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ON NONLINEAR INTEGRO-DIFFERENTIAL EQUATIONS BASED ON MAXWELL'S SYSTEM

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Abstract. Large time behavior of solutions and finite difference schemes of nonlinear integro-differential equations associated with the penetration of a magnetic field into a substance are studied. Two types of integro-differential equations are considered. Two initial-boundary value problems are investigated for each equation. The first with homogeneous conditions on whole boundary and the second with nonhomogeneous boundary data on one side of lateral boundary. The rates of convergence to steady-state solutions are established too. The convergence properties of the corresponding finite difference schemes are also given.

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1. Introduction

The nonlinear integro-differential equations and their systems describe various processes in physics, economics, chemistry, technology and so on. One type of integro-differential equation arises for mathematical modeling of the process of penetrating of magnetic field into a substance. In a quasistationary case the corresponding system of Maxwell's equations has the form [1]:

$$\frac{\partial H}{\partial t} = -rot(\nu_m rot H), \tag{1.1}$$

$$c_\nu \frac{\partial \theta}{\partial t} = \nu_m \left(rot H\right)^2, \tag{1.2}$$

where $H = (H_1, H_2, H_3)$ is a vector of the magnetic field, θ is temperature, c_ν and ν_m are coefficients that characterize physical properties of the substance. System (1.1) defines the process of diffusion of the magnetic field and equation (1.2), change of the temperature at the expense of Joule's heating without taking into account the heat conductivity.

If c_ν and ν_m depend on temperature θ, i.e., $c_\nu = c_\nu(\theta)$, $\nu_m = \nu_m(\theta)$, then the system (1.1), (1.2) can be rewritten in the following form [2, 3]:

$$\frac{\partial H}{\partial t} = -rot\left[a\left(\int_0^t |rotH|^2\, d\tau\right) rotH\right], \qquad (1.3)$$

where function $a = a(S)$ is defined for $S \in [0, \infty)$.

Note that the system (1.3) is complex. Equations and systems of (1.3) type still yield to the investigation for special cases. The model of (1.3) type was intensively studied by many authors and a large amount of literature is devoted to its investigation (see, for example, [2-14] and references therein).

The existence, uniqueness and asymptotic behavior of the solutions of the initial-boundary value problems for the equations of type (1.3) are studied in the works [2-10, 13]. The existence theorems, that are proved in [2-6, 8] are based on Galerkin's method and compactness arguments as in [15, 16] for nonlinear problems.

If the magnetic field has the form $H = (0, 0, U)$ and $U = U(x, t)$, then we have

$$rot(a(S)rotH) = \left(0,\ -\frac{\partial}{\partial x}\left(a(S)\frac{\partial U}{\partial x}\right),\ 0\right),$$

where

$$S(x,t) = \int_0^t \left(\frac{\partial U}{\partial x}\right)^2 d\tau,$$

and from the system (1.3) we obtain the following nonlinear integro-differential equation:

$$\frac{\partial U}{\partial t} = \frac{\partial}{\partial x}\left[a(S)\frac{\partial U}{\partial x}\right], \qquad (1.4)$$

where

$$S(x,t) = \int_0^t \left(\frac{\partial U}{\partial x}\right)^2 d\tau. \qquad (1.5)$$

In [7] some generalization of (1.4), (1.5) is proposed. In particular, assuming the temperature of the considered body to be constant throughout the material, i.e., depending on time, but independent of the space coordinates, the process of penetration of the magnetic field into the material is modeled by averaged integro-differential model, one-dimensional and scalar variant of which has the following form

$$\frac{\partial U}{\partial t} = a(S)\frac{\partial^2 U}{\partial x^2}, \qquad (1.6)$$

where

$$S(t) = \int_0^t \int_0^1 \left(\frac{\partial U}{\partial x}\right)^2 dx d\tau. \qquad (1.7)$$

Our aim is to study long time behavior of solutions of the first boundary value problems for the equations (1.4), (1.5) and (1.6), (1.7) with zero conditions in whole lateral boundary as well as the problem with non zero conditions on one side of lateral boundary. To investigate the corresponding difference schemes and their convergence properties are purpose of this note too.

2. Problem with nonhomogeneous Dirichlet conditions on one side of the lateral boundary

Consider following problem:

$$\frac{\partial U}{\partial t} = \frac{\partial}{\partial x}\left[a(S)\frac{\partial U}{\partial x}\right], \quad (x,t) \in Q = (0,1) \times (0,\infty), \tag{2.1}$$

$$U(0,t) = 0, \quad U(1,t) = \psi, \quad t \geq 0, \tag{2.2}$$

$$U(x,0) = U_0(x), \quad x \in [0,1], \tag{2.3}$$

where

$$S(x,t) = \int_0^t \left(\frac{\partial U}{\partial x}\right)^2 d\tau, \tag{2.4}$$

or

$$S(t) = \int_0^t \int_0^1 \left(\frac{\partial U}{\partial x}\right)^2 dx d\tau, \tag{2.5}$$

$\psi = const > 0$.

In this paper everywhere we assume that $a(S) = (1+S)^p$. Restrictions for the p will be concretized in the statements.

It should be noted that boundary conditions (2.2) are used by taking into account the physical problems considered in [14].

Following statement takes place [9].

Theorem 2.1. *Suppose that* $0 < p \leq 1$*,* $U_0 \in H^2(0,1)$*,* $U_0(0) = 0$*,* $U_0(1) = \psi$*. Then for the solution of problem (2.1)-(2.4) the following asymptotic relations hold as* $t \to \infty$*:*

$$\left|\frac{\partial U(x,t)}{\partial x} - \psi\right| \leq Ct^{-1-p}, \quad \left|\frac{\partial U(x,t)}{\partial t}\right| \leq Ct^{-1},$$

uniformly in x *on* $[0,1]$*.*

Everywhere in this paper we use usual $L_2(0,1)$ inner-product, corresponding norm and Sobolev spaces $H^k(0,1)$ and $H_0^k(0,1)$. As to symbols C, as well as C_i and c, in the sections 2 and 3, denote various positive constants, independent of t.

A series of lemmas is necessary in order to prove Theorem 2.1. We assume that conditions of the Theorem 2.1 hold.

Lemma 2.1. *For the solution of problem (2.1)-(2.4) the following estimate is true:*

$$\int_0^t\int_0^1\left(\frac{\partial U}{\partial\tau}\right)^2dxd\tau\le C.$$

Lemma 2.2. *For the function S following estimates hold:*

$$c\varphi^{\frac{1}{1+2p}}(t)\le 1+S(x,t)\le C\varphi^{\frac{1}{1+2p}}(t),$$

where

$$\varphi(t)=1+\int_0^t\int_0^1\sigma^2(x,t)dxd\tau \tag{2.6}$$

and $\sigma=(1+S)^p\partial U/\partial x$.

Lemma 2.3. *The following inequalities are true:*

$$c\varphi^{\frac{2p}{1+2p}}(t)\le\int_0^1\sigma^2(x,t)dx\le C\varphi^{\frac{2p}{1+2p}}(t).$$

Lemma 2.4. *The derivative* $\partial U/\partial t$ *satisfies the inequality*

$$\int_0^1\left(\frac{\partial U}{\partial t}\right)^2dx\le C\varphi^{-\frac{2}{1+2p}}(t).$$

Lemma 2.5. *For* $\partial S/\partial x$ *the following inequality is true*

$$\int_0^1\left|\frac{\partial S}{\partial x}\right|dx\le C\varphi^{-\frac{p}{1+2p}}(t).$$

It is not difficult to show that when $p>0$ Lemmas 2.2, 2.3 and 2.4 are also true for the solution of problem (2.1)-(2.3), (2.5). From these lemmas, according to the scheme used in [9] we get analogous theorem for problem (2.1)-(2.3), (2.5) (see [10]).

Theorem 2.2. *Suppose that* $p>0$, $U_0\in H^2(0,1)$, $U_0(0)=0$, $U_0(1)=\psi$. *Then for the solution of problem (2.1)-(2.3), (2.5) the following asymptotic relations hold as* $t\to\infty$*:*

$$\left|\frac{\partial U(x,t)}{\partial x}-\psi\right|\le Ct^{-1-p},\quad\left|\frac{\partial U(x,t)}{\partial t}\right|\le Ct^{-1},$$

uniformly in x *on* $[0,1]$.

Note that, to receive results, given in this section, the scheme similar to [17], in which the adiabatic shearing of incompressible fluids with temperature-dependent viscosity is studied, was used.

3. Problem with homogeneous Dirichlet boundary conditions

Consider following initial-boundary value problem:

$$\frac{\partial U}{\partial t} = \frac{\partial}{\partial x}\left[a(S)\frac{\partial U}{\partial x}\right], \quad (x,t) \in Q, \tag{3.1}$$

$$U(0,t) = U(1,t) = 0, \quad t \geq 0, \tag{3.2}$$

$$U(x,0) = U_0(x), \quad x \in [0,1], \tag{3.3}$$

where again

$$S(x,t) = \int_0^t \left(\frac{\partial U}{\partial x}\right)^2 d\tau, \tag{3.4}$$

or

$$S(t) = \int_0^t \int_0^1 \left(\frac{\partial U}{\partial x}\right)^2 dx d\tau. \tag{3.5}$$

It is easy to verify the following statement.

Lemma 3.1. *If $p > 0$, then for the solution of problems (3.1)-(3.4) and (3.1)-(3.3), (3.5) the following estimate is true*

$$\|U\| \leq C\exp(-t).$$

Therefore, Lemma 3.1 gives exponential stabilization of the solution of problems (3.1)-(3.4) and (3.1)-(3.3), (3.5) in the norm of the space $L_2(0,1)$. As it was shown in [12, 18] the stabilization take place in the norm of the space $H^1(0,1)$ as well. In particular, following statement hold.

Theorem 3.1. *Assume that $U_0 \in H^2(0,1) \cap H_0^1(0,1)$. If $0 < p \leq 1$ then for the solution of problem (3.1)-(3.4) and if $p > 0$ then for the solution of problem (3.1)-(3.3), (3.5) the following estimate is true as $t \to \infty$*

$$\left\|\frac{\partial U}{\partial x}\right\| + \left\|\frac{\partial U}{\partial t}\right\| \leq C\exp\left(-\frac{t}{2}\right).$$

Let's strengthen the Theorem 3.1. In particular, let us show that stabilization can be achieved in the stronger norm.

The main result of this section has the following form.

Theorem 3.2. *Suppose that $U_0 \in H^2(0,1) \cap H_0^1(0,1)$. If $0 < p \leq 1$ then for the solution of problem (3.1)-(3.4) and if $p > 0$ then for the solution of problem (3.1)-(3.3), (3.5) the following estimates hold as $t \to \infty$:*

$$\left|\frac{\partial U(x,t)}{\partial x}\right| \leq C\exp\left(-\frac{t}{2}\right), \quad \left|\frac{\partial U(x,t)}{\partial t}\right| \leq C\exp\left(-\frac{t}{2}\right),$$

uniformly in x on $[0,1]$.

Theorem 3.1 helps us to deduce that Lemma 2.2 holds also for the solution of problem (3.1)-(3.4) and (3.1)-(3.3), (3.5). Therefore using this lemma, (2.6) and again Theorem 3.1 we obtain

$$\frac{d\varphi(t)}{dt}=\int_0^1 (1+S)^{2p}\left(\frac{\partial U}{\partial x}\right)^2 dx \le C\varphi^{\frac{2p}{1+2p}}(t)\exp(-t).$$

After integrating this inequality, taking into account (2.6), we arrive at

$$1\le \varphi(t)\le C.$$

From this, keeping in mind Lemma 2.2, we get

$$1\le 1+S(x,t)\le C. \tag{3.6}$$

From (3.6) and Theorem 3.1, by taking into account identity

$$\begin{aligned}\sigma^2(x,t)&=\int_0^1 \sigma^2(y,t)dy+\int_0^1\int_y^x \frac{\partial\sigma^2(\xi,t)}{\partial\xi}d\xi dy\\ &=\int_0^1 \sigma^2(y,t)dy+2\int_0^1\int_y^x \sigma(\xi,t)\frac{\partial U(\xi,t)}{\partial t}d\xi dy,\end{aligned}$$

we get

$$\begin{aligned}\sigma^2(x,t)\le 2\int_0^1 (1+S)^{2p}\left(\frac{\partial U}{\partial x}\right)^2 dx\\ +\int_0^1\left(\frac{\partial U}{\partial t}\right)^2 dx\le C\exp(-t).\end{aligned} \tag{3.7}$$

At last if we remind definition of σ, from (3.7) it will be obvious validity of the first part of the Theorem 3.2.

Now let us estimate derivative $\partial U/\partial t$. For this differentiate equation (3.1) with respect to t

$$\frac{\partial^2 U}{\partial t^2}-\frac{\partial}{\partial x}\left[\frac{\partial(1+S)^p}{\partial t}\frac{\partial U}{\partial x}+(1+S)^p\frac{\partial^2 U}{\partial t\partial x}\right]=0. \tag{3.8}$$

Multiplying (3.8) by $\partial U/\partial t$ and carrying integration by parts we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_0^1\left(\frac{\partial U}{\partial t}\right)^2 dx+\int_0^1 (1+S)^p\left(\frac{\partial^2 U}{\partial t\partial x}\right)^2 dx\\ +p\int_0^1 (1+S)^{p-1}\left(\frac{\partial U}{\partial x}\right)^3\frac{\partial^2 U}{\partial t\partial x}dx=0.\end{aligned} \tag{3.9}$$

Identity (3.9) yields

$$\frac{d}{dt}\int_0^1 \left(\frac{\partial U}{\partial t}\right)^2 dx + \int_0^1 (1+S)^p \left(\frac{\partial^2 U}{\partial t \partial x}\right)^2 dx \\ \leq p^2 \int_0^1 (1+S)^{p-2} \left(\frac{\partial U}{\partial x}\right)^6 dx. \tag{3.10}$$

Let's multiply (3.10) scalarly by $\exp(2t)$ and integrate it on $(0,t)$. Using (3.6), Theorem 3.1 and the first part of the Theorem 3.2, after simple transformations we have

$$\int_0^t \exp(2\tau)\frac{d}{d\tau}\int_0^1 \left(\frac{\partial U}{\partial \tau}\right)^2 dx\,d\tau + \int_0^t \exp(2\tau)\int_0^1 (1+S)^p \left(\frac{\partial^2 U}{\partial x\,\partial \tau}\right)^2 dx\,d\tau$$

$$\leq p^2 \int_0^t \exp(2\tau)\int_0^1 (1+S)^{p-2} \left(\frac{\partial U}{\partial x}\right)^6 dx\,d\tau,$$

$$\int_0^t \exp(2\tau)\int_0^1 \left(\frac{\partial^2 U}{\partial x\,\partial \tau}\right)^2 dx\,d\tau \leq -\exp(2t)\int_0^1 \left(\frac{\partial U}{\partial t}\right)^2 dx + \int_0^1 \left(\frac{\partial U}{\partial t}\right)^2 dx\Bigg|_{t=0}$$

$$+2\int_0^t \exp(2\tau)\int_0^1 \left(\frac{\partial U}{\partial \tau}\right)^2 dx\,d\tau + C\int_0^t \exp(-\tau)\,d\tau$$

or

$$\int_0^t \exp(2\tau)\int_0^1 \left(\frac{\partial^2 U}{\partial x\,\partial \tau}\right)^2 dx\,d\tau \leq C\exp(t). \tag{3.11}$$

Multiplying (3.8) scalarly by $\exp(2t)\partial^2 U/\partial t^2$, using the first part of the Theorem 3.2 and a priori estimates (3.6), (3.11), we get

$$\int_0^t \exp(2\tau)\int_0^1 \left(\frac{\partial^2 U}{\partial \tau^2}\right)^2 dx d\tau + \frac{1}{2}\int_0^t\int_0^1 \exp(2\tau)(1+S)^p \frac{\partial}{\partial \tau}\left(\frac{\partial^2 U}{\partial \tau \partial x}\right)^2 dx d\tau$$

$$+p\int_0^t\int_0^1 \exp(2\tau)(1+S)^{p-1}\left(\frac{\partial U}{\partial x}\right)^3 \frac{\partial}{\partial \tau}\left(\frac{\partial^2 U}{\partial \tau \partial x}\right) dx d\tau = 0,$$

$$\frac{\exp(2t)}{2}\int_0^1 \left(\frac{\partial^2 U}{\partial t \partial x}\right)^2 dx \leq \frac{1}{2}\int_0^1 \left(\frac{\partial^2 U}{\partial t \partial x}\right)^2 dx\Bigg|_{t=0}$$

$$+\int_0^t\int_0^1 \exp(2\tau)(1+S)^p\left(\frac{\partial^2 U}{\partial\tau\partial x}\right)^2 dxd\tau$$

$$+\frac{p}{2}\int_0^t\int_0^1 \exp(2\tau)(1+S)^{p-1}\left(\frac{\partial U}{\partial x}\right)^2\left(\frac{\partial^2 U}{\partial\tau\partial x}\right)^2 dxd\tau$$

$$-\exp(2t)p\int_0^1 (1+S)^{p-1}\left(\frac{\partial U}{\partial x}\right)^3\frac{\partial^2 U}{\partial t\partial x}dx$$

$$+p\int_0^1\left(\frac{\partial U}{\partial x}\right)^3\frac{\partial^2 U}{\partial t\partial x}dx\bigg|_{t=0}+2p\int_0^t\int_0^1 \exp(2\tau)(1+S)^{p-1}\left(\frac{\partial U}{\partial x}\right)^3\frac{\partial^2 U}{\partial\tau\partial x}dxd\tau$$

$$+p(p-1)\int_0^t\int_0^1 \exp(2\tau)(1+S)^{p-2}\left(\frac{\partial U}{\partial x}\right)^5\frac{\partial^2 U}{\partial\tau\partial x}dxd\tau$$

$$+3p\int_0^t\int_0^1 \exp(2\tau)(1+S)^{p-1}\left(\frac{\partial U}{\partial x}\right)^2\left(\frac{\partial^2 U}{\partial\tau\partial x}\right)^2 dxd\tau$$

$$\le C_1+C_2\exp(t)+C_3\int_0^t \exp(2\tau)\exp(-\tau)\int_0^1\left(\frac{\partial^2 U}{\partial\tau\partial x}\right)^2 dxd\tau$$

$$+\frac{\exp(2t)}{4}\int_0^1\left(\frac{\partial^2 U}{\partial t\partial x}\right)^2 dx+C_4\exp(-t)+C_5\int_0^t\exp(2\tau)\int_0^1\left(\frac{\partial^2 U}{\partial\tau\partial x}\right)^2 dxd\tau$$

$$+C_5\int_0^t\exp(-\tau)d\tau+C_6\int_0^t\exp(2\tau)\int_0^1\left(\frac{\partial^2 U}{\partial\tau\partial x}\right)^2 dxd\tau+C_6\int_0^t\exp(-3\tau)d\tau$$

$$+C_7\int_0^t\exp(2\tau)\exp(-\tau)\int_0^1\left(\frac{\partial^2 U}{\partial\tau\partial x}\right)^2 dxd\tau,$$

i.e.,

$$\int_0^1\left(\frac{\partial^2 U}{\partial t\partial x}\right)^2 dx\le C\exp(-t). \tag{3.12}$$

Using Theorem 3.1 from (3.12), taking into account the relation

$$\frac{\partial U(x,t)}{\partial t}=\int_0^1\frac{\partial U(y,t)}{\partial t}dy+\int_0^1\int_y^x\frac{\partial^2 U(\xi,t)}{\partial t\partial\xi}d\xi dy,$$

we prove the second part of the Theorem 3.2.

Results of Theorems 2.1, 2.2, 3.1 and 3.2 show the difference between stabilization character of solutions with homogeneous and nonhomogeneous boundary conditions.

4. Finite difference schemes and numerical solution

Now, assume that $p = 1$ and rewrite systems (2.1), (2.4) and (2.1), (2.5) in the following forms

$$\frac{\partial U}{\partial t} = \frac{\partial}{\partial x}\left\{\left[1 + \int_0^t \left(\frac{\partial U}{\partial x}\right)^2 d\tau\right]\frac{\partial U}{\partial x}\right\} \tag{4.1}$$

and

$$\frac{\partial U}{\partial t} = \left[1 + \int_0^t \int_0^1 \left(\frac{\partial U}{\partial x}\right)^2 dx d\tau\right]\frac{\partial^2 U}{\partial x^2}. \tag{4.2}$$

For the equations (4.1) and (4.2) let us consider the following initial-boundary value problem:

$$U(0,t) = U(1,t) = 0, \quad t \geq 0, \tag{4.3}$$

$$U(x,0) = U_0(x), \quad x \in [0,1]. \tag{4.4}$$

On $[0,1] \times [0,T]$ let us introduce a net with mesh points denoted by $(x_i, t_j) = (ih, j\tau)$, where $i = 0, 1, ..., M$; $j = 0, 1, ..., N$, with $h = 1/M$, $\tau = T/N$. The initial line is denoted by $j = 0$. The discrete approximation at (x_i, t_j) is designed by u_i^j and the exact solution to problems (4.1), (4.3), (4.4) and (4.2)-(4.4) by U_i^j. We will use the following known notations:

$$r_{t,i}^j = \frac{r_i^{j+1} - r_i^j}{\tau}, \quad r_{\bar{t},i}^j = r_{t,i}^{j-1} = \frac{r_i^j - r_i^{j-1}}{\tau}.$$

For problem (4.1), (4.3), (4.4) let us consider the finite difference scheme:

$$\begin{gathered}\frac{u_i^{j+1} - u_i^j}{\tau} - \left\{\left[1 + \tau\sum_{k=1}^{j+1}(u_{\bar{x},i}^k)^2\right] u_{\bar{x},i}^{j+1}\right\}_x = 0,\\ i = 1, 2, ..., M-1; \quad j = 0, 1, ..., N-1,\\ u_0^j = u_M^j = 0, \quad j = 0, 1, ..., N,\\ u_i^0 = U_{0,i}, \quad i = 0, 1, ..., M,\end{gathered} \tag{4.5}$$

and the corresponding scheme for averaged problem (4.2)-(4.4):

$$\begin{gathered}\frac{u_i^{j+1}-u_i^j}{\tau}-\left[1+\tau h\sum_{i=1}^{M}\sum_{k=1}^{j+1}(u_{\bar{x},i}^k)^2\right]u_{\bar{x}x,i}^{j+1}=0,\\ i=1,2,...,M-1;\ \ j=0,1,...,N-1,\\ u_0^j=u_M^j=0,\ \ j=0,1,...,N,\\ u_i^0=U_{0,i},\ \ i=0,1,...,M.\end{gathered}\tag{4.6}$$

Theorem 4.1. *If problems (4.1), (4.3), (4.4) and (4.2)-(4.4) have sufficiently smooth solution $U=U(x,t)$, then the solutions $u^j=(u_1^j,u_2^j,\dots,u_{M-1}^j)$, $j=1,2,\dots,N$ of the difference schemes (4.5) and (4.6) tend to the solutions of continuous problems $U^j=(U_1^j,U_2^j,\dots,U_{M-1}^j)$, $j=1,2,\dots,N$ as $\tau\to0$, $h\to0$ and the following estimate is true*

$$\|u^j-U^j\|_h\le C(\tau+h),\quad j=1,2,\dots,N.$$

Note that, in Theorem 4.1 the positive constant C is independent of h and τ.

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