**Proceedings of I. Vekua Institute
of Applied Mathematics
Vol. 61-62, 2011-2012**

SOME ANALYTICAL AND GEOMETRICAL ASPECTS OF THE STABLE PARTIAL INDICES

Bojarski B., Giorgadze G.

Abstract. In this paper, main properties of the partial indices of the Riemann boundary value problem, introduced by Muskhelishvili and Vekua, are considered. This important invariant point of view gives a modern approach to two central problems of complex analysis: Riemann–Hilbert monodromy and boundary value problems.

Keywords and phrases: Index, partial indices, Chern number, vector bundle, monodromy, connection.

AMS subject classification (2000): 34M50, 34M35.

1. Introduction

We consider the impact of stability properties of the partial indices on the solvability conditions of two classical problems: 1) the Riemann boundary value problem, consisting in finding a piecewise holomorphic matrix function with some boundary condition and 2) the Riemann monodromy problem, consisting in the construction of a Fuchs type system of differential equations with given monodromy. Both problems are given in an unfinished work of Riemann [35]. There, Riemann considered the first problem as an auxiliary method for solving the second problem.

Both problems have their specific methods of investigation and in the scientific literature they are considered as independent central problems of different areas of the complex analysis, the first one concerned with boundary value problems from the theory of analytic functions and the second — with the study of the analytic theory of differential equations

Starting from the second half of the 19th century, differential equations with meromorphic coefficients were a subject of intensive research. In particular, L. Fuchs [17] proved regularity properties of n-th order differential equations. During this period fundamental results have been obtained by Hilbert, Poancaré, Schlesinger, Birkhoff (see [26]). In particular, the form of the fundamental matrix in a neighborhood of a regular singular point was established (Poincaré [33]), influence of the configuration of singular points of a differential equation on monodromy matrices was investigated (Shlesinger [38]), canonical form of systems of differential equations, in general case, in the neighborhood of a regular singular point was found (Birkhoff [3]). In the first decade of the 20th century investigation of regular systems was to an extent stimulated by the Hilbert 21st problem [27], whose motivation, according to Hilbert, was that after its solution the theory of analytic differential equations on the complex plane would acquire finalized form.

J. Plemelj in papers [32] successfully applied the Fredholm theory of integral equations developed by Hilbert to investigate behavior of analytic functions near boundary points and gave solution of the Riemann monodromy problem for regular systems of differential equations.

The works mentioned above, in particular [4],[32] contain certain defects, which were caused by the noncommutation properties of matrix functions. As it is well known today, not only proofs of theorems contained errors, but the theorems themselves were not true [10],[2].

Later for the Riemann boundary value problem Muskhelishvili and Vekua [29] introduced the concept of partial indices. This invariant of the boundary problem turned out to be the reason of the above imprecisions in [4],[32]. In particular, the case when partial indices are stable is "generic" and in this case the solution of the Riemann monodromy problem [12] and the Birkhoff standard form theorem are both correct [2]. Moreover, for stable partial indices the Riemann boundary value problem is of constructive character, i. e. exactly solvable, just as in the one dimensional case.

Muskhelishvili in [29] has several times remarked about imprecisions in reasonings of Plemelj and give absolutely new proof of the boundary value problem. He moreover noticed that without introduction of partial indices, solution of the problem cannot be considered complete. As later was shown by Bolibruch [12], the complete decision of the monodromy problem strictly depends on the partial indices.

Below we give detailed analysis of the relationship between the partial indices of the Riemann boundary value and monodromy problems.

1. Classical versions of the Riemann problems

1.1. The Riemann boundary value problem

Let Γ be a smooth closed positively oriented loop in CP^1 which separates CP^1 into two connected domains U_+ and U_-. Suppose $0 \in U_+$ and $\infty \in U_-$. Let us denote by Ω the space of all Hölder-continuous matrix functions $f : \Gamma \to GL_n(\mathbb{C})$ with the natural topology.

Problem I. Find a piecewise holomorphic vector function $\Phi(t)$ in $U_+ \cup U_-$, which admits continuous boundary values on Γ and Γ the boundary condition

$$\Phi^+(t) = f(t)\Phi^-(t), t \in \Gamma$$

and has finite order at ∞.

1.2. The monodromy problem

Let $s_1, ..., s_m \in \mathbb{CP}^{\,1}$ be some points, with no ∞ among them, and let $\varrho : \pi_1(\mathbb{CP}^{\,1} \setminus \{s_1, ..., s_m\}, z_0) \to \mathrm{GL_n}(\mathbb{C})$ be a representation.

The problem consists in the following:

Problem II. For the representation ϱ, find a Fuchs system

$$df = \left(\sum_{j=1}^{m} \frac{A_j}{z - s_j} dz \right) f, \tag{1}$$

such that its monodromy representation coincides with ϱ. In (1), the A_j are constant matrices satisfying the condition $\sum_{j=1}^{m} A_j = 0$.

Today the standard name of this problem is the Hilbert 21st problem.

1.3. Solution of the problem I

Let the matrix function $X(z)$ be a solution of problem I [30], [40]. It is called *canonical* if it has the form

$$\chi(z) = \chi_0(z) \text{ on } z \in U_+, \chi(z) = \chi_0(z) D^{-1}(z), \text{ on } z \in U^-,$$

where $\chi_0(z)$ is a holomorphic matrix function in $U^+ \cup U^-$, admitting a continuous inverse $\chi_0^{-1}(z)$ in $\overline{U}^+$ and $\overline{U}^-$, respectively, including the point $z = \infty$ and $\mathsf{det}\chi_0(\infty) = 1$. The matrix function $D(z)$ is diagonal $D(z) = \mathsf{diag}(z^{k_1}, z^{k_2}, ..., z^{k_n})$ and the integers $k_1, k_2, ..., k_n$ satisfy the inequalities

$$k_1 \geq k_2 \geq ... \geq k_n.$$

Theorem 1. *[29] For every $f(t) \in \Omega$ the canonical solution always exists. The integer valued vector $K = (k_1, ..., k_n)$ does not depend on the considered canonical solution.*

The integers $k_1, k_2, ..., k_n$ are called the partial indices of boundary problem 1 or of the matrix function $f(t)$. In [29] the following formula for the global index k of problem I is given:

$$k = k_1 + k_2 + ... + k_n \text{ with } k = \frac{1}{2\pi} \Delta_\Gamma \mathsf{argdet} G(t).$$

1.4. Factorization of the matrix function

Let

$\Omega^+ = \{f \in \Omega : f$ be the boundary value of the matrix function holomorphic in $U^+\}$.

$\Omega^- = \{f \in \Omega : f$ be the boundary value of the matrix function holomorphic in U$^-$ and is regular at infinity $f(\infty) = \mathbf{1}\}$.

Theorem 2. *Any matrix function $f \in \Omega$ can be represented as*

$$f(t) = f^-(t) d_K f^+(t), \tag{2}$$

where $f^\pm \in \Omega^\pm$ and d_K is a diagonal matrix $d_K = diag(t^{k_1}, ..., t^{k_n})$ satisfying the condition $k_1 \geq ... \geq k_n$.

The diagonal matrix d_K will be called the characteristic loop of the corresponding matrix function, $K = (k_1, k_2, ..., k_n)$ will be called the characteristic multi-index or partial indices of f. Two matrix functions $f, g \in \Omega$ will be called equivalent, if f and g have identical characteristic multi-indices.

For $K = (k_1, k_2, ..., k_n)$, denote by Ω_K the set of equivalence classes of loops Ω. The representation (2) is not unique, but if one fixes f^+ (or f^-) then f^- (respectively f^+) will be uniquely defined.

If the matrix function $f(t)$ admits the representation (2), then it is represented as

$$f(t) = \widetilde{f}^+(t)\widetilde{d}_{\widetilde{K}}\widetilde{f}^-(t), \tag{3}$$

where $\widetilde{d}_{\widetilde{K}} = diag(t^{\kappa_1}, t^{\kappa_2}, ..., t^{\kappa_2})$ and $\widetilde{f}^+(t), \widetilde{f}^-(t)$ are boundary values of functions holomorphic in U^+, U^- respectively. Thus $\sum_{j=1}^n k_j = \sum_{j=1}^n \kappa_j$ and for given n integer vectors $(k_1, k_2, ..., k_n)$, $(\kappa_1, \kappa_2, ..., \kappa_n)$ there exists a matrix function which admits the representation (2) and (3) with diagonal matrices $d_K = (k_1, k_2, ..., k_n)$ and $\widetilde{d}_{\widetilde{K}} = (\kappa_1, \kappa_2, ..., \kappa_n)$ respectively.

In general, there exist many different factorizations of the matrix function similar to (2) or (3). Seemingly the first such representation has been given by L. Sauvage [37] in the analytic theory of differential equations. In different areas of mathematics analogical factorizations were used by Schlesinger, Hilbert, Birkhoff, Wiener and Hopf, Grothendieck, Simonenko (see modern overview of the matrix factorization theory [22]).

1.5. Stability of the partial indices

The partial indices of a matrix function are called *stable*, if in its sufficiently small neighborhood all matrix functions have the same partial indices.

The topological space Ω decomposes into a countable number of open components

$$\Omega^k = \{f \in \Omega, \Delta_\Gamma argdetf(t) = 2\pi k\}, \Omega = \cup_k \Omega^k, k \in \mathbb{Z}.$$

One has $\Omega^k = \cup_K \Omega_K$ and Ω^k is connected.

Theorem 3. *[8], [21] The set of partial indices is stable iff* $|k_i - k_j| \leq 1$, $i, j = 1, 2, ..., n$.

The partial indices completely describe the solvability properties of problem I. They also determine solvability conditions for problem II (see below). The global index is the only topological invariant of problem I in the sense that each Ω^k is connected. Thus in view of theorem 1.3 one can say that generally the topological invariant completely describes the qualitative character of the solutions of problem I (see [7]).

The deformation $\Omega_{K'}$, $K' = (k_1', k_2', ..., k_n')$ of the strata Ω_K, $K = (k_1, k_2, ..., k_n)$, is called elementary, if $k_i' = k_i$ except for two indices p and q, $p < q$, for which we have $k_p' = k_p - 1$, $k_q' = k_q + 1$.

Theorem 4. *[8] The matrix functions $f_1(t)$ and $f_2(t)$ belong to the same Ω^k iff $f_1(t)$ and $f_2(t)$ are homotopic.*

From above theorems it follows, that in Ω^k, for every k, there exists a diagonal matrix with stable partial indices $(p+1, p+1, ..., p+1, p, p, ..., p)$, where $k = np + r$, $0 \leq r < n$ and every matrix function can be transformed into such stable diagonal matrix by elementary operations. Besides, the multi-index K as a function of $f \in \Omega^k$ has discontinuities only on the strata Ω_K.

Theorem 5. *[8] Let $0 < k < n$, then in Ω^k, among the strata Ω_K the only ones which are open and dense subspaces are the ones with $K = (1, 1, ..., 1, 0, 0, ...0)$, i. e. for such K, $\Omega^k \setminus \Omega_K$ does not contain interior points.*

In the particular case, when $k = np$ and $k = 0$ we have

Corollary 1. *1) If $k = np$, from the stability of the partial indices it follows, that $K = (p, p, ..., p)$.*

2) If $k = 0$ and $K = (k_1, ..., k_n)$ is stable, then $K = (0, 0, ..., 0)$.

The Banach Lie group $\Omega^+ \times \Omega^-$ acts analytically on Ω via

$$f \stackrel{\alpha}{\longmapsto} h_1 f h_2^{-1}, f \in \Omega, h_1 \in \Omega^+, h_2 \in \Omega^-.$$

It is clear, that the orbit of the diagonal matrix d_K by the action α is Ω_K.

The stability subgroup H_K of f under the action α consists of those pairs (h_1, h_2) of upper triangular matrix-functions where the (i, j)-th entry in h_1 is a polynomial in z of degree at most $(k_1 - k_2)$ and $f = h_1 f h_2^{-1}$; the space H_K has finite dimension

$$\dim H_K = \sum_{k_i \geq k_j} (k_i - k_j + 1).$$

The stratum Ω_K is a locally closed analytical submanifold of Ω and codimension of Ω_K in Ω is equal to

$$\dim \Omega/\Omega_K = \sum_{k_i \rangle k_j} (k_i - k_j - 1).$$

From the topological point of view the spaces $\Omega, \Omega^k, \Omega_K$ are considered in [7], [34], [25].

1.6. Connection between the problem I and problem II

Let $s_1, ..., s_m \in \Gamma$ and $M_1, ..., M_m \in GL_n(\mathbb{C})$. We will say that the piecewise constant matrix function $G(t)$ is induced from collections $s = \{s_1, ..., s_m\}$, $M = \{M_1, ..., M_m\}$ if it is constructed in the following manner

$$G(t) = M_j \cdot ... \cdot M_1, \text{ if } t \in [s_j, s_{j+1}),$$

where M_j are monodromy matrices, corresponding to small loops going around the singular points s_j. It is possible to reduce problem I for such matrix function to the boundary value problem with the continuous transmission function [40].

Let $s_1, ..., s_m \in \Gamma$ be the points of discontinuity and suppose there exist finite limits $G(s_j + 0) = \lim_{t\to s_1+0} G(t)$ and $G(s_j - 0) = \lim_{t\to s_1-0} G(t)$. The curve Γ is supposed to be a union of smooth nonintersecting arcs $\Gamma_1, \Gamma_2, ..., \Gamma_m$ with fixed orientations. The ends of arcs Γ_j (j=1,2,...,m) are s_j and s_{j+1}.

Suppose $M_j = G^{-1}(s_j+0)G(s_j-0)$ and $E_j = \frac{1}{2\pi i}\ln M_j$ so that if λ_j^i are eigenvalues of G^j, then $\mu_j^i = \frac{1}{2\pi i}\ln\lambda_j^i$. Denote $\rho_j^i = Re\mu_j^i$ and normalize the logarithm demanding that $0 \le \rho_j^i < 1$.

Consider the matrix-functions

$$\Omega_j^+(z) = A_jG(s_j+0)(z-s_j)^{E_j}, \Omega_j^-(z) = B_j\left(\frac{z-s_j}{z-z_0}\right)^{E_j},$$

where A_j, B_j are constant matrices:

$$A_1 = E, A_j = \left[\prod_{k=1}^{j-1}\Omega_k^+(s_j)\right]^{-1}, B_1 = E, B_j = \left[\prod_{k=1}^{j-1}\Omega_k^-(s_j)\right]^{-1}, j = 2, 3, ...m.$$

Functions $\Omega_j^\pm(z)$ are holomorphic, respectively, in $U^\pm$.

Proposition 1. *The matrix-function*

$$G_1(t) = \left(\prod_{j=1}^m \Omega_j^+(t)\right)^{-1} G(t)\prod_{j=1}^m \Omega_j^-(t)$$

is continuous at points $s_1, ..., s_m$.

According to [40] there exists a canonical solution $\chi(z)$ of problem I which satisfies the following conditions:

1. $det\chi(z) \neq 0$ on $\mathbb{C}$ with possible exception of points $s_1, s_2, ..., s_m$.
2. There exists a diagonal matrix-function d_K such that $\lim_{z\to\infty}\chi(z)d_K(z)$ is invertible at ∞.
3. If s_j is some singular point then

$$\lim_{z\to s_j}(z-s_j)^\varepsilon\chi(z) = 0,$$

for some real number $\varepsilon > 0$.

Let $\omega = d\chi\cdot\chi^{-1}$ be a holomorphic 1-form on $CP^1\setminus\{s_1, ..., s_m\}$.

Theorem 6. *[32], [10] The system of differential equations*

$$df = \omega f$$

is regular with singular points $s_1, ..., s_m$ *and given monodromy.*

This theorem gives a solution of problem II in the class of regular systems.

2. Algebraic-topological version of the problems I and II

In this section by given data of problem I (the transmission function $f(t)$) and problem II (the monodromy matrices $M_1, M_2, ..., M_m$) we construct a holomorphic vector bundle on CP^1 and in terms of the invariants of this bundle pose and solve the aforementioned problems.

2.1. The vector bundle induced from problem I

Consider the holomorphic vector bundle on CP^1 which is obtained by the covering of the Riemann sphere CP^1 by three open sets $\{U^+, U^-, U_3 = CP^1 \setminus \{0, \infty\}\}$, with transition functions

$$g_{13} = f^+ : U^+ \cap U_3 \to GL_n(C),$$

$$g_{23} = f^- d_K : U^- \cap U_3 \to GL_n(C).$$

It is denoted by $E \to CP^1$.

Theorem 7. *[23] Every holomorphic vector bundle splits into direct sum of the line bundles*

$$E \cong E(k_1) \oplus ... \oplus E(k_n). \tag{4}$$

The numbers k_1,...,k_n are the Chern numbers of the line bundles $E(k_1)$,. ..,$E(k_n)$ and satisfy the conditions $k_1 \geq ... \geq k_n$. The integer-valued vector $K = (k_1, ..., k_n) \in Z^n$ is called the splitting type of the holomorphic vector bundle E. It defines uniquely the holomorphic type of the bundle E.

Connection between partial indices of the boundary value problem, characteristic multi-index of the matrix-function $f \in \Omega$ and splitting type of the holomorphic vector bundle E are presented in the following summarizing theorem:

Theorem 8. *There is a one-to-one correspondence between the strata Ω_K and holomorphic vector bundles on CP^1.*

Denote by $O(E)$ the sheaf of germs of holomorphic sections of the bundle E, then the solutions of problem I are elements of the zeroth cohomology group $H^0(CP^1, O(E))$, therefore the number l of the linearly independent solutions of problem I is $\dim H^0(CP^1, O(E))$. Since the Chern number $c_1(E)$ of the bundle E is equal to the index of $\det G(t)$, we have obtained the known criterion of solvability of the problem I. In particular the following theorem is true:

Theorem 9. *The Riemann-Hilbert boundary problem has solutions if and only if $c_1(E) \geq 0$, and the number l of linearly independent solutions is*

$$l = \dim H^0(CP^1, O(E)) = \sum_{k_i > 0} k_i + 1.$$

Consider $H^1(CP^1; O(EndE))$ — the first cohomology group with coefficients in holomorphic sections of the bundle $End(E)$. Since $End(E) \cong E \otimes E^*$, the corresponding cocycle will be $\gamma \otimes \gamma^{-1} : S^1 \to GL_{n^2}(C)$, $\gamma \otimes \gamma^{-1} = \mathrm{diag}(\mathrm{z}^{\mathrm{k_1 - k_1}}, \mathrm{z}^{\mathrm{k_1 - k_2}}, \ldots, \mathrm{z}^{\mathrm{k_n - k_n}})$. Then $End(E) = \mathcal{O}(k_1 - k_1) \oplus \mathcal{O}(k_1 -$

$k_2) \oplus \cdots \oplus \mathcal{O}(k_{n-1} - k_n) \oplus O(k_n - k_n)$. Since $\dim H^1(CP^1; O(End(E)) = \sum \dim H^1(CP^1; O(k_i - k_j))$, and $\dim H^1(CP^1; O(k)) = |k| - 1$ for $k < 0$, whereas $H^0(CP^1; O(End(O(k)))) = 0$ for $k \geq 0$, using moreover the Riemann-Roch theorem, one obtains

$$\dim H^1(CP^1; O(EndE)) = \sum_{k_i > k_j} (k_i - k_j - 1).$$

Suppose $k_1 > \cdots > k_n$. Then, following Kodaira,

$$\dim H^1(CP^1; O(End(E))) = \sum_{i<j} (k_i - k_j) - \frac{n(n-1)}{2},$$

is called the moduli number and plays important role in the theory of deformations of complex structures of vector bundles (see [18]).

2.2. Vector bundle induced from the problem II

The idea of the constructions of holomorphic vector bundles by monodromy matrices belongs to H.Röhrl [36]. Röhrl applied this construction to the solution of a weak version of problem II for noncompact Riemann surfaces.

Let $S = \{s_1, s_2, ..., s_m\}$ be a set of marked points on CP^1. Denote by $X_m = CP^1 \setminus S$. Let $\tilde{X} \to X_m$ be the universal covering map of X_m, then it is a bundle with fibre $\pi_1(X_m, z_0)$, where $z_0 \in X_m$. $\pi_1(X_m, z_0)$ is isomorphic to the group of deck transformations of this covering and therefore acts on $\tilde{X}$.

Let

$$\rho : \pi_1(X_m, z_0) \to GL_n(\mathbb{C}) \tag{5}$$

be some representation.

Consider the trivial principal bundle $\tilde{X} \times GL_n(C) \to \tilde{X}$ (or vector bundle $\tilde{X} \times C^n \to \tilde{X}$). The quotient space $\tilde{X} \times GL_n(C)/\sim$ gives a locally trivial bundle on X_m, where $\sim$ is an equivalence relation identifying the pairs $(\tilde{x}, g)$ and $(\sigma\tilde{x}, \rho(\sigma)g)$, for every $\tilde{x} \in \tilde{X}, g \in GL_n(C)$ (or $g \in C^n$). Denote the obtained bundle by $P_\rho \to X_m$ (or $E_\rho \to X_m$) and call it the bundle associated with the representation ρ.

Obviously, this bundle according to the transition functions may be constructed in the following manner. Let $\{U_\alpha\}$ be a simple covering of X_m, i. e. every intersection $U_{\alpha_1} \cap U_{\alpha_2} \cap ... \cap U_{\alpha_k}$ is connected and simply connected. For each U_α, we choose a point $z_\alpha \in U_\alpha$ and join z_0 and z_α with a γ_α starting at z_0 and ending at z_α. For a point $z \in U_\alpha \cap U_\beta$ we choose a path $\tau_\alpha \subset U_\alpha$ which starts at z_α and ends at z. Consider

$$g_{\alpha\beta}(z) = \rho\left(\gamma_\alpha \tau_\alpha(z) \tau_\beta^{-1}(z) \gamma_\beta^{-1}\right). \tag{6}$$

We see that $g_{\alpha\beta}(z) = g_{\beta\alpha}(z)$ on $U_\alpha \cap U_\beta$ and $g_{\alpha\beta} g_{\beta\gamma}(z) = g_{\alpha\gamma}(z)$ on $U_\alpha \cap U_\beta \cap U_\gamma$.

The cocycle $\{g_{\alpha\beta}(z)\}$ does not depend on the choice of z. Hence $\{g_{\alpha\beta}\}$ are constants. It is known, that the holomorphic bundle can be equipped with a flat holomorphic connection iff the transition functions of the cocycle defined by the bundle are constant. Hence local 1-forms $\{\omega_\alpha = 0\}$ on U_α define a holomorphic connection on E_ρ. This follows from the identity

$$\omega_\alpha = g_{\alpha\beta}\omega_\beta g_{\beta\alpha} + dg_{\alpha\beta}g_{\beta\alpha}.$$

So, $\omega = \{\omega_\alpha\}$ is a holomorphic 1-form on X_m and therefore is a connection form of the bundle $P'_\rho \to X_m$. The corresponding connection is denoted by ∇'. We will extend the pair $\left(P'_\rho, \nabla'\right)$ to CP^1. As the required construction is of local character, we shall extend $P'_\rho \to X_m$ to the bundle $P''_\rho \to X_m \cup \{s_i\}$, where $s_i \in S$.

First consider the extension of the principal bundle $P'_\rho \to X_m$. Suppose a neighborhood V_i of the point s_i meets $U_{\alpha_1}, U_{\alpha_2}, ...U_{\alpha_k}$. As we noted when constructing the bundle from transition functions (6) only one of them is different from identity. Let us denote it by g_{1k}, then $g_{1k} = M_i$, where M_i is the monodromy which corresponds to the singular point s_i and is obtained from the representation (5). Mark a branch of the many-valued function $(\tilde{z} - s_i)^{E_i}$ containing the point $\tilde{s}_i \in \tilde{U}_i$, where $E_i = \frac{1}{2\pi i}\ln M_i$ with eigenvalues $\mu_j^1, \mu_j^2, ..., \mu_j^n$ satisfying the conditions $0 \le Re\mu_j^i < 1$. Thus the marked branch defines a function

$$g_{01} = (z - s_i)^{E_i}. \tag{7}$$

Denote by g_{02} the extension of g_{01} along the path which goes around s_i counterclockwise, and similarly for other points. At last on $V_i \cap U_{\alpha_k} \cap U_{\alpha_1}$ we shall have:

$$g_{0k}(z) = g_{01}(z)M_i = g_{01}(z)g_{0k}(z).$$

The function $g_{0k} : V_i \to GL_n(C)$ is the one defined at the point s_i, and takes there value coinciding with the monodromy matrix. It means, that we made extension of the bundle to the point s_i. In a neighborhood of s_i one will have

$$\omega_i = dg_{0k}g_{0k}^{-1} = E_i\frac{dz}{z - s_i}.$$

So we obtained the holomorphic principal bundle $P_\rho \to CP^1$ on the sphere CP^1. The vector bundle associated to $P_\rho \to CP^1$, which we denote by $E_\rho \to CP^1$ and call canonical, is not topologically trivial.

Proposition 2. *The Chern number $c_1(E_\rho)$ of $E_\rho \to X$ is equal to*

$$c_1(E_\rho) = \sum_{i=1}^{m} tr(E_i). \tag{8}$$

Denote by ∇ the connection of E_ρ. The holomorphic horizontal sections of E_ρ satisfy the equation

$$\nabla f = 0 \iff df = \omega f. \tag{9}$$

Theorem 10. *System (9) has regular singularity at points $s_1, s_2, ..., s_m$ and its monodromy representation coincides with the given representation.* This theorem is other formulation of the result of Plemelj [32].

2.3. Solution of the problem II

Let (9) be the regular system of differential equations which is induced by the representation (5). The fundamental matrix of solutions in a neighborhood of s_j is

$$\Phi_j\left(\tilde{z}\right) = U_j(z)(z - s_j)^{\Psi_j}(\tilde{z} - s_j)^{E_j}. \tag{10}$$

Here Ψ_j are exponents of the solution space R of system (9) and $E_j = \frac{1}{2\pi i}\ln M_j$, with eigenvalues $\mu_j^1, \mu_j^2, ..., \mu_j^n$ satisfying the conditions $0 \leq Re\mu_j^i < 1$. The numbers $\beta_j^i = \varphi_j^i + \mu_j^i$ will be called exponents of the solution space $\Re$ at the point s_j (or j-exponents).

Using the exponents β_j^i, the condition for a regular system on CP^1 to be of Fuchs type is given by the following proposition.

Proposition 3. *[10] A regular system $df = \omega f$ on CP^1 with singular points $s_1, ..., s_m$ is of Fuchs type if and only if the following condition is satisfied*

$$\beta = \sum_{j=1}^{m}\sum_{i=1}^{n}\beta_j^i = 0.$$

If we take in (7)

$$g_{01} = (z - s_i)^{\Psi_i}(\widetilde{z} - s_i)^{\widetilde{E}_i},$$

where Ψ_i are diagonal matrices with integer entries placed in increasing order and $\widetilde{E}_i$ are upper-triangular matrices, we obtain another extension of the bundle E_ρ with a meromorphic connection with singularities at the points $s_1, ..., s_m$, whose monodromy coincides with (5). Denote by E^{C,Ψ_i} all possible extensions of E'_ρ, where C is any collection of nondegenerate matrices $C_1, ..., C_n$ which transform monodromy matrices $M_1, ..., M_m$ into upper triangular form and Ψ_i are diagonal matrices described above.

For a holomorphic bundle $E'_\varrho \to CP^1 \setminus \{s_1, ..., s_m\}$ with connection ∇' consider such an extension $E^{C,\Psi}$ to CP^1 for which $\Psi = (0, ..., 0)$. Denote the corresponding vector bundle with connection by (E^0, ∇^0) and call it the canonical extension (see [14]).

Let us now formulate a condition for solvability of the 21st Hilbert problem.

Theorem 11. *[10] A representation ϱ is realizable as a monodromy representation of a Fuchs system with given singular points s_1,...,s_m if and only if among the bundles $E^{C,\Psi} \to CP^1$ there is a holomorphically trivial one, i. e. such that its splitting type is* $(0, ..., 0)$.

It is known that any irreducible representation ϱ is realizable by a Fuchsian system and every finitely generated irreducible subgroup of $GL_n(C)$ is the monodromy group of a Fuchsian system on the Riemann sphere, so that counterexamples must be sought among reducible representations. Thus irreducibility of the representation is a sufficient condition, any regular system with irreducible monodromy group is equivalent to Fuchsian one, although there exists a special class of reducible representations, the s. c. B-representations (the notation is in honor of A. Bolibruch, who was the first to distinguish this class of representations), which are realizable by Fuchsian systems. We will now make this assertion more precise.

2.4. The stability of holomorphic vector bundles

A concept of stability (semistability) of holomorphic vector bundles was introduced by D. Mumford (see [28]) for the classification of the holomorphic vector bundles on Riemann surfaces of genus $g > 1$. A criterion of stability for flat holomorphic bundles was obtained by A. Weil. A generalization of Weil's theorem is given by the Narasimhan-Seshadri theorem [31], which gives a criterion of stability for topologically nontrivial holomorphic vector bundles. The differential-geometric approach to the Narasimhan-Seshadri theorem is given in [15].

Let $E \to X$ be a holomorphic vector bundle on a Riemann surface X, with $\deg E = k$ and $\mathrm{rank} E = n$. The normalized Chern class of the vector bundle E is defined by $\mu(E) = \frac{k}{n}$.

A bundle E is called stable (resp. semistable) in the sense of Mumford, if for every subbundle $F \subset E$, we have

$$\mu(F) < \mu(E), \qquad (\mathrm{resp.}\mu(F) \leq \mu(E)).$$

If $E \to X$ is a holomorphic vector bundle over a Riemann surface of genus $g \geq 2$, then it does not necessarily split into the sum of line bundles but some analogous decompositions are still available [19].

On the Riemann sphere there do not exist vector bundles stable in the sense of Mumford, and a holomorphic vector bundle is semistable iff its splitting type is $(k, ..., k)$. This follows from the following statement (see [1]). Let

$$E = O(\beta_1) \oplus O(\beta_2) \oplus \ldots \oplus O(\beta_m)$$

and

$$F = O(\alpha_1) \oplus O(\alpha_2) \oplus \ldots \oplus O(\alpha_n)$$

be two vector bundles on the Riemann sphere, where $O(\alpha_j), O(\beta_j), j = 1, ..., n$ are sheaves of the germs of holomorphic sections of the line bundles with Chern numbers α_j, β_j, and $m > n$. The holomorphic vector bundle F to be isomorphic to a holomorphic subbundle of E if the following inequalities are satisfied:

$$\alpha_i \leq \beta_i, \quad i = 1, ..., n. \tag{11}$$

But on the Riemann sphere there exist holomorphic bundles (E, ∇) stabilized by the connection ∇. We will say that the subbundle $F \subset E$ is stabilized by the connection ∇, if the covariant derivative $\nabla_{\frac{d}{dx}}$ maps local holomorphic sections of F into sections of the same subbundle F. It means that $\nabla(\Gamma(F)) \subset \Gamma(\tau^*_{CP^1} \otimes F)$. The pair (E, ∇) is called stable (semistable), if for every subbundle $F \subset E$ stabilized by ∇, one has $\mu(F) < \mu(E)$ (resp. $\mu(F) \leq \mu(E)$.) For given irreducible monodromy, existence of the holomorphic semistable bundle is the condition of solvability of the problem II, but there exists a class of reducible representations for which the problem is known to be solvable independently of this condition.

The representation (5) is called a B-representation, if it is reducible and the Jordan normal form of every monodromy matirix M_i consists of only one Jordan block.

Theorem 12. *[12] For the B-representation* (5) *Problem II is solvable iff the vector bundle obtained from the cannonical extension of the bundle E_ρ is semistable in the sense of Mumford.*

The splitting type of the vector bundle obtained from the canonical extension of the bundle $E_\rho \to X_m$ coincides with the partial indices of the continuous matrix function constructed from monodromy matrices in the way which we describe in section 1.6.

From the theorem above and from corollary 1.1 we have

Corollary 2. The Chern number of the canonical bundle is equal to $c_1(E_\rho) = np$ and therefore the splitting type of E_ρ is (p,p,...,p).

3. The generalized analytic vectors

In this section we consider holomorphic vector bundles with L_p - connections from the viewpoint of the theory of generalized analytic vectors [8],[9]. By definition generalized analytic vectors, by analogy with the one dimensional case [39], are regular solutions of systems of $2n$ elliptic partial differential equations presented in the complex form

$$\partial_{\bar{z}} f(z) = A(z)f(z) + B(z)\overline{f(z)}, \tag{12}$$

where $A(z)$, $B(z)$ are bounded matrix functions on a domain $U \subset \mathbb{C}$ and $f(z) = (f^1(z), ..., f^n(z))$ is an unknown vector function. A solution of system (12) is called regular in U, if it does not have singular points, is single-valued and has partial derivatives in the sense of Sobolev.

Along with similarities between the one-dimensional and multi-dimensional cases, there also exist essential differences. One of them, as noticed in [8], is that there can exist solutions of system (16) for which there is no analogue of the Liouville theorem on the constancy of bounded entire functions.

We present first some necessary fundamental results of the theory of generalized analytic functions [39],[7],[5],[8] in the form convenient for our purposes.

Let $f \in \mathrm{L}^p(U)$, where U is a domain in $\mathbb{C}$. We write $f \in \mathrm{W}_p(U)$, if there exist functions θ_1 and θ_2 of class $\mathrm{L}^p(U)$ such that the equalities

$$\iint_U f\frac{\partial\varphi}{\partial\bar{z}}dU = -\iint_U \theta_1\varphi dU, \quad \iint_U f\frac{\partial\varphi}{\partial z}dU = -\iint_U \theta_2\varphi dU$$

hold for any function $\varphi \in \mathrm{C}^1(\mathrm{U})$.

Let us define two differential operators on $\mathrm{W}_p(U)$

$$\partial_{\bar z} : \mathrm{W}_p(U) \to \mathrm{L}_p(U), \quad \partial_z : \mathrm{W}_p(U) \to \mathrm{L}_p(U),$$

by setting $\partial_{\bar z} f = \theta_1$, $\partial_z f = \theta_2$. The functions θ_1 and θ_2 are called the generalized partial derivatives of f with respect to $\bar z$ and z respectively. Sometimes we will use a shorthand notation $f_{\bar z} = \theta_1$ and $f_z = \theta_2$. It is clear that ∂_z and $\partial_{\bar z}$ are linear operators satisfying the Leibnitz equality.

Define the following singular integral operator on the Banach space $\mathrm{L}_p(U)$:

$$T : \mathrm{L}_p(U) \to W_p(U),$$

$$T(\omega) = -\frac{1}{\pi} \iint_U \frac{\omega(t)}{t - z} dU, \omega \in L_p(U). \tag{13}$$

The integral (13) makes sense for all $\omega \in \mathrm{L}_p(U)$, almost all $z \in U$, and all $z \notin \bar U$ and (13) determines a function $\varphi(z) = T(\omega)$ on the whole $\mathbb{C}$. For $\omega \in \mathrm{L}_p(U)$ with $p > 2$, the function φ is continuous.

Any element of $\mathrm{W}_p(U)$ can be represented by an integral (13). In particular, if $f_{\bar z} = \omega$, then $f(z)$ can be represented in the form

$$f(z) = h(z) - \frac{1}{\pi} \iint_U \frac{\omega(t)}{t - z} dU,$$

where $h(z)$ is holomorphic in U. The converse is also true, i.e., if $h(z)$ is holomorphic in U and $\omega \in \mathrm{L}_p(U)$, then $h(z) - \frac{1}{\pi} \iint_U \frac{\omega(t)}{t-z} dU$ determines an element $f(z)$ of $\mathrm{W}_p(U)$ satisfying the equality $f_{\bar z} = \omega$.

As we saw, the generalized derivative with respect to $\bar z$ of the integral (13) is ω. Similarly, there exists a generalized derivative of this integral with respect to z. It equals

$$-\frac{1}{\pi} \iint_U \frac{\omega(t)}{(t - z)^2} dU. \tag{14}$$

The integral (14) is understood in the sense of the Cauchy principal value.

It is known [39] that in one dimensional case, if $B = 0$, a solution of (12) can be represented as

$$\Phi(z) = F(z) \exp(\omega(z)), \tag{15}$$

where F is a holomorphic function in U, and $\omega = -\frac{1}{\pi} \int\int_U \frac{A(z)}{\xi - z} dU$.

Consider a matricix elliptic system of the form:

$$\partial_{\bar z} \Phi(z) = A(z) \Phi(z). \tag{16}$$

In this case an analogue of factorization (15) is given by the following theorem

Theorem 13. *[7] Each solution of the matricial equation (16) in U can be represented as*

$$\Phi(z) = F(z)V(z), \tag{17}$$

where $F(z)$ is an invertible holomorphic matrix function in U, and $V(z)$ is a single-valued matrix function invertible outside $\overline{U}$.

We will use the representation of the solution of system of (16) in the form (17) for the construction of a holomorphic vector bundle on the Riemann sphere.

We recall some properties of solutions of (16).

Let $C(z)$ be a holomorphic matrix function, then $[C(z), \partial_{\overline{z}}] = 0$. Indeed,

$$[C(z), \partial_{\overline{z}}]\Phi(z) = C(z)\partial_{\overline{z}}\Phi(z) - \partial_{\overline{z}}C(z)\Phi(z) = C(z)\partial_{\overline{z}}\Phi(z) - C(z)\partial_{\overline{z}}\Phi(z) = 0.$$

Here we have used that $\partial_{\overline{z}}C(z) = 0$.

Two systems $\partial_{\overline{z}}\Phi(z) = A(z)\Phi(z)$ and $\partial_{\overline{z}}\Phi(z) = B(z)\Phi(z)$ are called gauge equivalent if there exists a nondegenerate holomorphic matrix function $C(z)$ such that $B(z) = C(z)A(z)C(z)^{-1}$.

Let the matrix function $\Psi(z)$ be a solution of the system $\partial_{\overline{z}}\Phi(z) = A(z)\Phi(z)$ and let $\Phi_1(z) = C(z)\Phi(z)$, where $C(z)$ is a nonsingular holomorphic matrix function. Then $\Phi(z)$ and $\Phi_1(z)$ are solutions of the gauge equivalent systems. The converse is also true: if $\Phi(z)$ and $\Phi_1(z)$ satisfy systems of equations

$$\partial_{\overline{z}}\Phi(z) = A(z)\Phi_(z), \partial_{\overline{z}}\Phi_1(z) = B(z)\Phi_1(z)$$

and $A(z) = C^{-1}(z)B(z)C(z)$, then $\Phi_1 = D(z)\Phi(z)$ for some holomorphic matrix function $D(z)$.

Indeed, as we proved above, $C(z)\partial_{\overline{z}}\Phi_1(z) = A(z)C(z)\Phi_1(z)$, and therefore $\Phi_1(z)$ satisfies the equation $\partial_{\overline{z}}\Phi_1(z) = C^{-1}(z)A(z)C(z)\Phi_1(z)$. To prove the converse let us substitute in $\partial_{\overline{z}}\Phi(z) = A(z)\Phi_(z)$, instead of $A(z)$ the expression of the form $C^{-1}(z)B(z)C(z)$ and consider

$$\partial_{\overline{z}}\Phi_1(z) = C^{-1}(z)B(z)C(z)\Phi(z);$$

then it follows, that

$$C(z)\partial_{\overline{z}}\Phi(z) = B(z)C(z)\Phi(z).$$

But for the left hand side of the last equation we have $C(z)\partial_{\overline{z}}\Phi(z) = \partial_{\overline{z}}C(z)\Phi(z)$, therefore

$$\partial_{\overline{z}}(C(z)\Phi(z)) = B(z)(C(z)\Phi(z)).$$

From this it follows that Φ and $C\Phi$ are the solutions of equivalent systems, which means that $\Phi_1 = D\Phi$.

The above arguments for solutions of (16) are of a local nature, so they are applicable for an arbitrary compact Riemann surface X, which enables us to construct a holomorphic vector bundle on X. Moreover using the solutions of system (16) one can construct a matrix 1-form $\Omega = D_{\overline{z}} F F^{-1}$ on X which is analogous to holomorphic 1-forms on Riemann surfaces.

Let X be a Riemann surface. Denote by $L_p^{\alpha,\beta}(X)$ the space of L_p-forms of type $(\alpha, \beta), \alpha, \beta = 0, 1$. Denote by $W_p(U) \subset L_p(U)$ the subspace of functions which have generalized derivatives.

We define the operators

$$D_z = \frac{\partial}{\partial z} : W_p(U) \to L_p^{1,0}(U), f \mapsto \omega_1 dz = \partial_z f dz,$$

$$D_{\bar{z}} = \frac{\partial}{\partial \bar{z}} : W_p(U) \to L_p^{0,1}(U), f \mapsto \omega_2 d\,\bar{z} = \partial_{\overline{z}} f d\overline{z}.$$

It is clear that $D_{\bar{z}}^2 = 0$, hence the operator $D_{\bar{z}}$ can be used to construct the de Rham cohomology.

Let us denote by $CL_p^1(X)$ the complexification of $L_p^1(X)$, i.e. $CL_p^1(X) = L_p^1(X) \otimes C$. Then we have the natural decomposition

$$CL_p^1(X) = L_p^{1,0}(X) \oplus L_p^{0,1}(X) \tag{18}$$

according to the eigenspaces of the Hodge operator $* : L_p^1(X) \to L_p^1(X)$, $* = -\imath$ on $L_p^{1,0}(X)$ and $* = \imath$ on $L_p^{0,1}(X)$.

The decomposition (18) splits the operator $D : L_p^0(X) \to L_p^0(X)$ in the sum $D = D_z + D_{\bar{z}}$.

Next, let $E \to X$ be a C^∞-vector bundle on X, let $L_p(X, \mathcal{E})$ be the sheaf of germs of L_p-sections of $\mathcal{E}$ and let $\Omega \in L_p^1(X, E) \otimes GL_n(C)$ be a matrix valued 1-form on X. If the above arguments are applied to the complex $L_p^*(X, \mathcal{E})$ with covariant derivative $\bigtriangledown_\Omega$, we obtain again the decomposition of the space $CL_p^1(X, E)$ and the operator $\bigtriangledown_\Omega$:

$$CL_p^1(X, E) = L_p^{1,0}(X, E) \oplus L_p^{0,1}(X, E),$$

$$\bigtriangledown_\Omega = \bigtriangledown'_\Omega + \bigtriangledown''_\Omega.$$

Locally, on the domain U, we have $\bigtriangledown_\Omega^U = d_U + \Omega$, where $\Omega \in L_p^1(X, U) \otimes GL_n(C)$ is a 1-form. Therefore $\bigtriangledown_\Omega^U = (D_z + \Omega_1) + (D_{\bar{z}} + \Omega_2)$, where Ω_1 and Ω_2 are the matrix valued 1-forms on U. We say that a W_p-section f of the bundle $\mathcal{E}$ with L_p-connection is pseudoholomorphic if it satisfies the system of equations

$$\partial_{\bar{z}} f(z) = A(z) f(z), \tag{19}$$

where $A(z)$ is a $n \times n$ matrix-function with entries in $L_p^0(X) \otimes GL_n(C)$ and $f(z)$ is a vector function $f(z) = (f_1(z), f_2(z), ..., f_n(z))$, or

$$D_{\overline{z}} f = \Omega f,$$

where $\Omega \in L_p^1(X) \otimes GL_n(C)$.

Theorem 14. *There exists a one-to-one correspondence between the space of gauge equivalent systems (16) and the space of holomorphic structures on the bundle $E \to X$.*

Consider the analogue of problem I for the system (12):

Problem III. Find a piecewise-regular solution of system (12) on the whole plane $\mathbb{C}$ equal to zero at infinity and satisfying on Γ the following boundary condition

$$W^{+} = G(t)W^{-}, \tag{20}$$

where $detG(t) \neq 0$ on Γ.

This problem is solvable using the methods of the singular integral equations [39] and it is known, that the number of linearly independent solutions on R is finite [9]. Denote this number by l. Let $k = \frac{1}{2\pi}\Delta_{\Gamma}\mathtt{argdet}G(t)$ as above be the index of the boundary value problem III. It is clear (see [9]), that $l \geq max(0, 2k)$ and it is possible to choose a matrix function $G(t)$, such that $l(G) = s$, for every given number $s \geq max(0, 2k)$, and therefore it is possible to consider the number l as a function of $G(t)$. The index of the problem is a topological invariant and in the one dimensional case it is a complete invariant. It is known also, that in the multi-dimensional case the index is not a complete invariant, but in the stable case, the index defines all invariants of the problem. The number l as the function of $G(t)$ is called stable, if $l(G) = l(G_1)$ for all nondegenerate matrix functions on Γ, which are sufficiently close to $G(t)$.

Theorem 15. *[9] The number l is stable iff $l = max(0, 2k)$.*

Let $C(t)$ be any matrix function on Γ and $C(t) \in \Omega$, which has a holomorphic extension to U^{+}, not necessarily nonsingular everywhere, and let $\frac{1}{2\pi}\Delta_{\Gamma}\mathtt{argdet}(G^{-1}C) = 0$, then there exists an extension of $G^{-1}C$ to U^{+}. Denote by $P(z)$ this extension and let $\Phi(z)$ be some holomorphic solution of problem I. Consider the substitution

$$w(z) = P(z)\Psi(z) \text{ on } z \in U^{+}; w(z) = \Psi(z) \text{ on } z \in U^{-}.$$

Proposition 4. *The matrix function $\Psi(z)$ is holomorphic in $U^{+} \cup U^{-}$ iff w is a solution of the system*

$$\partial_{\bar{z}}w = Aw, \tag{21}$$

where $A(z) = \partial_{\bar{z}}PP^{-1}$, for $z \in U^{+}$ and $A(z) = 0$, for $z \in U^{-}$.

Let the index of the problem be k and let $C(t)$ be a diagonal matrix function with diagonal entries $diagC(t) = (t^p, ..., t^p)$.

Theorem 16. *[7] The matrix function $G(t) \in \Omega^k$ iff the Liouville theorem holds for the system (21).*

Proof of this theorem follows from the following arguments from the theory of singular integral equations. The fulfilment of the Liouville theorem for the solution of the system (21) is equivalent to the existence of a solution of the following matricial system of singular integral equations

$$B(z) + \frac{1}{\pi}\int\int_{U}\frac{WB}{t-z}dU_t = E, \tag{22}$$

where E is the identity matrix. On the other hand $G(t) \in \Omega^k$ iff the system (22) is solvable with respect to $B(z)$.

4. Conclusion

It is possible to pose the problems I and II for any compact Riemann surfaces [19], [16], [20], [12]. In this case as well as in section 2, from the problem data one can construct a holomorphic vector bundle on the Riemann surfaces. However in this case one does not have theorems of type 1.2 and 2.1 anymore. For the classification of holomorphic vector bundles discrete invariants are not sufficient, since there appear moduli spaces of holomorphic vector bundles (see [18]). Besides, classical statement of problem II demands some specification related to apparent singular points of the constructed system of differential equations, which necessarily arise and their number depends on the genus of the Riemann surface.

For the Riemann surfaces it is also possible to pose problem II in the following form: construct a semistable holomorphic vector bundle of degree zero with a logarithmic connection, which has prescribed singular points and monodromy.

Under such formulation of problem II it is known, that for irreducible representations problem II has a solution iff the pair (E, ∇) constructed by the given data is stable.

For the Riemann surfaces the majority of results from section 3 may be generalized and in our opinion they will be adequate tools for the enrichment of methods of Riemann problems (see [24]).

REFERENCES

1. Atiyah M.F., Bott R. The Yang-Mills equation on the Riemann surface. *Philos.Trans. R. Soc. London A*, **308**, (1982), 523-615.

2. Balser W., Bolibruch A.A. Transformation of reducible equations to Birkhoff standard form. *In: Ulmer Seminare Euber Funktionalanalysis und Differentialgleichungen, Heft*, **2** (1997), 73-81.

3. Birkhoff G A theorem on matrics of analytic functions. *Math. Ann.*, **74** (1913), 122-133.

4. Birkhoff G. Equivalent singular points of differential equations. *Math. Ann.*, **74** (1913), 134-139.

5. Bojarski B. Generalized solutions of the PD system of the first order and elliptic type with discontinuous coefficients. *Mat. Sbornik.*, **43**, (1957), 451-503.

6. Bojarski B. A theory of generalized annalytic vector. *Annales Polonici Mathematici*, **XVII** (1966), 281-320.

7. Bojarski B. Connection between complex and global analysis: analytical and geometrical aspects of the Riemann-Hilbert transition problem. *In book "Complex analysis, methods, application, Berlin, A.V.*, 1983.

8. Bojarski B. Stability of the Hilbert problem for a holomorphic vector. *Bull. Georgian Acad. Sci.*, **21** (1958), 391-398.

9. Bojarski B. Some boundary value problems for $2n$ system of elliptic differential equations on plane. *Doklady Mathematics*, **124**, 1 (1959).

10. Bolibruch A.A. The Riemann-Hilbert problem. *Russian Math. surveys*, **45**, 2 (1990), 1-47.

11. Bolibruch A.A. Stable vector bundles with logarithmic connections and Riemann-Hilbert problem. *Doklady Akademii Nauk*, **381**, 1 (2001), 10-13.

12. Bolibruch A.A. The Riemann-Hilbert problem of the compact Riemann surfaces. *Proc. Steklov Ins. Math.*, **238** (2002), 47-60.

13. Bolibruch A.A. Inverse monodromy problems of the Analytic theory of differential equations. In book "Mathematical Events of the Twentieth Century", *Springer-Verlag*, 2006.

14. Deligne P. Equations différentiales á points singuliers réguliers. *Lect. Notes Math.*, 1970.

15. Donaldson S.K. A new proof of a theorem of Narasimhan and Seshadri. *J. Differential Geometry*, **18** (1983), 269-277.

16. Esnault H. Viehweg E. Semistable vector bundles on curves and irreducible representations of the fundamental group. *Contemporary Mathematics*, **241** (1999), 129-138.

17 Fuchs L. Zur Theorie der linearen Diferentalgleichungen. Sitzungsberichte der K.preuss. Akademie der Wissenschaften zu Berlin. **1-7** (1888), 1115-1126; **8-15** (1888), 1273-1290; **16-21** (1889), 713-726; **23-31** (1890), 21-38.

18. Giorgadze G. Regular systems on Riemann surfaces. *Journal of Math. Sci.*, **118**, 5 (2003), 5347-5399.

19. Giorgadze G. On connections with a regular singularity. *Reports of Enlar. Session of the Seminar of I. Vekua Istitute of Appl. Math.*, **9**, 1 (1994), 1-3.

20. Gantz C., Steer B. Gauge fixing for logarithmic connections over curves and Riemann-Hilbert problem. *J. London Math. Soc.*, **59**, 2 (1999), 479-490.

21. Gohberg I., Krein M.G. On the stability of a system of partial indices of the Hilbert problem for several unknown functions. *Dokl. Akad. Nauk SSSR*, **119** (1958), 854-857.

22. Gohberg I., Kasshoek M., Spitkovski I. An overview matrix factorization theory and operator application. *Operator theory: Advances and Applications*, **141** (2003), 1-102.

23. Grothendieck A. Sur la classification des fibrés olomorphes sur la sphére de Riemann. *Amer. j. Math.*, **79** (1957), 121-138.

24. Its A., Kapaev A., Novokshenov V., Fokas A. Penleve transcendentes. Methods of Riemann problems. *R&C Dynamics, Moscow*, 2005.

25. Khimshiashvili G.N. Geometric aspects of Riemann-Hilbert problems. *Mem. Differ. Equations Math. Phys.*, **27** (2002), 1-114.

26. Hartman Ph. Ordinary differentail equations. *John Wiley & Sons, New York-London, Sydney*, 1964.

27. Hilbert D. Mathematische Probleme. *Archiv für Math. und Phys., 3 Reihe*, Bd.1, pp. 44-63, pp. 213-237, 1901.

28. Mumford D. Geometric invariant theory. *Berlin, Springer-Verlag*, 1965.

29. Muskhelishvili N., Vekua N. Riemann boundary value problem for several unknown functions and its application for system singular integral equations. *Travaux de Institut Mathematique de Tbilisi*, **12** (1943), 1-46.

30. Muskhelishvili N.I. Singular Integral Equations. *Noordhoff, Groningen*, 1953.

31. Narasimhan M.S., Seshadri T.R. Stable and unitary vector bundles on a compact Riemann surface. *Ann. of Math.*, **82** (1965), 540-564.

32. Plemelj J. Riemanische Functionenscharen mit gegebener Monodromiegruppe. *Motanshefte für Math.*, **XIX** Jahrgang, (1908), 211-245.

33. Poincare H. Mémoire sur les fonctions zetafuchsiennes. *Acta Math.*, **5** (1984), 209-278.

34. Pressley A., Segal G. Loop Groups. *Clarendon Press, Oxford*, 1984.

35. Riemann B. Theorie der Abel'schen Funktionen. *J. Reine angew. Math.*, **54** (1857), 115-155.

36. Röhrl H. Das Riemann-Hilbertishe Problem der theorie der linear differentialgleichungen. *Math. Ann.*, **133** (1957), 1-25.

37. Sauvage L. Sur les solutions réguliére d'un system. *Ann. Sci. École Norm. Sup.*, **3**, 3 (1886), 391-404.

38. Schlesinger L. Über der Lösungen gewisser linear differentalgleichungen als Funktionen der singulären Punkte. *Journal für reine and angew. Math.*, **129** (1905), 287-294.

39. Vekua I.N. Generalized Analytic Functions. *Pergamon Press, London*, 1962.

40. Vekua N.P. Systems of Singular Integral Equations and Certain Boundary Value Problems. *Moscow*, 1970.

Received 18.08.2012; revised 24.09.2012; accepted 23.11.2012.

Authors' addresses:

B. Bojarski
Institute of Mathematics
Polish Academy of Sciences
8, ul. Sniadeckich, Warszawa 00-956
Poland
E-mail: B.Bojarski@impan.pl

G. Giorgadze
I. Vekua Institute of Applied Mathematics of
Iv. Javakhishvili Tbilisi State University
2, University St., Tbilisi 0186
Georgia
E-mail: gia.giorgadze@tsu.ge