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EXPLICIT SOLUTIONS OF BVPs OF 2D THEORY OF TERMOELASTICITY WITH MICROTEMPERATURES FOR THE HALF-PLANE

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Abstract. The present paper deals with a two-dimensional version of statics of the linear theory of elastic materials with inner structure whose particles, in eddition to the classical displacement and temperature fields, possess microtemperatures. Using the Fourier integrals, some basic boundary value problems are solved explicitly (in quadratures) for the half-plane.

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1. Introduction

The linear theory for elastic materials with inner structure whose particles, in addition to the classical displacement and temperature fields, possess microtemperatures was constructed by Iesan and Quintanilla [1] in 2000. The fundamental solutions of the equations of the three-dimensional (3D) theory of thermoelasticity with microtemperatures were constructed by Svanadze [2] in 2004. The representations of the Galiorkin type and general solutions of the system of statics of the above theory were obtained by Scalia, Svanadze, and Tracina [3] in 2010. The linear theory for microstretch elastic materials with microtemperatures was constructed by Iesan [4] in 2001, where the uniqueness and existence theorems in the dynamical case for isotropic materials are proved. The fundamental solutions of the equations of the two-dimensional (2D) theory of thermoelasticity with microtemperatures were constracted by Basheleishvili, Bitsadze, and Jaiani [5] in 2011.

In the present paper, using the Fourier transform, the two-dimensional boundary value problems (BVPs) of statics for the linear theory of thermoelasticity with microtemperatures for the half-plane are solved explicitly.

2. Basic equations. Boundary value problems

We consider an isotropic elastic material with microtemperatures. Let R^2_+ denote the upper half-plane $x_2 > 0$. The boundary of R^2_+ which is x_1-axis will be denoted by S : Let $\mathbf{x} := (x_1, x_2) \in R^2_+$, $\quad \partial\mathbf{x} := \left(\dfrac{\partial}{\partial x_1}, \dfrac{\partial}{\partial x_2}\right)$.

The governing homogeneous (i.e., body forces are neglected) system of the theory of thermoelasticity with microtemperatures has the form [1]-[3]

$$\mu\Delta\mathbf{u} + (\lambda+\mu) graddiv\mathbf{u}\text{-}\beta grad\theta = 0, \qquad (1)$$

$$k_6\Delta\mathbf{w}+(k_4+k_5)graddiv\mathbf{w}-k_3grad\theta-k_2\mathbf{w}=0 \tag{2}$$

$$k\Delta\theta+k_1div\mathbf{w}=0, \tag{3}$$

where $\mathbf{u}:=(u_1,u_2)$ is the displacement vector, $\mathbf{w}:=(w_1,w_2)$ is the microtemperature vector, θ is the temperature measured from the constant absolute temperature T_0 ($T_0>0$) by the natural state (i.e. by the state of the absence of loads), $\lambda,\ \mu,\ \beta,\ k,\ k_j,\ j=1,...,6$ are constitutive coefficients, Δ is the 2D Laplace operator.

Here we state the BVPs and solved in this paper.

Find a solution $\boldsymbol{U}=(\boldsymbol{u},\boldsymbol{w},\theta)\in C^2(R^2_+)$ *to the system (1-3) in* R^2_+ *, satisfying one of the following boundary conditions (BCs) on* S *:*

Problem 1.

$$(\mathbf{u})^+=\boldsymbol{\varphi}(x_1),\quad (\mathbf{w})^+=\mathbf{f}(x_1),\quad (\theta)^+=f_3(x_1).$$

Problem 2.

$$(\mathbf{u})^+=\boldsymbol{\varphi}(x_1),\quad (\mathbf{w})^+=\mathbf{f}(x_1),\quad \left(k_1w_2+k\frac{\partial\theta}{\partial x_2}\right)^+=f_3(x_1).$$

Problem 3.

$$(\mathbf{u})^+=\boldsymbol{\varphi}(x_1),(w_1)^+=f_1(x_1),\left(\mathbf{T}^{(2)}(\partial\mathbf{z},\mathbf{n})\mathbf{w}\right)^+_2=f_2(x_1),(\theta)^+=f_3(x_1).$$

Problem 4.

$$(\mathbf{u})^+=\boldsymbol{\varphi}(x_1),\quad (w_2)^+=f_1(x_1),\quad \left(\mathbf{T}^{(2)}(\partial\mathbf{z},\mathbf{n})\mathbf{w}\right)^+_1=f_2(x_1),$$

$$\left(k_1w_2+k\frac{\partial\theta}{\partial x_2}\right)^+=f_3(x_1).$$

The symbol $(.)^+$ denotes the limit on S from R^2_+, the vector-functions $\boldsymbol{\varphi}(x_1):=(\varphi_1,\varphi_2)$, $\mathbf{f}(x_1):=(f_1,f_2)$, and function f_3, are prescribed, $\mathbf{n}:=(0,1)$ is a unit normal vector, $\mathbf{T}^{(2)}(\partial\mathbf{x},\mathbf{n})\mathbf{w}$ is the microtemperature stress vector,

$$\mathbf{T}^{(2)}(\partial\mathbf{x},\mathbf{n}):=\begin{pmatrix} k_6\dfrac{\partial}{\partial x_2} & k_5\dfrac{\partial}{\partial x_1}\\ k_4\dfrac{\partial}{\partial x_1} & k_7\dfrac{\partial}{\partial x_2}\end{pmatrix},\quad k_7:=k_4+k_5+k_6.$$

Note that BVPs for the system (2),(3), which contain only $\mathbf{w}$ and θ, can be investigated separately. Then supposing θ as known we can study BVPs for the system (1) with respect to $\mathbf{u}$. Combining the results obtained we arrive at explicit solution for BVPs for the system (1)-(3). First we assume that $\theta(\mathbf{x})$ is known, when $\mathbf{x}\in R^2_+$, then for $\mathbf{u}$ we get the following nonhomogeneous equation

$$\mu\Delta\mathbf{u}+(\lambda+\mu)graddiv\mathbf{u}=\beta grad\theta. \tag{4}$$

It is known that the volume potential $\mathbf{u}_0$ [6]

$$\mathbf{u}_0 = -\frac{1}{\pi}\int\limits_{R_+^2} \mathbf{\Gamma}(\mathbf{x}\text{-}\mathbf{y})grad\theta ds, \tag{5}$$

where

$$\mathbf{\Gamma}(\mathbf{x}\text{-}\mathbf{y}) = \begin{pmatrix} \dfrac{\lambda+3\mu}{2a\mu}\ln r - \dfrac{\lambda+\mu}{2a\mu}\left(\dfrac{\partial r}{\partial x_1}\right)^2, & -\dfrac{\lambda+\mu}{2a\mu}\dfrac{\partial r}{\partial x_1}\dfrac{\partial r}{\partial x_2} \\ -\dfrac{\lambda+\mu}{2a\mu}\dfrac{\partial r}{\partial x_1}\dfrac{\partial r}{\partial x_2}, & \dfrac{\lambda+3\mu}{2a\mu}\ln r - \dfrac{\lambda+\mu}{2a\mu}\left(\dfrac{\partial r}{\partial x_2}\right)^2 \end{pmatrix},$$

is a particular solution of (4). In (5) $grad\theta$ is an exponentially vanishing at infinity (see (22) below) continuous vector in R_+^2 along with its first order derivatives.

Thus, the general solution of equation (4) is representable in the form $\mathbf{u} = \mathbf{V} + \mathbf{u}_0$, where

$$\mu\Delta\mathbf{V} + (\lambda+\mu)graddiv\mathbf{V} = 0.$$

The last equation is the equation of an isotropic elastic body. So, we have reduced solving of basic BVPs under consideration to the solution of the basic BVPs for the equation of an isotropic elastic body.

The solution of the BVP under BC $(\mathbf{V})^+ = \mathbf{f}$ can be given in the form [6]

$$\mathbf{V}(\mathbf{x}) = \frac{1}{\pi}\int\limits_S \mathbf{N}(\partial\mathbf{y},\mathbf{n})\mathbf{\Gamma}(\mathbf{x}\text{-}\mathbf{y})\mathbf{f}(\mathbf{y})ds,$$

where

$$\mathbf{N}(\partial\mathbf{y},\mathbf{n})\mathbf{\Gamma}(\mathbf{x}\text{-}\mathbf{y}) = \begin{pmatrix} 1 + \dfrac{\lambda+\mu}{\lambda+3\mu}\cos 2\theta, & \dfrac{\lambda+\mu}{\lambda+3\mu}\sin 2\theta \\ \dfrac{\lambda+\mu}{\lambda+3\mu}\sin 2\theta, & 1 - \dfrac{\lambda+\mu}{\lambda+3\mu}\cos 2\theta \end{pmatrix}\frac{\partial\theta}{\partial s},$$

$$\theta = \arctan\frac{x_2}{y_1 - x_1}, \quad \frac{\partial}{\partial s} = n_1\frac{\partial}{\partial x_2} - n_2\frac{\partial}{\partial x_1}.$$

So, it remains to solve BVPs for the system (2),(3).

2.1. Expansion of regular solutions

In 2D space "rot" is defined as a scalar

$$rot\phi = \frac{\partial\phi_2}{\partial x_1} - \frac{\partial\phi_1}{\partial x_2}$$

for a vector $\phi := (\phi_1, \phi_2)$ and as a vector

$$rot\psi := \left(\frac{\partial\psi}{\partial x_2}, -\frac{\partial\psi}{\partial x_1}\right)$$

for a scalar ψ (see [7]).

Let further $\mathbf{W}=(\mathbf{w},\theta)$, where $\mathbf{w}=(w_1,w_2)$ is the microtemperature vector and θ the temperature, be a regular solution of homogeneous equations (2),(3).

Theorem 1. *The regular solution $\boldsymbol{W}=(\boldsymbol{w},\theta)$ of systems (2),(3) admits in the domain of regularity a representation*

$$\boldsymbol{W}=(\overset{\mathbf{1}}{\mathbf{w}}+\overset{\mathbf{2}}{\mathbf{w}},\theta),$$

where $\overset{\mathbf{1}}{\mathbf{w}}$ and $\overset{\mathbf{2}}{\mathbf{w}}$ are the regular vectors, satisfying the conditions

$$\Delta(\Delta-s_1^2)\overset{\mathbf{1}}{\mathbf{w}}=0,\quad rot\overset{\mathbf{1}}{\mathbf{w}}=0,\quad (\Delta-s_1^2)div\overset{\mathbf{1}}{\mathbf{w}}=0,$$
$$(\Delta-s_2^2)\overset{\mathbf{2}}{\mathbf{w}}=0,\quad div\overset{\mathbf{2}}{\mathbf{w}}=0,\quad \Delta(\Delta-s_1^2)\theta=0,$$

and the constants s_1^2 and s_2^2 are determined by the formulas

$$s_2^2:=\frac{k_2}{k_6}>0,\quad s_1^2:=\frac{k_2k-k_1k_3}{k_7k}>0.$$

Proof. Let $\mathbf{W}=(\mathbf{w},\theta)$ be a regular solution of equations (2),(3).

Taking into account the identity

$$\Delta\mathbf{w}=graddiv\mathbf{w}-rotrot\mathbf{w}, \tag{6}$$

where

$$rotrot\mathbf{w}:=\left(\frac{\partial}{\partial x_2}\left(\frac{\partial w_2}{\partial x_1}-\frac{\partial w_1}{\partial x_2}\right),-\frac{\partial}{\partial x_1}\left(\frac{\partial w_2}{\partial x_1}-\frac{\partial w_1}{\partial x_2}\right)\right),$$

from (2) we obtain

$$\mathbf{w}=\frac{k_7}{k_2}graddiv\mathbf{w}-\frac{k_6}{k_2}rotrot\mathbf{w}-\frac{k_3}{k_2}grad\theta,$$

Let

$$\overset{\mathbf{1}}{\mathbf{w}}:=\frac{k_7}{k_2}graddiv\mathbf{w}-\frac{k_3}{k_2}grad\theta, \tag{7}$$

$$\overset{\mathbf{2}}{\mathbf{w}}:=-\frac{k_6}{k_2}rotrot\mathbf{w}. \tag{8}$$

Acting with the operator rot on (7) and considering the identity $rotgrad\equiv 0$ and with the operator div on (8) we have

$$rot\overset{\mathbf{1}}{\mathbf{w}}=0,\quad and\quad div\overset{\mathbf{2}}{\mathbf{w}}=0, \tag{9}$$

respectively. Taking into account the last equalities and (6), from (8) we get

$$(\Delta-s_2^2)\overset{\mathbf{2}}{\mathbf{w}}=0. \tag{10}$$

Applying the operator *div* to equation (2) and taking into account the identity $divgrad \equiv \Delta$, we obtain

$$(k_7\Delta - k_2)div\mathbf{w} - k_3\Delta\theta = 0. \tag{11}$$

Substitution of the value $div\mathbf{w} = -\frac{k}{k_1}\Delta\theta$ from (3) into (9) gives

$$\Delta(\Delta - s_1^2)\theta = 0. \tag{12}$$

From (7),(11), and (12), according to (9), we have

$$\Delta(\Delta - s_1^2)\overset{\mathbf{1}}{\mathbf{w}} = 0, \quad (\Delta - s_1^2)div\overset{\mathbf{1}}{\mathbf{w}} = 0. \tag{13}$$

Formulas (9),(10),(12),(13) prove the theorem.

Theorem 2. *In the domain of regularity the regular solution of the system (2),(3) can be represented in the form*

$$\boldsymbol{W} = \overset{\mathbf{1}}{\mathbf{V}} + \overset{\mathbf{2}}{\mathbf{V}} + \overset{\mathbf{3}}{\mathbf{V}}, \tag{14}$$

where

$$\overset{\mathbf{1}}{\mathbf{V}} := (\overset{\mathbf{1}}{\mathbf{v}}, \theta_1), \quad \overset{\mathbf{2}}{\mathbf{V}} := (\overset{\mathbf{2}}{\mathbf{v}}, \theta_2), \quad \overset{\mathbf{3}}{\mathbf{V}} := (\overset{\mathbf{3}}{\mathbf{v}}, 0), \tag{15}$$

and

$$\Delta\overset{\mathbf{1}}{\mathbf{v}} = 0, \quad (\Delta - s_2^2)\overset{\mathbf{3}}{\mathbf{v}} = 0, \quad (\Delta - s_1^2)\overset{\mathbf{2}}{\mathbf{v}} = 0,$$

$$rot\overset{\mathbf{1}}{\mathbf{v}} = 0, \quad rot\overset{\mathbf{2}}{\mathbf{v}} = 0 \quad div\overset{\mathbf{3}}{\mathbf{v}} = 0.$$

$$(\Delta - s_1^2)div\overset{\mathbf{2}}{\mathbf{v}} = 0, \quad \Delta\theta_1 = 0, \quad (\Delta - s_1^2)\theta_2 = 0.$$

Proof. Let

$$\overset{\mathbf{1}}{\mathbf{v}} := -\frac{(\Delta - s_1^2)\overset{\mathbf{1}}{\mathbf{w}}}{s_1^2}, \quad \overset{\mathbf{2}}{\mathbf{v}} := \frac{\Delta\overset{\mathbf{1}}{\mathbf{w}}}{s_1^2}, \quad \theta_1 := -\frac{(\Delta - s_1^2)\theta}{s_1^2}, \quad \theta_2 := \frac{\Delta\theta}{s_1^2}. \tag{16}$$

By virtue of (13),(16) it follows that

$$\overset{\mathbf{1}}{\mathbf{v}} + \overset{\mathbf{2}}{\mathbf{v}} = \overset{\mathbf{1}}{\mathbf{w}}, \quad \Delta\overset{\mathbf{1}}{\mathbf{v}} = 0, \quad (\Delta - s_1^2)\overset{\mathbf{2}}{\mathbf{v}} = 0.$$

Since θ is a solution of equation (11) which is of the type of equation $(12)_1$ satisfied by the vector $\overset{\mathbf{1}}{\mathbf{w}}$, similarly ,

$$\theta = \theta_1 + \theta_2, \quad \Delta\theta_1 = 0, \quad (\Delta - s_1^2)\theta_2 = 0.$$

Now, it is clear that if we take $\overset{\mathbf{3}}{\mathbf{v}} = \overset{\mathbf{2}}{\mathbf{w}}$, the theorem will be proved by virtue of (14),(15). Thus,

$$\begin{aligned} &\overset{\mathbf{1}}{\mathbf{w}} = \overset{\mathbf{1}}{\mathbf{v}} + \overset{\mathbf{2}}{\mathbf{v}}, \quad \theta = \theta_1 + \theta_2, \quad rot\overset{\mathbf{1}}{\mathbf{w}} = 0, \quad div\overset{\mathbf{2}}{\mathbf{w}} = 0, \\ &\Delta\overset{\mathbf{1}}{\mathbf{v}} = 0, \quad (\Delta - s_1^2)\overset{\mathbf{2}}{\mathbf{v}} = 0, \\ &\Delta\theta_1 = 0, \quad (\Delta - s_1^2)\theta_2 = 0, \quad (\Delta - s_2^2)\overset{\mathbf{2}}{\mathbf{w}} = 0. \end{aligned} \tag{17}$$

Substituting $\mathbf{w}=\overset{1}{\mathbf{w}}+\overset{2}{\mathbf{w}}$ into (2),(3)and replacing $\overset{1}{\mathbf{w}}$ and θ by their values from (16), we obtain

$$\begin{aligned}&k_7 s_1^2\overset{2}{\mathbf{v}}-k_2(\overset{1}{\mathbf{v}}+\overset{2}{\mathbf{v}})=k_3 grad(\vartheta_1+\vartheta_2),\\&k\Delta\theta_2+k_1 div(\overset{1}{\mathbf{v}}+\overset{2}{\mathbf{v}})=0.\end{aligned}\tag{18}$$

Equation(18) is satisfied by

$$\overset{1}{\mathbf{v}}=-\frac{k_3}{k_2}grad\vartheta_1,$$

$$\overset{2}{\mathbf{v}}=-\frac{k}{k_1}grad\vartheta_2.$$

So, we have proved the following

Lemma. *If*

$$\overset{1}{\mathbf{v}}=-\frac{k_3}{k_2}grad\vartheta_1,\tag{19}$$

$$\overset{2}{\mathbf{v}}=-\frac{k}{k_1}grad\vartheta_2.\tag{20}$$

and they satisfy the conditions

$$\Delta\overset{1}{\mathbf{v}}=0,\quad (\Delta-s_1^2)\overset{2}{\mathbf{v}}=0,$$

then

$$\boldsymbol{W}=(\overset{1}{\mathbf{w}}+\overset{2}{\mathbf{w}},\theta),$$

where

$$\overset{1}{\mathbf{w}}=\overset{1}{\mathbf{v}}+\overset{2}{\mathbf{v}},\quad \theta=\theta_1+\theta_2$$

and $\overset{2}{\mathbf{w}}$, θ_1, θ_2 *satisfy the equations*

$$(\Delta-s_2^2)\overset{2}{\mathbf{w}}=0,\quad div\overset{2}{\mathbf{w}}=0,\quad \Delta\theta_1=0,\quad (\Delta-s_1^2)\theta_2=0,\tag{21}$$

is a solution of equations (2),(3).

2.2. Solution of problem I for a half-plane

The solution of the problem ($\mathbf{w}^+=\mathbf{f}(x_1)$, $\theta^+=f_3(x_1)$) is sought in the form

$$\begin{aligned}\overset{1}{\mathbf{v}}(\mathbf{x})&=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\boldsymbol{\alpha}^{(1)}(\xi)\exp(-x_2|\xi|)\exp(ix_1\xi)d\xi,\\\overset{2}{\mathbf{v}}(\mathbf{x})&=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\boldsymbol{\alpha}^{(2)}(\xi)\exp(-x_2r_1)\exp(ix_1\xi)d\xi,\\\overset{2}{\mathbf{w}}(\mathbf{x})&=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\boldsymbol{\alpha}^{(3)}(\xi)\exp(-x_2r_2)\exp(ix_1\xi)d\xi,\end{aligned}\tag{22}$$

$$\theta_1(\mathbf{x}) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{+\infty} \alpha_4(\xi) \exp(-x_2|\xi|) \exp(ix_1\xi) d\xi,$$

$$\theta_2(\mathbf{x}) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{+\infty} \alpha_5(\xi) \exp(-x_2 r_1) \exp(ix_1\xi) d\xi,$$

$$r_l^2 = \xi^2 + s_l^2, \quad l = 1,2 \quad \boldsymbol{\alpha}^{(j)} = (\alpha_1^{(j)}, \alpha_2^{(j)})^T, j = 1,2,3.$$

where $\boldsymbol{\alpha}^{(j)}$ and α_4, α_5 are absolutely integrable on S unknown vectors and functions, respectively; besides, according to $(21)_2$

$$\alpha_1^{(3)} i\xi - r_2\alpha_2^{(3)} = 0. \tag{23}$$

Let us note ([8],[9]) that if vectors $\mathbf{F}$ and $\widehat{\mathbf{F}}(\xi)$ are absolutely integrable over the entire S, F is bounded and continuous there, then there exists the Fourier transform

$$\widehat{\mathbf{F}}(x_1) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{+\infty} \mathbf{F}(\xi) \exp(-ix_1\xi) d\xi$$

and the inversion formula

$$\mathbf{F}(\xi) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{+\infty} \widehat{\mathbf{F}}(x_1) \exp(ix_1\xi) dx_1$$

is valid.

In what follows we assume,that $\mathbf{f}$, *and* f_3, are absolutely integrable, bounded, and continuous on S, moreover $\widehat{\mathbf{f}}$ and $\widehat{f_3}$ are absolutely integrable on S.

Taking into account the boundary conditions and (19),(20), (23) for determining the unknown vector functions $\boldsymbol{\alpha}^{(j)}$, $j = 1,2,3$, and functions α_4, α_5 we obtain the following system of algebraic equations

$$\boldsymbol{\alpha}^{(1)} + \boldsymbol{\alpha}^{(2)} + \boldsymbol{\alpha}^{(3)} = \widehat{\mathbf{f}}, \quad \alpha_4 + \alpha_5 = \widehat{f_3}, \tag{24}$$

$$\boldsymbol{\alpha}^{(1)} = \frac{k_3}{k_2} \begin{pmatrix} -i\xi \\ |\xi| \end{pmatrix} \alpha_4, \quad \boldsymbol{\alpha}^{(2)} = \frac{k}{k_1} \begin{pmatrix} -i\xi \\ r_1 \end{pmatrix} \alpha_5, \tag{25}$$

$$\alpha_1^{(3)} i\xi - r_2\alpha_2^{(3)} = 0. \tag{26}$$

From (24)-(26) we obtain the following system of algebraic equations

$$\begin{aligned} & -\frac{k_3}{k_2} i\xi\alpha_4 - \frac{k}{k_1} i\xi\alpha_5 + \alpha_1^{(3)} = \hat{f}_1, \\ & \frac{k_3}{k_2}|\xi|\alpha_4 + \frac{k}{k_1} r_1\alpha_5 + \frac{i\xi}{r_2}\alpha_1^{(3)} = \hat{f}_2, \\ & \alpha_4 + \alpha_5 = \hat{f}_3. \end{aligned} \tag{27}$$

It is easy to show that the determinant of system (27) has the form

$$D_1=\frac{k}{k_1}\left[\frac{\xi^2}{r_2}-r_1\right]-\frac{k_3}{k_2}\left[\frac{\xi^2}{r_2}-|\xi|\right]=\left[\frac{k}{k_1}-\frac{k_3}{k_2}\right]\left[\frac{\xi^2}{r_2}-|\xi|\right]$$

$$+\frac{k}{k_1}\left[|\xi|-r_1\right]=-\frac{ks_1^2}{k_1}\frac{1}{r_2}\left[\frac{k_7}{k_6}\frac{|\xi|}{|\xi|+r_2}+\frac{r_2}{r_1+|\xi|}\right]\neq 0,$$

since k_6 and k_7 are positive [see[3]).

From (27) we find

$$\begin{aligned}
\alpha_4&=-\frac{1}{D_1}\left[\frac{i\xi}{r_2}\widehat{f}_1-\widehat{f}_2-\frac{k}{k_1}\left(\frac{\xi^2}{r_2}-r_1\right)\widehat{f}_3\right],\\
\alpha_5&=\frac{1}{D_1}\left[\frac{i\xi}{r_2}\widehat{f}_1-\widehat{f}_2-\frac{k_3}{k_2}\left(\frac{\xi^2}{r_2}-|\xi|\right)\widehat{f}_3\right],\\
\alpha_1^{(3)}&=\frac{1}{D_1}\left[\left(\frac{k_3}{k_2}|\xi|-\frac{k}{k_1}r_1\right)\widehat{f}_1+\left(\frac{k_3}{k_2}-\frac{k}{k_1}\right)i\xi\widehat{f}_2-i\xi\frac{kk_3}{k_1k_2}(r_1-|\xi|)\widehat{f}_3\right],
\end{aligned}$$

By their means, according to (25),(26) we find

$$\boldsymbol{\alpha}^{(1)}=-\frac{k_3}{D_1k_2}\begin{pmatrix}-i\xi\\|\xi|\end{pmatrix}\left[\frac{i\xi}{r_2}\widehat{f}_1-\widehat{f}_2-\frac{k}{k_1}\left(\frac{\xi^2}{r_2}-r_1\right)\widehat{f}_3\right],$$

$$\boldsymbol{\alpha}^{(2)}=\frac{k}{D_1k_1}\begin{pmatrix}-i\xi\\r_1\end{pmatrix}\left[\frac{i\xi}{r_2}\widehat{f}_1-\widehat{f}_2-\frac{k_3}{k_2}\left(\frac{\xi^2}{r_2}-|\xi|\right)\widehat{f}_3\right],$$

$$\alpha_2^{(3)}=\frac{i\xi}{r_2}\frac{1}{D_1}\left[\left(\frac{k_3}{k_2}|\xi|-\frac{k}{k_1}r_1\right)\widehat{f}_1+\left(\frac{k_3}{k_2}-\frac{k}{k_1}\right)i\xi\widehat{f}_2-i\xi\frac{kk_3}{k_1k_2}(r_1-|\xi|)\widehat{f}_3\right].$$

Substituting the obtained values in (22), we obtain the desired solution of the BVP in quadratures.

2.3. Solution of problem 2 for a half-plane

A solution is sought in the form (22). Keeping in mind BCs (i.e. $(\mathbf{w})^+=\mathbf{f}(x_1)$, $\left(k_1w_2+k\frac{\partial\theta}{\partial x_2}\right)^+=f_3(x_1)$) and (19),(20),(23), after passing to the limit, as $x_2\to 0$, we get the following system of algebraic equations

$$\boldsymbol{\alpha}^{(1)}+\boldsymbol{\alpha}^{(2)}+\boldsymbol{\alpha}^{(3)}=\widehat{\mathbf{f}},\quad k_1[\alpha_2^{(1)}+\alpha_2^{(2)}+\alpha_2^{(3)}]-k[\alpha_4|\xi|+r_1\alpha_5]=\widehat{f}_3,$$

$$\boldsymbol{\alpha}^{(1)}=\frac{k_3}{k_2}\begin{pmatrix}-i\xi\\|\xi|\end{pmatrix}\alpha_4,\quad \boldsymbol{\alpha}^{(2)}=\frac{k}{k_1}\begin{pmatrix}-i\xi\\r_1\end{pmatrix}\alpha_5,\quad \alpha_1^{(3)}i\xi-r_2\alpha_2^{(3)}=0. \tag{28}$$

From here we obtain the following system of algebraic equations

$$
\begin{aligned}
&-\frac{k_3}{k_2} i\xi\alpha_4 - \frac{k}{k_1} i\xi\alpha_5 + \alpha_1^{(3)} = \hat{f}_1,\\
&\frac{k_3}{k_2}|\xi|\alpha_4 + \frac{k}{k_1} r_1\alpha_5 + \frac{i\xi}{r_2}\alpha_1^{(3)} = \hat{f}_2,\\
&-\frac{kk_7s_1^2}{k_2}|\xi|\alpha_4 + \frac{i\xi}{r_2} k_1\alpha_1^{(3)} = \hat{f}_3.
\end{aligned} \tag{29}
$$

The determinant of the system (29) has the form

$$
D_2 = -|\xi|\frac{k}{k_2}\left[-k_3 r_1\left(\frac{|\xi|}{r_2} - 1\right) + \frac{k_2 k}{k_1}\left(\frac{\xi^2}{r_2} - r_1\right)\right]. \tag{30}
$$

By elementary calculation, from (30) we obtain

$$
D_2 = -\frac{ks_1^2}{k_1}\frac{|\xi|}{r_2}\left[\frac{|\xi|}{|\xi| + r_1} + \frac{k_7}{k_6}\frac{r_1}{|\xi| + r_2}\right].
$$

Clearly, $D_2(0) = 0$; $\quad D_2(\xi) \neq 0, \quad \xi \neq 0$ and from (29) we have

$$
\alpha_4 = \frac{k}{D_2}\left[\frac{i\xi r_1}{r_2}\widehat{f}_1 - \frac{\xi^2}{r_2}\widehat{f}_2 + \frac{1}{k_1}\left(\frac{\xi^2}{r_2} - r_1\right)\widehat{f}_3\right],
$$

$$
\alpha_5 = \frac{|\xi|}{D_2}\left[\frac{i\xi}{r_2}\left(k + \frac{k_1k_3}{k_2}\right)\widehat{f}_1 + \left(\frac{k_1k_3}{k_2}\frac{|\xi|}{r_2} + kk_7s_1^2\right)\widehat{f}_2 - \frac{k_3}{k_2}\left(\frac{|\xi|}{r_2} - 1\right)\widehat{f}_3\right],
$$

$$
\alpha_1^{(3)} = \frac{k}{k_1k_2D_2}\left[kk_7s_1^2|\xi|(r_1\widehat{f}_1 + i\xi\widehat{f}_2) - k_3 i\xi(r_1 - |\xi|)\widehat{f}_3\right],
$$

$$
\boldsymbol{\alpha}^{(1)} = \frac{k_3}{k_2}\begin{pmatrix} -i\xi \\ |\xi| \end{pmatrix}\frac{k}{D_2}\left[\frac{i\xi r_1}{r_2}\widehat{f}_1 - \frac{\xi^2}{r_2}\widehat{f}_2 + \frac{1}{k_1}\left(\frac{\xi^2}{r_2} - r_1\right)\widehat{f}_3\right],
$$

$$
\boldsymbol{\alpha}^{(2)} = \frac{k}{k_1}\begin{pmatrix} -i\xi \\ r_1 \end{pmatrix}
$$

$$
\times\frac{|\xi|}{D_2}\left[\frac{i\xi}{r_2}\left(k + \frac{k_1k_3}{k_2}\right)\widehat{f}_1 + \left(\frac{k_1k_3}{k_2}\frac{|\xi|}{r_2} + kk_7s_1^2\right)\widehat{f}_2 - \frac{k_3}{k_2}\left(\frac{|\xi|}{r_2} - 1\right)\widehat{f}_3\right],
$$

$$
\alpha_2^{(3)} = \frac{i\xi}{r_2}\frac{k}{k_1k_2D_2}\left[kk_7s_1^2|\xi|(r_1\widehat{f}_1 + i\xi\widehat{f}_2) - k_3 i\xi(r_1 - |\xi|)\widehat{f}_3\right].
$$

Substituting the obtained values in (22), we obtain the solution of the

BVP 2 in quadratures. We assume that $\widehat{f}_3(0)=0$. i.e. $\int\limits_{-\infty}^{+\infty} f_3(\xi)d\xi=0$.

2.4. Solution of problem 3 for a half-plane

A solution of Problem 3 is sought in the form (22). Keeping in mind BCs (i.e, $(w_1)^+=f_1(x_1)$, $\left(\mathbf{T}^{(2)}(\partial\mathbf{z},\mathbf{n})\mathbf{w}\right)_2^+=f_2(x_1)$, $(\theta)^+=f_3(x_1)$) and passing to the limit, as $x_2\to 0$, we have the following system of algebraic equations

$$\alpha_1^{(1)}+\alpha_1^{(2)}+\alpha_1^{(3)}=\widehat{f}_1,\quad \alpha_4+\alpha_5=\widehat{f}_3,$$

$$i\xi k_4[\alpha_1^{(1)}+\alpha_1^{(2)}+\alpha_1^{(3)}]-k_7[\alpha_2^{(1)}|\xi|+\alpha_2^{(2)}r_1+\alpha_2^{(3)}r_2]=\widehat{f}_2,$$

$$\boldsymbol{\alpha}^{(1)}=\frac{k_3}{k_2}\begin{pmatrix}-i\xi\\ |\xi|\end{pmatrix}\alpha_4,\quad \boldsymbol{\alpha}^{(2)}=\frac{k}{k_1}\begin{pmatrix}-i\xi\\ r_1\end{pmatrix}\alpha_5,\quad \alpha_1^{(3)}i\xi-r_2\alpha_2^{(3)}=0.$$

It is easily seen that the determinant of the system for $\alpha_4,\ \alpha_5,\ \alpha_1^{(3)}$

$$D_3=-\frac{kk_7s_1^2}{k_1}\neq 0$$

and

$$\alpha_4=-\frac{1}{D_3}\left[i\xi(k_5+k_6)\widehat{f}_1+\widehat{f}_2+\frac{kk_7s_1^2}{k_1}\widehat{f}_3\right],$$

$$\alpha_5=\frac{1}{D_3}\left[i\xi(k_5+k_6)\widehat{f}_1+\widehat{f}_2\right],$$

$$\alpha_1^{(3)}=\frac{1}{k_2}\left[((k_5+k_6)\xi^2+k_2)\widehat{f}_1-i\xi\widehat{f}_2+k_3i\xi\widehat{f}_3\right],$$

$$\boldsymbol{\alpha}^{(1)}=-\frac{k_3}{k_2}\begin{pmatrix}-i\xi\\ |\xi|\end{pmatrix}\frac{1}{D_3}\left[i\xi(k_5+k_6)\widehat{f}_1+\widehat{f}_2+\frac{kk_7s_1^2}{k_1}\widehat{f}_3\right],$$

$$\boldsymbol{\alpha}^{(2)}=\frac{k}{k_1}\begin{pmatrix}-i\xi\\ r_1\end{pmatrix}\frac{1}{D_3}\left[i\xi(k_5+k_6)\widehat{f}_1+\widehat{f}_2\right],$$

$$\alpha_2^{(3)}=\frac{i\xi}{k_2r_2}\left[((k_5+k_6)\xi^2+k_2)\widehat{f}_1-i\xi\widehat{f}_2+k_3i\xi\widehat{f}_3\right].$$

Substituting the obtained values in (22) and taking into account the following formulas [8]

$$\frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{+\infty}\exp[-x_2|\xi|]\exp[i\xi(x_1-y_1)]d\xi=\sqrt{\frac{2}{\pi}}\frac{x_2}{r^2},$$

$$\frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{+\infty}\exp(-x_2 r_\alpha)\exp[i\xi(x_1-y_1)]\frac{1}{r_\alpha}d\xi=\sqrt{\frac{\pi}{2}}iH_0^{(1)}(is_\alpha r),$$

where $H_0^{(1)}(is_\alpha r)$ is the first kind Hankel function of zero order,

$$r^2=(x_1-y_1)^2+x_2^2,\quad r_\alpha^2=s_\alpha^2+\xi^2,\quad \alpha=1,2,$$

we obtain

$$\mathbf{w}=-\frac{k_3}{k_2}grad\theta_1-\frac{k}{k_1}grad\theta_2+\overset{2}{\mathbf{w}},$$

$$\overset{2}{\mathbf{w}}(\mathbf{x})=\frac{i}{2k_2}\int\limits_{-\infty}^{+\infty}\begin{pmatrix}(k_5+k_6)\dfrac{\partial^3 H_0^{(1)}(is_2r)}{\partial x_1^2\partial x_2}-k_2\dfrac{\partial H_0^{(1)}(is_2r)}{\partial x_2}, \\ -(k_5+k_6)\dfrac{\partial^3 H_0^{(1)}(is_2r)}{\partial x_1^3}+k_2\dfrac{\partial H_0^{(1)}(is_2r)}{\partial x_1}\end{pmatrix}f_1(y)dy$$

$$+\frac{i}{2k_2}\int\limits_{-\infty}^{+\infty}\begin{pmatrix}-\dfrac{\partial^2 H_0^{(1)}(is_2r)}{\partial x_1\partial x_2} \\ \dfrac{\partial^2 H_0^{(1)}(is_2r)}{\partial x_1^2}\end{pmatrix}[-f_2(y)+k_3f_3]dy,$$

$$\theta_1(\mathbf{x})=\frac{-1}{\pi D_3}\int\limits_{-\infty}^{+\infty}\left\{(k_5+k_6)\frac{\partial^2\ln r}{\partial x_1\partial x_2}f_1(y)+\frac{\partial\ln r}{\partial x_2}\left[f_2(y)+\frac{kk_7s_1^2}{k_1}f_3(y)\right]\right\}dy,$$

$$\theta_2(\mathbf{x})=\frac{-i}{2D_3}\int\limits_{-\infty}^{+\infty}\left\{(k_5+k_6)\frac{\partial^2 H_0^{(1)}(is_1r)}{\partial x_1\partial x_2}f_1(y)+\frac{\partial H_0^{(1)}(is_1r)}{\partial x_2}f_2(y)\right\}dy.$$

2.5. Solution of problem 4 for a half-plane

Analogously, we obtain a solution of Problem 4 with BCs

$$(w_2)^+=f_1(x_1),\quad \left(\mathbf{T}^{(2)}(\partial\mathbf{z},\mathbf{n})\mathbf{w}\right)_1^+=f_2(x_1),\quad \left(k_1w_2+k\frac{\partial\theta}{\partial x_2}\right)^+=f_3(x_1)$$

for a half-plane:

$$\mathbf{w} = -\frac{k_3}{k_2} grad\theta_1 - \frac{k}{k_1} grad\theta_2 + \overset{\mathbf{2}}{\mathbf{w}},$$

$$\overset{\mathbf{2}}{\mathbf{w}}(\mathbf{x}) = \frac{i}{2k_2}\int_{-\infty}^{+\infty} \begin{pmatrix} (k_5+k_6)\dfrac{\partial^3 H_0^{(1)}(is_2r)}{\partial x_2^2 \partial x_1}, \\ \\ -(k_5+k_6)\dfrac{\partial^3 H_0^{(1)}(is_2r)}{\partial x_2 \partial x_1^2} \end{pmatrix} f_1(y)dy$$

$$-\frac{i}{2k_2}\int_{-\infty}^{+\infty} \begin{pmatrix} \dfrac{\partial^2 H_0^{(1)}(is_2r)}{\partial x_2^2} \\ \\ -\dfrac{\partial^2 H_0^{(1)}(is_2r)}{\partial x_1 \partial x_2} \end{pmatrix} f_2(y)dy, \quad D_4 = -\frac{k^2 k_7 s_1^2}{k_1}|\xi|\frac{r_1}{r_2},$$

$$\theta_1(\mathbf{x}) =$$
$$\frac{k_1}{\pi k k_7 s_1^2}\int_{-\infty}^{+\infty}\left\{(k_5+k_6)\frac{\partial^2 \ln r}{\partial x_2^2} f_1(y) + \frac{\partial \ln r}{\partial x_1} f_2(y) + \frac{k_2}{k_1}\ln r f_3(y)\right\} dy,$$

$$\theta_2(\mathbf{x}) = \frac{ik_1}{2kk_7 s_1^2}\int_{-\infty}^{+\infty}\left\{-(k_5+k_6)\frac{\partial^2 H_0^{(1)}(is_1r)}{\partial x_1^2} + k_7 s_1^2 H_0^{(1)}(is_1r)\right\} f_1(y)dy$$

$$+\frac{ik_1}{2k^2k_7 s_1^2}\int_{-\infty}^{+\infty}\left\{k\frac{\partial H_0^{(1)}(is_1r)}{\partial x_1} f_2(y) + k_3 H_0^{(1)}(is_1r) f_3(y)\right\} dy.$$

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