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OPTIMIZATION OF DELAY CONTROLLED SYSTEMS WITH NON-FIXED INITIAL MOMENT AND MIXED INITIAL CONDITION

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Abstract. For the controlled differential equation with variable delays in phase coordinates and variable commensurable delays in controls, optimal control problem with general boundary conditions and functional is considered.Necessary optimality conditions are obtained: for the optimal initial function and control in the form of maximum principles;for the initial and final moments in the form of equalities and inequalities. One of them,the essential novelty is necessary condition of optimality for the initial moment,which contains the effect of the mixed initial condition.

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Let R^n_x be an n-dimensional vector space of points $x = (x^1, ..., x^n)^T$ and $O \subset R^k_y, G \subset R^e_z, V \subset R^r_u$ be open sets , $x = (y, z)^T, n = k + e$; let the function $f(t, y_1, ..., y_s, z_1, ..., z_m, u_1, ..., u_\nu)$ be continuous on the set $I \times O^s \times G^m \times V^\nu$ and continuously differentiable with respect to $y_i, i = \overline{1, s}, z_j, j = \overline{1, m}$. Further, let scalar-valued functions $\tau_i(t), i = \overline{1, s}, \sigma_j(t), j = \overline{1, m}, t \in R^1_t$ be absolutely continuous and satisfy the following conditions: $\tau_i(t) \le t, \dot{\tau}_i(t) > 0, \sigma_j(t) \le t, \dot{\sigma}_j(t) > 0$; the functions $\theta_i(t), i = \overline{1, \nu}, t \in R^1_t$ satisfy the commensurability condition, i.e., there exists an absolutely continuous function $\theta(t), t \in R^1_t$, such that $\theta(t) < t, \dot{\theta}(t) > 0$, and $\theta_i(t) = \theta^{k_i}(t), i = \overline{1, \nu}$, where $k_\nu > ... > k_1 \ge 0$ are integers, $\theta^i(t) = \theta(\theta^{i-1}(t))$, and $\theta^0(t) = t; E_\varphi = E_\varphi(I_1, R^k_y)$ be space of piecewise-continuous functions $\varphi : I_1 = [\tau, b] \to R^k_y$ with a finite number of discontinuity points of first type, $\tau = \min(\tau_1(a), ..., \tau_s(a), \sigma_1(a), ..., \sigma_m(a)); \Delta_1 = \{\varphi \in E_\varphi : \varphi(t) \in M, cl\varphi(I_1) \subset O\}, \Delta_2 = \{g \in E_g = E_g(I_1, R^e_z) : g(t) \in N, clg(I_1) \subset G\}$ are sets of initial functions,where $M \subset G, N \subset G$ are convex sets; $\Omega = \{u \in E_u = E_u([\theta_\nu(a), b], R^r_u) : u(t) \in U, clu([\theta_\nu(a), b]) \subset V\}$, is a set of control functions, where $U \subset V$ is an arbitrary set; scalar-valued functions $q^i(t_0, t_1, y_0, z_0, x), i = \overline{1, l}$ are continuously differentiable on the set $I^2 \times O \times$

$G \times (O, G)^T$, where

$$(O, G)^T = \Big\{ x = (y, z)^T \in R^n_x : y \in O, z \in G \Big\}.$$

To each element $w = (t_0, t_1, y_0, \varphi, g, u) \in W = I^2 \times G \times \Delta_1 \times \Delta_2 \times \Omega$ we set in correspondence the equation

$$\dot{x}(t) = (\dot{y}(t), \dot{z}(t))^T = f(t, x(\cdot), u(\cdot)), t \in [t_0, t_1] \subset I \tag{1}$$

with a mixed initial condition

$$x(t) = (y(t), z(t))^T = (\varphi(t), g(t))^T, t \in [\tau, t_0), x(t_0) = (y_0, g(t_0))^T. \tag{2}$$

Here

$$f(t, x(\cdot), u(\cdot)) = f(t, y(\tau_1(t)), ..., y(\tau_s(t)), z(\sigma_1(t)), ..., z(\sigma_m(t)), u(\theta_1(t)),$$
$$..., u(\theta_\nu(t))).$$

The condition (2) is said to be a mixed initial condition, it consists of two parts: the first part is $y(t) = \varphi(t), t \in [\tau, t_0), y(t_0) = y_0$, the discontinuous part since in general $\varphi(t_0) \neq y_0$; the second part is $z(t) = g(t), t \in [\tau, t_0]$, the continuous part since always $z(t_0) = g(t_0)$.

Definition 1. Let $w = (t_0, t_1, y_0, \varphi, g, u) \in W$ and $t_0 < t_1$. A function $x(t) = x(t; w) \in (O \times G)^T, t \in [\tau, t_1]$ is called a solution (trajectory) corresponding to the element w, if it satisfies condition (2), is absolutely continuous on the interval $[t_0, t_1]$ and satisfies Eq.(1) everywhere on $[t_0, t_1]$.

Definition 2. The element $w \in W$ is called admissible, if the corresponding solution satisfies the boundary conditions

$$q^i(t_0, t_1, y_0, g(t_0), x(t_1)) \leq 0, i = \overline{1, l_1}, q^i(t_0, t_1, y_0, g(t_0), x(t_1)) = 0,$$
$$i = \overline{l_1 + 1, l}. \tag{3}$$

We denote the set of admissible elements by W_0.

Definition 3. The element $w_0 = (t_{00}, t_{10}, y_{00}, \varphi_0, g_0, u_0) \in W_0$ is called optimal, if for any $w \in W$ the following inequality

$$q^0(t_{00}, t_{10}, y_{00}, g_0(t_{00}), x_0(t_{10})) \leq q^i(t_0, t_1, y_0, g(t_0), x(t_1)) \tag{4}$$

is fulfilled, where $x_0(t) = (y_0(t), z_0(t))^T = x(t; w_0), x(t) = x(t; w)$. We called (1)-(4) optimal control problem with the mixed initial condition. The aim is to fined the optimal element w_0.

Introduce the following notations: $\gamma_i(t) = \tau_i^{-1}(t), \gamma_i = \gamma_i(t_{00}), \dot{\gamma}_i^- = \dot{\gamma}(t_{00}-), \hat{\gamma}_0^- = 1, \hat{\gamma}_i^- = \dot{\gamma}_i^-, i = \overline{1, p}, \hat{\gamma}_{p+1}^- = 0, \rho_j(t) = \sigma_j^{-1}(t)$;

$$F_i^- = f(t_{00}, \underbrace{y_{00}, ..., y_{00}}_{i}, \underbrace{\varphi_0(t_{00}-), ..., \varphi_0(t_{00}-)}_{p-i}, \varphi_0(\tau_{p+1}(t_{00}-)), ..., \varphi_0(\tau_s(t_{00}-)),$$

$$g_0(\sigma_1(t_{00}-)),...,g_0(\sigma_m(t_{00}-)),u_0(\theta_1(t_{00}-)),...,u_0(\theta_\nu(t_{00}-))),i=\overline{0,p};$$

$$F_i^-[y]=f(\gamma_i,y_0(\tau_1(\gamma_i)),...,y_0(\tau_{i-1}(\gamma_i)),y,\varphi_0(\tau_{i+1}(\gamma_i-)),...,\varphi_0(\tau_s(\gamma_i-)),$$

$$z_0(\sigma_1(\gamma_i-)),...,z_0(\sigma_m(\gamma_i-)),u_0(\theta_1(\gamma_i-)),...,u_0(\theta_\nu(\gamma_i-))),i=\overline{p+1,s};$$

$$F_{s+1}^-=f(t_{10},y_0(\tau_1(t_{10})),...,y_0(\tau_s(t_{10})),z_0(\sigma_1(t_{10}-)),\ldots,z_0(\sigma_m(t_{10}-)),$$

$$u_0(\theta_1(t_{00}-)),...,u_0(\theta_\nu(t_{10}-))).$$

Similarly we can define: $\hat{\gamma}_i^+,i=\overline{0,p+1};F_i^+,i=\overline{0,p};F_i^+[y],i=\overline{p+1,s+1}$.

Theorem 1. *Let $w_0=(t_{00},t_{10},y_{00},\varphi_0,g_0,u_0)$ be an optimal element, $t_{00}>a$ and $x_0(t)$ be a corresponding trajectory. Let the following conditions hold:*

1) $\gamma_i=t_{00},i=\overline{1,p},\gamma_{p+1}<...<\gamma_s<t_{10}$;

2) $\dot{\gamma}_i^-<\infty,i=\overline{1,p}$, and $\dot{\gamma}_p^-<...<\dot{\gamma}_1^-$;

3) the function $g_0(t)$ is absolutely continuous on the interval $(t_{00}-\delta,t_{00}]$, where $\delta>0$ and there exists the finite limit $\dot{g}_0^-=\dot{g}_0(t_{00}-)$.

Then the exist a vector $\pi=(\pi_0,...,\pi_l)\neq 0,\pi_i\leq 0,i=\overline{0,l_1}$ and a solution $\Psi(t)=(\psi(t),\chi(t))=(\psi_1(t),...,\psi_k(t),\chi_1(t),...,\chi_e(t))$ of the equation

$$\begin{cases}\dot{\psi}(t)=-\sum_{i=1}^s\Psi(\gamma_i(t))f_{y_i}(\gamma_i(t),x_0(\cdot),u_0(\cdot))\dot{\gamma}_i(t),\\ \dot{\chi}(t)=-\sum_{j=1}^m\Psi(\rho_j(t))f_{z_j}(\rho_j(t),x_0(\cdot),u_0(\cdot))\dot{\rho}_j(t),\\ t\in[t_{00},t_{10}],\Psi(t)=0,t>t_{10},\end{cases}\tag{5}$$

such that the following conditions hold:

4) the integral maximum principle for the control:

$$\int_{t_{00}}^{t_{10}}\Psi(t)f(t,x_0(\cdot),u_0(\cdot))dt=\max\nolimits_{u(\cdot)\in\Omega}\int_{t_{00}}^{t_{10}}\Psi(t)f(t,x_0(\cdot),u(\cdot))dt;$$

5) the integral maximum principle for the initial function:

$$\sum_{i=p+1}^s\int_{\tau_i(t_{00})}^{t_{00}}\Psi(\gamma_i(t))f_{y_i}(\gamma_i(t),x_0(\cdot),u_0(\cdot))\dot{\gamma}_i(t)\varphi_0(t)dt$$

$$=\max\nolimits_{\varphi(\cdot)\in\Delta_1}\sum_{i=p+1}^s\int_{\tau_i(t_{00})}^{t_{00}}\Psi(\gamma_i(t))f_{y_i}(\gamma_i(t),x_0(\cdot),u_0(\cdot))\dot{\gamma}_i(t)\varphi(t)dt,$$

$$\sum_{j=1}^m\int_{\sigma_j(t_{00})}^{t_{00}}\Psi(\rho_j(t))f_{z_j}(\rho_j(t),x_0(\cdot),u_0(\cdot))\dot{\rho}_j(t)g_0(t)dt$$

$$=\max\nolimits_{g(\cdot)\in\Delta_2}\sum_{j=1}^m\int_{\sigma_j(t_{00})}^{t_{00}}\Psi(\rho_j(t))f_{z_j}(\rho_j(t),x_0(\cdot),u_0(\cdot))\dot{\rho}_j(t)g(t)dt;$$

$$(\pi Q_{0z_0}+\chi(t_{00}))g_0(t_{00})=\max\nolimits_{g\in N}(\pi Q_{0z_0}+\chi(t_{00}))g;$$

6) the conditions for the function $\Psi(t)$ and the vector π :

$$\Psi(t_{10})=\pi Q_{0x},\ \psi(t_{10})=-\pi Q_{0y_0},$$

$$\pi_i q^i(t_{00}, t_{10}, y_{00}, g_0(t_{00}), x_0(t_{10})) = 0,\ i = \overline{1, l_1};$$

7) the conditions for the instants t_{00} and t_{10} :

$$\pi Q_{0t_0} \geq -(\pi Q_{0z_0} + \chi(t_{00}))\dot{g}_0^- - \Psi(t_{00}) \sum_{i=0}^{p} (\hat{\gamma}_{i+1}^- - \hat{\gamma}_i^-) F_i^-$$

$$+ \sum_{i=p+1}^{s} \Psi(\gamma_i) \Big\{ F_i^-[y_{00}] - F_i^-[\varphi_0(t_{00}-)] \Big\} \dot{\gamma}_i^-, \tag{6}$$

$$\pi Q_{0t_1} \geq -\Psi(t_{10}) F_{s+1}^-.$$

Here

$$Q = (q^0, ..., q^l)^T, Q_{0x} = Q_x(t_{00}, t_{10}, y_{00}, g_0(t_{00}), x_0(t_{10})).$$

Some comments. The expression

$$-(\pi Q_{0z_0} + \chi(t_{00}))\dot{g}_0^- - \Psi(t_{00}) \sum_{i=0}^{p} (\hat{\gamma}_{i+1}^- - \hat{\gamma}_i^-) F_i^-$$

$$+ \sum_{i=p+1}^{s} \Psi(\gamma_i) \Big\{ F_i^-[y_{00}] - F_i^-[\varphi_0(t_{00}-)] \Big\} \dot{\gamma}_i^-$$

in the condition (6) is the effect of the mixed initial condition (2).

The expression

$$-\Psi(t_{00}) \sum_{i=0}^{p} (\hat{\gamma}_{i+1}^- - \hat{\gamma}_i^-) F_i^- + \sum_{i=p+1}^{s} \Psi(\gamma_i) \Big\{ F_i^-[y_{00}] - F_i^-[\varphi_0(t_{00}-)] \Big\} \dot{\gamma}_i^-$$

is the effect of the discontinuous part of the condition (2).

The expression

$$-(\pi Q_{0z_0} + \chi(t_{00}))\dot{g}_0^-$$

is the effect of the continuous part of the condition (2).

Theorem 2. *Let $w_0 = (t_{00}, t_{10}, y_{00}, \varphi_0, g_0, u_0)$ be an optimal element, $t_{10} < b$ and let condition 1) and the following conditions hold:*

8) $\dot{\gamma}_i^+ < \infty, i = \overline{1, p}$, and $\dot{\gamma}_1^+ < ... < \dot{\gamma}_p^+$;

9) the function $g_0(t)$ is absolutely continuous on the interval $[t_{00}, t_{00} + \delta)$, where $\delta > 0$ and there exists the finite limit $\dot{g}_0^+ = \dot{g}_0(t_{00}+)$.

Then the exist a vector $\pi = (\pi_0, ..., \pi_l) \neq 0, \pi_i \leq 0, i = \overline{0, l_1}$, and a solution $\Psi(t) = (\psi(t), \chi(t))$ of Eq.(5) such that conditions 4)-6) hold. Moreover,

$$\pi Q_{0t_0} \leq -(\pi Q_{0z_0} + \chi(t_{00}))\dot{g}_0^+ - \Psi(t_{00}) \sum_{i=0}^{p} (\hat{\gamma}_{i+1}^+ - \hat{\gamma}_i^+) F_i^+$$

$$+\sum_{i=p+1}^{s}\Psi(\gamma_i)\Big\{F_i^+[y_{00}]-F_i^+[\varphi_0(t_{00}+)]\Big\}\dot\gamma_i^+,$$

$$\pi Q_{0t_1}\le -\Psi(t_{10})F_{s+1}^+.$$

Theorem 3. *Let $w_0=(t_{00},t_{10},y_{00},\varphi_0,g_0,u_0)$ be an optimal element, $t_{00},t_{10}\in(a,b)$, the function $g_0(t)$ is continuously differentiable in a neighborhood of the point t_{00} and let conditions 1),2),8) hold. Moreover,*

$$\sum_{i=0}^{p}(\hat\gamma_{i+1}^- -\hat\gamma_i^-)F_i^- =\sum_{i=0}^{p}(\hat\gamma_{i+1}^+ -\hat\gamma_i^+)F_i^+ \overset{def}{=} F_{01},$$

$$\Big\{F_i^-[y_{00}]-F_i^-[\varphi_0(t_{00}-)]\Big\}\dot\gamma_i^- =\Big\{F_i^+[y_{00}]-F_i^+[\varphi_0(t_{00}+)]\Big\}\dot\gamma_i^+ \overset{def}{=} F_{1i}$$

$$F_{s+1}^- =F_{s+1}^+ \overset{def}{=} F_{s+1}.$$

Then the exist a vector $\pi=(\pi_0,...,\pi_l)\neq 0,\pi_i\le 0, i=\overline{0,l_1}$ and a solution $\Psi(t)=(\psi(t),\chi(t))$ of Eq.(5) such that conditions 4)-6) hold. Moreover,

$$\pi Q_{0t_0}=-(\pi Q_{0z_0}+\chi(t_{00}))\dot g_0(t_{00})-\Psi(t_{00})F_{01}+$$

$$\sum_{i=p+1}^{s}\Psi(\gamma_i)F_{1i},$$

$$\pi Q_{0t_1}=-\Psi(t_{10})F_{s+1}.$$

Theorems 1,2 and 3, on the basis of the variation formulas [1], are proved by a method given in [2].

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R E F E R E N C E S

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