMI*, **3** (2002), 1-129.

8. Makhover E. Bending of a plate of variable thickness with a cusped edge. (Russian) *Scien- tific Notes of Leningrad State Ped. Institute*, **17**, 2 (1957), 28-39.

9. Nikolskii S.M., Lizorkin P.I., Miroshin N.V. Weighted functional spaces and their applications to the investigation of boundary value problems for degenerate elliptic equations. *Izvestia Vysshikh Uchebnykh Zavedenii*, **8**, 315 (1988), 4-30.

10. Triebel H. Interpolation Theory, Function spaces, Differential Operators. *Berlin, "DVW"*, 1978.

11. Vekua I.N. On a way of calculating of prismatic shells. (Russian) *Proceedings of A. Razmadze Institute of Mathematics of Georgian Academy of Sciences*, **21** (1955), 191-259.

12. Vekua I.N. Shell Theory: General Methods of Construction. Pitman Advanced Publishing Program, Boston-London-Melbourne, 1985.

13. Vishik M.I. Boundary value problems for elliptic equations degenerating of the boundary of domain. (Russian) *Math. Sb.*, **35(77)**, 3 (1954), 513-568.

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