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ON VARIATIONAL FORMULATION AND DECOMPOSITION METHODS FOR BITSADZE-SAMARSKII NONLOCAL BOUNDARY VALUE PROBLEM FOR TWO-DIMENSIONAL SECOND ORDER ELLIPTIC EQUATIONS

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Abstract. Bitsadze-Samarskii nonlocal boundary value problem for two-dimensional second order elliptic equations is considered. Solving of this problem by using domain and operator decomposition methods are given. Variational formulation for Poisson's equation is done and studied.

Keywords and phrases: Bitsadze-Samarskii nonlocal boundary value problem, domain decomposition, operator decomposition, variational formulation.

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1. Introduction

In applied sciences, such as plasma physics, shell and elastisity theories, hydrodynamics, oceanography etc. different problems with nonlocal boundary conditions arise very often.

One of the first works devoted to nonlocal boundary value problems belongs to the beginning of the 20^{th} century [1]. Modern investigation of nonlocal elliptic boundary value problems originates from A. Bitsadze and A. Samarskii work [2], in which by means of the method of integral equations the theorems of existence and uniqueness of a solution for the second order multi-dimensional elliptic equations in rectangle domains are proved. There are given some classes of problems for which the proposed method works. Many works are devoted to the investigation of the problem given in [2] and to some of its generalizations. One of the first among them was the work [3] where the iterative method of proving the existence of a solution for Laplace equation was proposed.

It should be noted that the usage of this method gives not only existence of a solution, but also allows to found effective algorithms for numerical resolution of such problems. By the approach proposed in work [3], the nonlocal problem reduces to classical Dirichlet problems, that yields the possibility to apply the elaborated effective methods for numerical resolution of these problems. After this work many scientists have been investigating nonlocal problems by using the same or different methods for elliptic equations and, among them, nonlinear models as well (see, for example, [4-25] and references therein).

It is well known that, in order to find the approximate solutions, it is important to construct useful economical algorithms. For constructing of such algorithms, the method of domain decomposition has a great importance. There are several reasons why the domain decomposition techniques might be attractive. Applying this method the whole problem can be reduced to relative subproblems on the domains which are comparatively less in size, than the one considered at the beginning. At the same time, we should note that, together with the sequential count algorithm on each of these domains, it is frequently possible to apply parallel count algorithm, too. In the works [5-7, 15, 20, 24] domain decomposition method based on Schwarz alternative method [26] are displayed for study of nonlocal problems for Laplace and nonlinear elliptic equations [6, 7, 20].

To solve two or more dimensional problems by using operator decomposition method is important as well. Naturally, among them the research of nonlocal tasks is essential. Note that, combination of domain as well as operator decomposition is very important too. In this direction, for nonlocal problems some results are already done (see, for example, [13]).

It is known, how great role takes place variational formulation of boundary problems in modern mathematics. This question for nonlocal elliptic problems is in the beginning of study yet (see, for example, [23, 25]).

In the present work we give some results, devoted to the domain decomposition and Schwarz-type iterative methods for Bitsadze-Samarskii nonlocal boundary value problem. Operator decomposition is done as well. More attention is paid on study of possibility of variational formulation.

Results of this paper are partly published in the works [23, 24].

The outline of this paper is as follows. In section 2, for the Poisson equation in a rectangle, we state Bitsadze-Samarskii nonlocal problem. In section 3 the convergence of the Schwarz-type iterative sequential algorithm as well as the same question for parallel algorithm is studied. The operator decomposition combining with domain ones is given too. In section 4 variational formulation is stated and discussed.

2. Formulation of the problem

In the plane Oxy, let us consider the rectangle $G = \{(x, y) | -a < x < 0, 0 < y < b\}$, a and b are the given positive constants. By ∂G we denote the boundary of the rectangle G, and by Γ_t the intersection of the line $x = t$ with the set $\overline{G} = G \cup \partial G$.

Consider the following nonlocal Bitsadze-Samarskii boundary value problem [2]:

$$-\Delta u(x, y) = f(x, y), \quad (x, y) \in G,$$

$$u(x,y)\big|_{\Gamma} = 0, \qquad (2.1)$$
$$u(x,y)\big|_{\Gamma_{-\xi}} = u(x,y)\big|_{\Gamma_0},$$

where $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is the Laplace operator, $\Gamma = \partial G \backslash \Gamma_0$, $\xi \in (0,a)$, $f(x,y)$ is a given function $f(x,y) \in C(\overline{G})$ and $u(x,y) \in C(\overline{G}) \cap C^{(2)}(G)$ is an unknown function.

Let's notice that the uniqueness of the solution of problem (2.1) follows from the extremum principle. It is known, that if f is continuous function on $\overline{G}$, then there exists unique regular solution of problem (2.1) $u(x) \in C^2(G) \cap C(\overline{G})$ [2].

3. Decomposition algorithms

For the investigation of the problem (2.1) stated in the previous section, let's consider the following sequential iterative procedure:

$$-\Delta u_1^n(x,y) = f(x,y), \quad (x,y) \in G_1,$$
$$u_1^n(x,y)|_{\Gamma^1} = 0, \quad u_1^n(x,y)|_{\Gamma_{-\xi_1}} = u_2^{n-1}(x,y)\big|_{\Gamma_{-\xi_1}}, \qquad (3.1)$$
$$n = 1,2,...;$$

$$-\Delta u_2^n(x,y) = f(x,y), \quad (x,y) \in G_2,$$
$$u_2^n(x,y)|_{\Gamma^2} = 0, \quad u_2^n(x,y)|_{\Gamma_{-\xi}} = u_2^n(x,y)|_{\Gamma_0} = u_1^n(x,y)|_{\Gamma_{-\xi}}, \qquad (3.2)$$
$$n = 1,2,... \ .$$

Here we utilize the following notations:

$$G_1 = \{-a < x < -\xi_1, \ 0 < y < b\}, \quad G_2 = \{-\xi < x < 0, \ 0 < y < b\},$$

where $-\xi_1$ is a fixed point of the interval $(-\xi, 0)$, $\Gamma^1 = \partial G_1 \setminus \Gamma_{-\xi_1}$, $\Gamma^2 = \partial G_2 \setminus (\Gamma_{-\xi} \cup \Gamma_0)$ and $u_2^0(-\xi_1, y) \equiv 0$.

The iterative procedure (3.1), (3.2) reduces our nonlocal nonclassical problem (2.1) to the sequence of classical Dirichlet boundary value problems on every step of the iteration.

The following statement is true [5].

Theorem 3.1. *The sequential iterative process (3.1), (3.2) converges to a solution of problem (2.1) uniformly in the domain $\overline{G}$, and the following estimations are valid:*

$$|u(x,y) - u_1^n(x,y)| \le Cq^{n-1}, \quad (x,y) \in \overline{G}_1,$$

$$|u(x,y) - u_2^n(x,y)| \le Cq^{n-1}, \quad (x,y) \in \overline{G}_2,$$

where $q \in (0,1)$ and C are constants independent of functions $u(x,y)$, $u_1^n(x,y)$, $u_2^n(x,y)$.

As we have already noted, algorithm (3.1), (3.2) for the solution of the problem (2.1) has a sequential form. Now, let us consider one more approach to the solution of the problem (2.1). In this case the searching of approximate solutions on domains $\overline{G}_1$ and $\overline{G}_2$ will be carried out not by means of a sequential algorithm, but in a parallel way.

Consider the following overlapping parallel iterative process:

$$\begin{gathered} -\Delta u_1^n(x,y) = f(x,y), \quad (x,y) \in G_1, \\ u_1^n(x,y)|_{\Gamma^1} = 0, \quad u_1^n(x,y)|_{\Gamma_{-\xi_1}} = u_2^{n-1}(x,y)\big|_{\Gamma_{-\xi_1}}, \\ n = 1,2,...; \end{gathered} \tag{3.3}$$

$$\begin{gathered} -\Delta u_2^n(x,y) = f(x,y), \quad (x,y) \in G_2, \\ u_2^n(x,y)|_{\Gamma^2} = 0, \quad u_2^n(x,y)|_{\Gamma_{-\xi}} = u_2^n(x,y)|_{\Gamma^0} = u_1^{n-1}(x,y)\big|_{\Gamma_{-\xi}}, \\ n = 1,2,\dots\ , \end{gathered} \tag{3.4}$$

where $u_1^0(-\xi,0) \equiv u_2^0(-\xi_1,0) \equiv 0$.

The following statement takes place [15].

Theorem 3.2. *The parallel iterative process (3.3), (3.4) converges to a solution of the problem (2.1) uniformly in the domain $\overline{G}$, and the following estimations are valid:*

$$|u(x,y) - u_1^n(x,y)| \le Cq^{\frac{n}{2}-1}, \quad (x,y) \in \overline{G}_1,$$

$$|u(x,y) - u_2^n(x,y)| \le Cq^{\frac{n}{2}-1}, \quad (x,y) \in \overline{G}_2,$$

where $q \in (0,1)$ and C are constants independent of functions $u(x,y)$, $u_1^n(x,y)$, $u_2^n(x,y)$.

Remark 3.1. The theorems analogical to theorems above are also true for Bitsadze-Samarskii boundary value problem for the following nonlinear equation

$$F(x,y,u,p,q,r,s,t) = 0,$$

where F is the analitic function of its arguments, $u = u(x,y)$, $p = u_x$, $q = u_y$, $r = u_{xx}$, $s = u_{xy}$, $t = u_{yy}$, and:

$$4F_rF_t - F_s^2 \ge const > 0, \quad F_u \le 0.$$

Remark 3.2. Bitsadze-Samarskii nonlocal boundary value problem for the abovementioned nonlinear equation by using iterative process analogical to [3] at first was studied in [4] and by domain decomposition method with Schwarz alternative algorithm in [6, 7, 20].

Remark 3.3. Theorems analogical to theorems above are valid for the sequential as well as parallel algorithms for multi-grid domain decomposition case.

Some developing in investigation of nonlocal problems happened after publishing of work [8]. In particular, by using specially defined scalar product [8], uniqueness of solution of various problems, for which extremum principle do not takes place, are shown.

In work [13] proof of convergence of corresponding operator and domain decomposition iterative processes for nonlocal value problem for Poisson equation are given.

For solving problem (2.1) the following iterative precess is considered:

1. On $\overline{G}$ arbitrary continuous function $q^0(x,y)$ is taken;
2. Solutions of following one-dimensional problems are found:

$$\begin{gathered}
-\frac{\partial^2 u_1^n}{\partial x^2} = q^n(x,y) + f_1(x,y), \quad (x,y) \in G, \\
u_1^n(-a,y) = 0, \quad u_1^n(0,y) = u_1^n(-\xi,y), \quad y \in (0,b), \\
n = 0,1,2,\ldots;
\end{gathered} \tag{3.5}$$

$$\begin{gathered}
-\frac{\partial^2 u_2^n}{\partial y^2} = -q^n(x,y) + f_2(x,y), \quad (x,y) \in G, \\
u_2^n(x,0) = 0, \quad u_2^n(x,b) = 0, \quad x \in (-a,0), \\
n = 0,1,2,\ldots,
\end{gathered} \tag{3.6}$$

where f_1 and f_2 are continuous functions such that $f_1 + f_2 = f$.

3. The new approximations are defined as follows:

$$q^{n+1}(x,y) = q^n(x,y) + \rho_n \left[u_1^n(x,y) - u_2^n(x,y)\right], \quad (x,y) \in G, \tag{3.7}$$

where ρ_n are parameters of iteration.

In the work [13] following scalar product is used

$$(v,w) = \int\limits_0^b \int\limits_{-\xi}^0 \int\limits_{-a}^x v(s,y)\, w(s,y)\, ds\, dx\, dy, \tag{3.8}$$

which is the same (to within a constant multiplayer) as scalar product given in [8] for multidimensional parallelepiped. For the corresponding norm of scalar product (3.8) we will use following notation $|[\cdot]|$.

Let us consider following differences:

$$\begin{gathered}
q(x,y) = -\frac{\partial^2 u}{\partial x^2} - f(x,y), \quad z_1^n(x,y) = u_1^n(x,y) - u(x,y), \\
z_2^n(x,y) = u_2^n(x,y) - u(x,y), \quad Q^n(x,y) = q^n(x,y) - q(x,y).
\end{gathered}$$

The following statement takes place [13].

Theorem 3.3. *If parameters ρ_n of the iterative process (3.5)-(3.7) satisfy the following conditions $0 < \rho_0 < \rho_n < 1$, then for all n the following relations take place:*

$$||[Q^{n+1}]|| \leq ||[Q^n]||, \quad \lim_{n\to\infty} \left|\left| \left[\frac{\partial z_1^n}{\partial x} \right] \right|\right| = 0,$$

$$\lim_{n\to\infty} \left|\left| \left[\frac{\partial z_2^n}{\partial y} \right] \right|\right| = 0, \quad \lim_{n\to\infty} ||[z_1^n - z_2^n]|| = 0.$$

Let us consider one more parallel (nonoverlapping) iterative process, corresponding to domain decomposition [13]. Divided domain G in two subdomains Ω_1 and Ω_2: $\Omega_1 = (-a, -\xi) \times (0, b)$, $\Omega_2 = (-\xi, 0) \times (0, b)$. Let's denote $\bar{\Gamma}^1 = \partial\Omega_1 \backslash \Gamma_{-\xi}$, $\bar{\Gamma}^2 = \partial\Omega_2 \backslash (\Gamma_{-\xi} \cup \Gamma_0)$ and construct sequences of functions $p^n(y)$, $u_1^n(x, y)$, $u_2^n(x, y)$ in the following way:

1. Initial approximations is taken as follows $p^0(y)$ and $u_2^0|_{\Gamma_{-\xi}} = 0$;
2. We solve following problems on subdomains:

$$-\Delta u_1^n(x, y) = f(x, y), \quad (x, y) \in \Omega_1,$$

$$u_1^n(x, y)|_{\bar{\Gamma}^1} = 0, \quad \left.\frac{\partial u_1^n(x, y)}{\partial x}\right|_{\Gamma_{-\xi}} = p^n(y), \tag{3.9}$$

$$n = 1, 2, \ldots;$$

and

$$-\Delta u_2^n(x, y) = f(x, y), \quad (x, y) \in \Omega_2,$$

$$u_2^n(x, y)|_{\bar{\Gamma}^2} = 0, \quad \left.\frac{\partial u_2^n(x, y)}{\partial x}\right|_{\Gamma_{-\xi}} = p^n(y),$$

$$u_2^n(x, y)\big|_{\Gamma_0} = u_2^{n-1}(x, y)\big|_{\Gamma_{-\xi}}, \tag{3.10}$$

$$n = 1, 2, \ldots;$$

3. The new approximations are defined as follows:

$$p^{n+1}(y) = p^n(y) - \rho_n \left[u_2^n(x, y) - u_1^n(x, y)\right]\big|_{\Gamma_{-\xi}}.$$

Theorem 3.4. *If $p^0 \in L^2(0, b)$, then parameter $\rho_n = \rho$ can be found such that sequences constructed from (3.9), (3.10) satisfy relations:*

$$u_i^n(x, y) \to u_i(x, y) = u(x, y)|_{\Omega_i} \quad \text{strongly in} \quad H^1(G_i),$$

$$u_i^n(x, y)|_{\Gamma_{-\xi}} \to u(x, y)|_{\Gamma_{-\xi}} \quad \text{strongly in} \quad L^2(0, 1),$$

$$\left.\frac{\partial u_i^n(x, y)}{\partial x}\right|_{\Gamma_{-\xi}} \to \left.\frac{\partial u(x, y)}{\partial x}\right|_{\Gamma_{-\xi}} \quad \text{strongly in} \quad L^2(0, 1), \quad i = 1, 2.$$

4. Variational formulation

Regarding to investigation of nonlocal boundary problems study of their variational formulation is important. In this direction the main difficulty is asymmetry of corresponding operators of nonlocal boundary problems (using scalar product (3.8), inequality of positively definiteness can be received [8]). To solve this problem, modification of formula from [8] can be used. This modification is connected to the function of symmetrically continuances of operator. Such type modification, firstly was done in [22]. In this work positively definiteness of corresponding operator for Bitsadze-Samarskii problem for the second order ordinary differential equation was shown on the specially defined lineal of functions. The main difficulty occurs, when the selection of parameters which are included in structure of functional is necessary for finding of a variational equivalent of a considered problem, and this selection in case of satisfaction the special conditions for coefficients which are included in the equation can be achieved [23]. The mentioned method for showing of positively definiteness of operator of nonlocal boundary problems for some elliptic equations can be extended. Problem of finding corresponding coordinating function-parameter for these issues is still difficult for this method and represents subject of future investigations. Below we consider mentioned method for Bitsadze-Samarskii nonlocal problem (2.1).

Let us denote by $D(\overline{G})$ the lineal of all real functions satisfying the following conditions:

1. $v(x,y)$ is defined almost everywhere on $\overline{G}$, and the boundary value $v(0,y)$ is defined almost everywhere on Γ_0.

2. $v(x,y) \in L_2(G)$, $v(0,y) \in L_2(0,b)$.

Two functions $v_1(x,y)$ and $v_2(x,y)$ are assumed as the same element of $D(\overline{G})$ if $v_1(x,y) = v_2(x,y)$ almost everywhere on $\overline{G}$ and $v_1(0,y) = v_2(0,y)$ almost everywhere on Γ_0.

Let $\overline{Q}$ be the closed rectangle $\big\{(x,y)\big|\, 0 < x < \xi,\, 0 < y < b\big\}$ and $\mathcal{T}$ be the operator which extends elements of $D(\overline{G})$ as follows

$$\mathcal{T}\, v(x,y) = \begin{cases} v(x,y), & \text{if} \quad (x,y) \in \overline{G}, \\ -v(-x,y) + 2v(0,y), & \text{if} \quad (x,y) \in \overline{Q}. \end{cases}$$

Let us note that operator $\mathcal{T}$ associates to every function $v(x,y)$ of the lineal $D(\overline{G})$ the function $\widetilde{v}(x,y) = \mathcal{T}\, v(x,y)$ in such a way that the function $\widetilde{v}(x,y) - v(0,y)$ is the odd function with respect to the variable x almost everywhere for the almost all $y \in [0,b]$.

For two arbitrary functions $v(x,y)$ and $w(x,y)$ from the lineal $D(\overline{G})$ we define the scalar product

$$[v,w] = \int_0^b \int_{-\xi}^{\xi} \int_{-a}^{x} \widetilde{v}(s,y)\, \widetilde{w}(s,y)\, ds\, dx\, dy. \tag{4.1}$$

After the introduction of the scalar product (4.1) the lineal $D(\overline{G})$ becomes the Hilbert space, which we denote by $H(\overline{G})$. The norm originated from the scalar product (4.1) in $H(\overline{G})$ we denote by $\|\cdot\|_H$:

$$\|v\|_H^2 = \int_0^b \int_{-\xi}^{\xi} \int_{-a}^{x} \widetilde{v}^2(s,y)\,ds\,dx\,dy.$$

The following statements take place [23].

Theorem 4.1. *The norm defined in $H(\overline{G})$ by the formula*

$$\|v\|^2 = \|v(x,y)\|_{L_2(G)}^2 + \|v(0,y)\|_{L_2(0,b)}^2$$

is equivalent to the norm $\|\cdot\|_H$.

Theorem 4.2. *Space $H(\overline{G})$ is complete with the metric $\rho(v,w) = \|v - w\|_H$.*

Let, the area of definition of the operator $A = -\Delta$ is the lineal $D_A(\overline{G})$ with $H(\overline{G})$ for elements $v(x,y)$ of which the following conditions are fulfilled:

1. $v(x,y) \in C^{(\infty)}(\overline{G}), \quad \dfrac{\partial^k v(0,y)}{\partial x^k} = 0, \quad y \in [0,b], \quad k = 1,2,\dots;$
2. $v(x,y)\big|_{\Gamma} = 0, \quad v(x,y)\big|_{\Gamma_{-\xi}} = v(x,y)\big|_{\Gamma_0}.$

Theorem 4.3. *The lineal $D_A(G)$ is dense in the space $H(\overline{G})$.*

Proof. It is enough to show that for the $\varepsilon > 0$ and for arbitrary function $v(x,y) \in H(\overline{G})$ function $V(x,y) \in D_A(\overline{G})$ can be found which satisfies following inequality

$$\|v(x,y) - V(x,y)\| < \varepsilon.$$

Assume, that ε_0 is arbitrary positive number. We have $v(0,y) \in L_2(0,b)$, so can be found such function $g(y) \in C_0^{(\infty)}(0,b)$ that satisfies following inequality

$$\|v(0,y) - g(y)\|_{L_2(0,b)} < \varepsilon_0. \tag{4.2}$$

Let function $\varphi(x,y) \in C^{(\infty)}(\overline{G})$ such that $\varphi(x,y) = 1$ if $(x,y) \in \overline{G}_2 = [-\xi,0] \times [0,b]$ and $\varphi(-a,y) = 0$, for all $y \in [0,b]$. Note that $v_1(x,y) = g(y)\varphi(x,y) \in D_A(\overline{G})$ and $v_1(0,y) = g(y)$. From (4.2) we get

$$\|v(0,y) - v_1(0,y)\|_{L_2(0,b)} < \varepsilon_0. \tag{4.3}$$

Let us take function $v_2(x,y) \in C_0^{(\infty)}(\overline{G}_2)$ such that

$$\|v(x,y) - v_1(x,y) - v_2(x,y)\|_{L_2(G_2)} < \varepsilon_0. \tag{4.4}$$

We assume that function $v_2(x,y)$ is zero on the $\overline{G}_1 = [-a,-\xi] \times [0,b]$. It easy to see that $v_2(x,y) \in D_A(\overline{G})$.

Analogously, for the difference $v(x,y) - [v_1(x,y) + v_2(x,y)]$ function $v_3(x,y) \in C_0^{(\infty)}(\overline{G}_1)$ can be found such that

$$\|v(x,y) - v_1(x,y) - v_2(x,y) - v_3(x,y)\|_{L_2(G_1)} < \varepsilon_0. \tag{4.5}$$

Let us denote $V(x,y)=v_1(x,y)+v_2(x,y)+v_3(x,y)$. It is obvious that $V(x,y)\in D_A(\overline{G})$ and from (4.4) and (4.5) we get

$$\|v(x,y)-V(x,y)\|^2_{L_2(G)}=\|v(x,y)-V(x,y)\|^2_{L_2(G_1)}$$

$$+\|v(x,y)-V(x,y)\|^2_{L_2(G_2)}=\|v(x,y)-V(x,y)\|^2_{L_2(G_1)}$$

$$+\|v(x,y)-v_1(x,y)-v_2(x,y)\|^2_{L_2(G_2)}<2\,\varepsilon_0^2,$$

i.e.,

$$\|v(x,y)-V(x,y)\|^2_{L_2(G)}<2\,\varepsilon_0^2$$

and using (4.3) we have also

$$\|v(0,y)-V(0,y)\|^2_{L_2(0,b)}<\varepsilon_0^2.$$

Taking into account these inequalities we get

$$\|v(x,y)-V(x,y)\|<\sqrt{3}\,\varepsilon_0 \tag{4.6}$$

and if we take $\varepsilon_0=\dfrac{\varepsilon}{\sqrt{3}}$, from (4.6) we get validity of the Theorem 4.3.

Thus, the operator A acts from the denser lineal $D_A(\overline{G})$ of the Hilbert space $H(\overline{G})$ to the space $H(\overline{G})$.

Theorem 4.4. *Operator A is positively defined on the lineal $D_A(\overline{G})$.*

Remark 4.1. To show the symmetry of the operator A we use the following two lemmas:

Lemma 4.1. *For an arbitrary function $v(x,y)$ of the lineal $D_A(\overline{G})$ the following identity is valid*

$$\mathcal{T}Av=A\mathcal{T}v.$$

Lemma 4.2. *For two arbitrary functions $v(x,y)$ and $w(x,y)$ of the lineal $D_A(\overline{G})$ we have*

$$\int_{-\xi}^{\xi}\frac{\partial\widetilde{v}(x,y)}{\partial x}\,\widetilde{w}(x,y)\,dx=0,\quad y\in[0,b].$$

The scalar product given by (4.1) could be represented in the form

$$[v,w]=\int_0^b\int_{-\xi}^0\int_{-a}^x v(s,y)\,w(s,y)\,ds\,dx\,dy+\xi\int_0^b\int_{-a}^0 v(s,y)\,w(s,y)\,dx\,dy$$

$$+\int_0^b\int_0^\xi\int_{-x}^0\big(2v(0,y)-v(s,y)\big)\big(2w(0,y)-w(s,y)\big)\,ds\,dx\,dy.$$

In the case of the scalar product (3.8) we have the positively defined operator A, but it is not symmetric.

As A is positive definite operator defined on the lineal $D_A(G)$ which is dense in the space $H(\overline{G})$, for the problem (2.1) we can use the standart way of the variational formulation [27].

Let us introduce the new scalar product on $D_A(\overline{G})$

$$\big[v, w\big]_A = [Av, w] = \int\limits_0^b \int\limits_{-\xi}^{\xi} \int\limits_{-a}^{x} \bigg(\frac{\partial \widetilde{v}(s,y)}{\partial s} \, \frac{\partial \widetilde{w}(s,y)}{\partial s}$$

$$+ \frac{\partial \widetilde{v}(s,y)}{\partial y} \, \frac{\partial \widetilde{w}(s,y)}{\partial y} \bigg) \, ds \, dx \, dy.$$

Denote by $\big\| \cdot \big\|_A$ the corresponding norm and by $\rho_A(v,w)$ the corresponding metric. By $H_A(\overline{G})$ we denote the energetic space obtained after completion of $D_A(\overline{G})$ by the metric $\rho_A(v,w)$.

Theorem 4.5. *The function $v(x,y) \in H(\overline{G})$ belongs to the space $H_A(\overline{G})$ if and only if the following relations are fulfilled:*

$$v(x,y) \in H^1(G), \quad v(0,y) \in \overset{\circ}{H}{}^1(0,b),$$

$$v(x,y)\big|_{\Gamma} = 0, \quad v(x,y)\big|_{\Gamma_{-\xi}} = v(x,y)\big|_{\Gamma_0} = v(0,y).$$

Thus, for the functions of the space $H_A(\overline{G})$ the boundary value conditions are conserved. For every function $f(x,y) \in H(\overline{G})$ there exists a unique function $u(x,y)$ in the space $H_A(\overline{G})$, which minimizes the quadratic functional

$$F(v) = \big\| v \big\|_A^2 - 2\,[f, v].$$

For any function $v(x,y) \in H_A(\overline{G})$ the following relation is fulfilled

$$[u, v]_A = [f, v].$$

If the function $u(x,y)$ is sufficiently smooth then $u(x,y)$ is a solution in a classical sense of problem (2.1).

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