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ON SOME CLASSES OF SPECIAL FUNCTIONS

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Abstract. The present paper deals with some classes of special functions which play a crucial part in investigation of weighted boundary value problems for the degenerate elliptic Euler-Poisson-Darboux equation and iterated one.

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1. Introduction

The present paper is devoted to a class of special functions represented in the following integral form

$$M_k(a,b,j,m) := y^{b+m-k-1} \int_{-\infty}^{+\infty} (\xi - x)^k \frac{\partial^m \theta^j e^{a\theta} \rho^{-b}}{\partial y^m} d\xi, \tag{1}$$

where

$$\theta = \arg(z-\xi), \quad \rho = |z-\xi|^{\frac{1}{2}}, \quad j,k,m \in \mathbb{N}^0, \quad z \in \mathbb{R}^2_+, \quad \xi \in \mathbb{R}^1,$$

a and b are complex constants, $\mathbb{R}^2_+$ is an upper half-plane of the complex plane of the variable $z = x + iy$, $\mathbb{R}^1$ is the axis of the real numbers,

$$\theta \in [0,\pi], \quad \mathbb{N}^0 := \mathbb{N} \cup \{0\},$$

$\mathbb{N}$ is the set of the natural numbers. $\mathbb{N}_1$ and $\mathbb{N}_2$ denote the sets of the odd and even natural numbers, respectively. $\mathbb{N}^0_2 := \mathbb{N} \cup \{0\}$.

The above class of special functions plays a crucial part in investigation of weighted boundary value problems for the degenerate elliptic Euler-Poisson-Darboux equation [1,2]

$$E^{(a,b)}u := y\,(u_{xx} + u_{yy}) + au_x + bu_y = 0$$

and iterated one [1,2]

$$\left(\prod_{k=0}^{n-1} E^{(a_k,b)}\right) u = 0,$$

where b, a_k, $k = 0, 1, \ldots, n-1$, are, in general, complex constants. When $y = 0$, the above equations have an order degeneration.

2. Main Theorem

Theorem 1. *The function* $M_k(a,b,j,m)$ *is defined [i.e., the integral (1) exists] and is independent of* x, y:
when

$$Re\, b+m-k-1>0 \tag{2}$$

and either $a\neq 0$, $m\in\mathbb{N}^0$, *or* $a=0$, $j\neq 0$, $m\in\mathbb{N}^0$, *or* $a=j=m=0$, *or* $a=j=0$, $b\neq 0,-2,\ldots,-2\left(m-\left[\frac{m}{2}\right]-1\right)$, $m\in\mathbb{N}_2$;
or when

$$Re\, b+m-k>0 \tag{3}$$

and $a=j=0$, $b\neq 0,-2,\ldots,-2\left(m-\left[\frac{m}{2}\right]-1\right)$, $m\in\mathbb{N}_1$.

If $a=j=0$, *and either* $b\in\left\{0,-2,\ldots,-2\left(m-\left[\frac{m}{2}\right]-1\right)\right\}$, $m\in\mathbb{N}$, *or condition (3) is fulfilled when* $m,k\in\mathbb{N}_1$, *or (2) is fulfilled when* $k\in\mathbb{N}_1$, $m\in\mathbb{N}_2^0$, *then*

$$M_k(0,b,0,m)=0. \tag{4}$$

Proof. Using the method of mathematical induction, we prove that

$$\frac{\partial^m e^{a\theta}\rho^{-b}}{\partial y^m}=\sum_{\kappa=1}^{\left[\frac{m}{2}\right]+1} B_\kappa\left(b,m;a(x-\xi),y\right)e^{a\theta}\rho^{-b-2(m-\kappa+1)}, \tag{5}$$

where

$$B_1\left(b,m;a(x-\xi),y\right)=\prod_{i=1}^{m}\left\{a(x-\xi)-\left[b+2(l-1)\right]y\right\}, \tag{6}$$

$$\begin{aligned} &B_\kappa\left(b,m;a(x-\xi),y\right)\\ &\quad=\sum_{\alpha_{\kappa-1}=2\kappa-3}^{m-1}\left(\prod_{j=1}^{\kappa-2}\sum_{\alpha_j=2j-1}^{\alpha_{j+1}-2}\right)\left\{\prod_{k=1}^{\kappa-1}\left[b+2\left(\alpha_k-k\right)\right]\left(\alpha_k-m\right)\right.\\ &\quad\left.\times\prod_{\substack{l=1\\ l\neq\alpha_i-i+1\\ i=1,2,\ldots,\kappa-1}}^{m-\kappa+1}\left\{a(x-\xi)-\left[b+2(l-1)\right]y\right\}\right\},\\ &\qquad\qquad \kappa=2,\ldots,\left[\frac{m}{2}\right]+1, \end{aligned} \tag{7}$$

$$\prod_{j=1}^{\kappa-2}\sum_{\alpha_j=2j-1}^{\alpha_{j+1}-2}:=\sum_{\alpha_{\kappa-2}=2\kappa-5}^{\alpha_{\kappa-1}-2}\ \sum_{\alpha_{\kappa-3}=2\kappa-7}^{\alpha_{\kappa-2}-2}\cdots\sum_{\alpha_2=3}^{\alpha_3-2}\sum_{\alpha_1=1}^{\alpha_2-2},\ \prod_{j=l}^{l-1}(\cdot)\equiv 1. \tag{8}$$

The last product in (7) we take equal to 1 if none of l are admissible.

Indeed, it is easy to see that (5) is true for $m=1,\ldots,6$. Now assuming that it takes place for $m=n-1$ and $m=n$, we prove its validity for

$m = n+1$. Evidently,

$$\frac{\partial^{n+1} e^{a\theta}\rho^{-b}}{\partial y^{n+1}} = \frac{\partial^{n}\left[a(x-\xi)-by\right] e^{a\theta}\rho^{-b-2}}{\partial y^{n}}$$

$$\left[a(x-\xi)-by\right]\frac{\partial^{n} e^{a\theta}\rho^{-b-2}}{\partial y^{n}} - nb\frac{\partial^{n+1} e^{a\theta}\rho^{-b-2}}{\partial y^{n+1}}$$

$$= \sum_{\kappa=1}^{\left[\frac{n}{2}\right]+1}\left[a(x-\xi)-by\right] B_\kappa\left(b+2,n;a(x-\xi),y\right) e^{a\theta}\rho^{-b-2-2(n-\kappa+1)}$$

$$- \sum_{\kappa=1}^{\left[\frac{n-1}{2}\right]+1} nbB_\kappa\left(b+2,n-1;a(x-\xi),y\right) e^{a\theta}\rho^{-b-2-2(n-\kappa)}$$

$$= \sum_{\kappa=1}^{\left[\frac{n}{2}\right]+1}\left[a(x-\xi)-by\right] B_\kappa\left(b+2,n;a(x-\xi),y\right) e^{a\theta}\rho^{-b-2-2(n-\kappa+1)}$$

$$- \sum_{\kappa=2}^{\left[\frac{n+1}{2}\right]+1} nbB_{\kappa-1}\left(b+2,n-1;a(x-\xi),y\right) e^{a\theta}\rho^{-b-2-2(n-\kappa+1)}$$

$$= \sum_{\kappa=2}^{\left[\frac{n}{2}\right]+1}\left\{\left[a(x-\xi)-by\right] B_\kappa\left(b+2,n;a(x-\xi),y\right)\right.$$

$$\left.-nbB_{\kappa-1}\left(b+2,n-1;a(x-\xi),y\right) e^{a\theta}\rho^{-b-2-2(n-\kappa+2)}\right.$$

$$+\left[a(x-\xi)-by\right] B_1\left(b+2,n;a(x-\xi),y\right) e^{a\theta}\rho^{-b-2(n+1)}$$

$$+\begin{cases} 0 \ \text{ for } \ n\in\mathbb{N}_2 \\ -nbB_{\frac{n+1}{2}}\left(b+2,n-1;a(x-\xi),y\right) e^{a\theta}\rho^{-b-n-1}, \ \text{ for } n\in\mathbb{N}_1. \end{cases} \tag{9}$$

By virtue of (6),

$$\left[a(x-\xi)-by\right] B_1\left(b+2,n;a(x-\xi),y\right)$$

$$= \left[a(x-\xi)-by\right]\prod_{l=0}^{n}\left\{a(x-\xi)-\left[b+2+2(l-1)\right]y\right\}$$

$$= \prod_{l=0}^{n}\left[a(x-\xi)-(b+2l)y\right] = \prod_{l=0}^{n+1}\left\{a(x-\xi)-\left[b+2(l-1)\right]y\right\}$$

$$= B_1\left(b,n+1;a(x-\xi),y\right). \tag{10}$$

In view of (7), (8), for $n\in\mathbb{N}_1$ we have

$$-nbB_{\frac{n+1}{2}}\left(b+2,n-1;a(x-\xi),y\right)$$

$$= -nb\sum_{\alpha_{\frac{n+1}{2}-1}=n-2}^{n-2}\ \sum_{\alpha_{\frac{n+1}{2}-2}=n-4}^{n-4}\cdots\sum_{\alpha_2=3}^{3}\sum_{\alpha_2=1}^{1}\left\{\prod_{k=1}^{\frac{n+1}{2}-1}\left[b+2\right.\right.$$

$$+2\left(\alpha_k-k\right)\right]\left(\alpha_k-n+1\right) \prod_{\substack{l=1\\ l\neq\alpha_i-i+1\\ i=1,2,\ldots,\frac{n+1}{2}-1}}^{\frac{n-1}{2}} \left\{a(x-\xi)-\left[b+2(l-1)\right]y\right\}\Bigg\}$$

$$=(-b)n(-b-2)(n-2)(-b-4)(n-4)\cdots(-b-n+1)$$

$$=(-1)^{\frac{n+1}{2}}n!!\prod_{k=1}^{\frac{n+1}{2}}\left[b+2(k-1)\right]. \tag{11}$$

On the other hand, because of (7) for $m=n+1\in\mathbb{N}_2$ we have

$$B_{\frac{n+1}{2}+1}\left(b,n+1;a(x-\xi),y\right)$$

$$=\sum_{\alpha_{\frac{n+1}{2}}=n}^{n}\left(\prod_{j=1}^{\frac{n+1}{2}-1}\sum_{\alpha_j=2j-1}^{\alpha_{j+1}-2}\right)\left\{\prod_{k=1}^{\frac{n+1}{2}}\left[b+2\left(\alpha_k-k\right)\right]\left(\alpha_k-n-1\right)\right.$$

$$\left.\times\prod_{\substack{l=1\\ l\neq\alpha_i-i+1\\ i=1,2,\ldots,\frac{n+1}{2}}}^{\frac{n+1}{2}}\left\{a(x-\xi)-\left[b+2(l-1)\right]y\right\}\right\}$$

$$=(-1)^{\frac{n+1}{2}}n!!\prod_{k=1}^{\frac{n+1}{2}}\left[b+2(k-1)\right], \tag{12}$$

since

$$\alpha_{\frac{n+1}{2}-1}=n-2,\ldots,\ \alpha_2=3,\ \alpha_1=1.$$

From the equality of the right hand sides of (11) and (12) there follows the equality of the left hand sides

$$-nbB_{\frac{n+1}{2}}\left(b+2,n-1;a(x-\xi),y\right)=B_{\frac{n+1}{2}+1}\left(b,n+1;a(x-\xi),y\right). \tag{13}$$

According to (6) we have

$$\left[a(x-\xi)-by\right]B_\kappa\left(b+2,n;a(x-\xi),y\right)-nbB_{\kappa-1}\left(b+2,n-1;a(x-\xi),y\right)$$

$$=\left[a(x-\xi)-by\right]\sum_{\alpha_{\kappa-1}=2\kappa-3}^{n-1}\left(\prod_{j=1}^{\kappa-2}\sum_{\alpha_j=2j-1}^{\alpha_{j+1}-2}\right)\left\{\prod_{k=1}^{\kappa-1}\left[b+2+2\left(\alpha_k-k\right)\right]\left(a_k-n\right)\right.$$

$$\left.\times\prod_{\substack{l=1\\ l\neq\alpha_i-i+1\\ i=1,2,\ldots,\kappa-1}}^{n-\kappa-1}\left\{a(x-\xi)-\left[b+2+2(l-1)\right]y\right\}\right\}$$

$$-nb\sum_{\alpha_{k-2}=2\kappa-5}^{n-2}\left(\prod_{j=1}^{\kappa-3}\sum_{\alpha_j=2j-1}^{\alpha_{j+1}-2}\right)\left\{\prod_{k=1}^{\kappa-2}\left[b+2+2\left(\alpha_k-k\right)\right]\left(a_k-n+1\right)\right.$$

$$\left.\times\prod_{\substack{l=1\\ l\neq\alpha_i-i+1\\ i=1,2,\ldots,\kappa-2}}^{n-\kappa+1}\left\{a(x-\xi)-[b+2+2(l-1)]y\right\}\right\},\tag{14}$$

$$\kappa=2,3,\ldots,\left[\frac{n}{2}\right]+1$$

(we assume that $\prod_{\substack{l=p\\ l\neq\alpha_i-i+1\\ i=k,k+1,\ldots,m}}^{q}(\cdot)\equiv\prod_{l=p}^{q}(\cdot)$ for $m<k$ and $\sum\left(\prod_{j=l}^{l-2}(\cdot)\right)\{\cdot\}\equiv\{\cdot\}$).

It is easy to see that

$$\begin{aligned}&\prod_{k=1}^{\kappa-1}\left[b+2+2\left(\alpha_k-k\right)\right]\left(a_k-n\right)\\&=\prod_{k=1}^{\kappa-1}\left[b+2\left(\alpha'_k-k\right)\right]\left(\alpha'_k-n-1\right),\quad \alpha'_k=\alpha_k+1;\end{aligned}\tag{15}$$

$$\begin{aligned}&[a(x-\xi)-by]\prod_{\substack{l=1\\ l\neq\alpha_i-i+1\\ i=1,2,\ldots,\kappa-1}}^{n-\kappa+1}\left\{a(x-\xi)-[b+2+2(l-1)]\,y\right\}\\&=[a(x-\xi)-by]\prod_{\substack{l=2\\ l\neq\alpha_i-i+2\\ i=1,2,\ldots,\kappa-1}}^{n+1-\kappa+1}\left\{a(x-\xi)-[b+2(l-1)]\,y\right\}\\&=\prod_{\substack{l=1\\ l\neq\alpha_i-i+1\\ i=1,2,\ldots,\kappa-1}}^{n+1-\kappa+1}\left\{a(x-\xi)-[b+2(l-1)]\,y\right\}\end{aligned}\tag{16}$$

$$\begin{aligned}&-nb\prod_{k=1}^{\kappa-2}\left[b+2+2\left(\alpha_k-k\right)\right]\left(\alpha_k-n+1\right)\\&=-nb\prod_{k=2}^{\kappa-1}\left[b+2+2\left(\alpha_{k-1}-k+1\right)\right]\left(\alpha_{k-1}-n+1\right)\\&=-nb\prod_{k=2}^{\kappa-1}\left[b+2\left(\alpha''_k-k\right)\right]\left(\alpha''_k-n-1\right),\quad \alpha''_{k+1}=\alpha_k+2;\end{aligned}\tag{17}$$

$$\prod_{\substack{l=1\\ l\neq\alpha_i-i+1\\ i=1,2,\ldots,\kappa-2}}^{n-\kappa+1}\{a(x-\xi)-[b+2+2(l-1)]\,y\}$$

$$=\prod_{\substack{l=2\\ l\neq\alpha_i-i+2\\ i=1,2,\ldots,\kappa-2}}^{n+1-\kappa+1}\{a(x-\xi)-[b+2(l-1)]\,y\}$$

$$=\prod_{\substack{l=2\\ l\neq\alpha^{"}_{i+1}-1\\ i=1,2,\ldots,\kappa-2}}^{n+1-\kappa+1}\{a(x-\xi)-[b+2(l-1)]\,y\}$$

$$=\prod_{\substack{l=2\\ l\neq\alpha^{"}_{i+1}-1\\ i=2,\ldots,\kappa-1}}^{n+1-k+1}\{a(x-\xi)-[b+2(l-1)]\,y\}\,. \tag{18}$$

Substituting (15)-(18) in (14), we get

$$[a(x-\xi)-by]\,B_\kappa\,(b+2,n;a(x-\xi),y)-nbB_{\kappa-1}\,(b+2,n-1;a(x-\xi),y)$$

$$=\sum_{\alpha'_{\kappa-1}=2\kappa-2}^{n-1}\left(\prod_{j=1}^{\kappa-2}\sum_{\alpha'_j=2j}^{\alpha'_{j+1}-2}\right)\left\{\prod_{k=1}^{\kappa-1}[b+2\,(\alpha'_k-k)]\,(a'_k-n-1)\right.$$

$$\left.\times\prod_{\substack{l=1\\ l\neq\alpha'_i-i+1\\ i=1,2,\ldots,\kappa-1}}^{n+1-\kappa+1}\{a(x-\xi)-[b+2(l-1)]y\}\right\}$$

$$+\sum_{\alpha''_{\kappa-2}=2\kappa-3}^{n}\left(\prod_{j=1}^{\kappa-3}\sum_{\alpha''_{j+1}=2j+1}^{\alpha''_{\kappa+2}-2}\right)\left\{-nb\prod_{k=2}^{\kappa-1}\left[b+2\left(\alpha''_k-k\right)\right]\left(a''_k-n-1\right)\right.$$

$$\left.\times\prod_{\substack{l=2\\ l\neq\alpha''_i-i+1\\ i=2,3,\ldots,\kappa-1}}^{n+1-\kappa+1}\{a(x-\xi)-[b+2(l-1)]y\}\right\}, \tag{19}$$

$$\kappa=2,3,\ldots,\left[\frac{n}{2}\right]+1$$

On the other hand, by virtue of (7), if we separate the sum corresponding to $\alpha_1 = 1$, then for $m = m+1$ we have

$$B_\kappa\left(b, n+1; a(x-\xi), y\right) = \sum_{\alpha_{\kappa-1}=2\kappa-2}^{n} \left(\prod_{j=1}^{\kappa-2} \sum_{\alpha_j=2j}^{\alpha_{j+1}-2}\right) \left\{\prod_{k=1}^{\kappa-1} \left[b+2\left(\alpha_k-k\right)\right]\left(\alpha_k-n-1\right) \right.$$

$$\left. \times \prod_{\substack{l=1 \\ l\neq\alpha_i'-i+1 \\ i=1,2,\ldots,\kappa-1}}^{n+1-\kappa+1} \left\{a(x-\xi)-\left[b+2(l-1)\right]y\right\}\right\}$$

$$+ \sum_{\alpha_{\kappa-1}=2\kappa-3}^{n} \left(\prod_{j=2}^{\kappa-2} \sum_{\alpha_j=2j-1}^{\alpha_{j+1}-2}\right) \left\{-nb\prod_{k=2}^{\kappa-1} \left[b+2\left(\alpha_k-k\right)\right]\left(\alpha_k-n-1\right) \right.$$

$$\left. \times \prod_{\substack{l=2 \\ l\neq\alpha_i-i+1 \\ i=2,3,\ldots,\kappa-1}}^{n+1-\kappa+1} \left\{a(x-\xi)-\left[b+2(l-1)\right]y\right\}\right\}, \tag{20}$$

$$\kappa = 2, 3, \ldots, \left[\frac{n+1}{2}\right]+1,$$

since in the first group of sums none of equalities

$$\alpha_j = 2j-1, \quad j = 2, 3, \ldots, \kappa-1,$$

are possible, otherwise we would obtain that $\alpha_1 = 1$ but such terms we have separated in the second group. Let us note that the last product in (20) begins from $l = 2$ because of $l \neq \alpha_1 - 1 + 1 = \alpha_1 = 1$.

If we compare (19) (where α_j' and $\alpha_j^{"}$ we can denote by α_j) and (20) and take into account that

$$\prod_{j=2}^{\kappa-2} \sum_{\alpha_j=2j-1}^{\alpha_{j+1}-2} \equiv \prod_{j=1}^{\kappa-3} \sum_{\alpha_{j+1}=2j+1}^{\alpha_{j+1}-2}, \quad \kappa \geq 4,$$

from the equality of the right-hand sides there follows the equality of the left-hand sides

$$\begin{aligned} &\left[a(x-\xi)-by\right] B_\kappa\left(b+2, n; a(x-\xi), y\right) \\ &\qquad -nbB_{\kappa-1}\left(b+2, n-1; a(x-\xi), y\right) \\ &\qquad = B_\kappa\left(b, n+1; a(x-\xi), y\right), \quad \kappa = 2, 3, \ldots, \left[\frac{n}{2}\right]+1. \end{aligned} \tag{21}$$

Substituting (10), (13), and (21) in (9), we get

$$\frac{\partial^{n+1} e^{a\theta}\rho^{-b}}{\partial y^{n+1}} - \sum_{\kappa=1}^{\left[\frac{n+1}{2}\right]+1} B_\kappa\left(b, n+1; a(x-\xi), y\right) e^{a\theta}\rho^{-b-2(n-\kappa+2)}.$$

So, equality (5) is proved.

It is well-known (s. [3], pp. 235-236), that

$$\frac{d^m \arctan\ \tau}{d\tau^m} = (-1)^{m-1}(m-1)\left(1-\tau^2\right)^{-\frac{m}{2}} \sin\left(m\ \arctan\frac{1}{\tau}\right), \quad \tau \neq 0. \tag{22}$$

But since,

$$\operatorname{arccot}\frac{1}{\tau} = \begin{cases} \arctan\ \tau & \text{for} \quad \tau > 0; \\ \arctan\ \tau + \pi & \text{for} \quad \tau < 0, \end{cases}$$

we have

$$\frac{d^m \operatorname{arccot}\frac{1}{\tau}}{d\tau^m} = \frac{d^m \arctan\ \tau}{d\tau^m}, \quad \tau \neq 0.$$

Introducing y by the relation

$$\tau = \frac{y}{x-\xi}, \quad y > 0, \quad x \neq \xi,$$

where $x, \xi \in \mathbb{R}^1$ are parameters, in view of (22), we get

$$\begin{aligned} \frac{\partial^m \theta}{\partial y^m} &= (x-\xi)^{-m} \left.\frac{\partial^m \theta}{\partial \tau^m}\right|_{\tau=\frac{y}{x-\xi}} = (x-\xi)^{-m} \left.\frac{\partial^m \arctan\ \tau}{\partial \tau^m}\right|_{\tau=\frac{y}{x-\xi}} \\ &= (-1)^{m-1}(m-1)!(x-\xi)^{-m}\left[1+\frac{y^2}{(x-\xi)^2}\right]^{-\frac{m}{2}} \sin\left(m\ \arctan\frac{x-\xi}{y}\right) \\ &= (-1)^{m-1}(m-1)!\rho^{-m}\left[\operatorname{sign}(x-\xi)\right]^{-m} \sin\left(m\ \arctan\frac{x-\xi}{y}\right). \end{aligned} \tag{23}$$

Using (23), by means of the mathematical induction with respect to j we can prove that

$$\begin{aligned} \frac{\partial^m \theta^j}{\partial y^m} &= (-1)^{m-j}\left[\operatorname{sign}(x-\xi)\right]^{-m} \rho^{-m} \sum_{\kappa_j=0}^{m} \left(\prod_{k=1}^{j-2} \sum_{\kappa_{k+1}=0}^{\kappa_{k+2}}\right) \left\{ \binom{m}{\kappa_j} \right. \\ &\times \prod_{k=1}^{j-2} \binom{\kappa_{k+2}}{\kappa_{k+1}} (\kappa_2-1)!\,(m-\kappa_j-1)! \\ &\times \prod_{k=1}^{j-2} (\kappa_{k+2}-\kappa_{k+1}-1)! \sin\left(\kappa_2 \arctan\frac{x-\xi}{y}\right) \sin\left[(m-\kappa_j) \arctan\frac{x-\xi}{y}\right] \\ &\left. \times \prod_{k=1}^{j-2} \sin\left[(\kappa_{k+2}-\kappa_{k+1}) \arctan\frac{x-\xi}{y}\right] \right\}, \quad j \geq 2. \end{aligned} \tag{24}$$

According to the Leibnitz formula,

$$\frac{\partial^m \theta^j e^{a\theta}\rho^{-b}}{\partial y^m} = \sum_{\kappa=0}^{m} \left(\begin{array}{c} m \\ \kappa \end{array} \right) \frac{\partial^\kappa \theta^j}{\partial y^\kappa} \frac{\partial^{m-\kappa} e^{a\theta}\rho^{-b}}{\partial y^{m-\kappa}} \tag{25}$$

By virtue of (5)-(7), and (24), it is easy to see that for a fixed z belonging to the closure of an arbitrary bounded domain lying inside $\mathbb{R}^2_+$, we have

$$\frac{\partial^m e^{a\theta}\rho^{-b}}{\partial y^m} = O\left(|x-\xi|^{-\mathrm{Re}b-m}\right), \quad |\xi| \to +\infty, \quad a \neq 0, \tag{26}$$

$$\frac{\partial^m \rho^{-b}}{\partial y^m} = \begin{cases} O\left(|x-\xi|^{-\mathrm{Re}b-2\left(m-\left[\frac{m}{2}\right]\right)}\right), & |\xi| \to +\infty, \\ 0, b \in \left\{0,-2,\ldots,-2\left(m-\left[\dfrac{m}{2}\right]-1\right)\right\}, & m \in \mathbb{N}, \end{cases} \tag{27}$$

$$\frac{\partial^m \theta^j}{\partial y^m} = O\left(|x-\xi|^{-m}\right), \quad |\xi| \to +\infty, \quad j \in \mathbb{N}. \tag{28}$$

After substitution $\xi = x + yt$ formulas (26)-(28) we can rewrite in the following forms

$$\left.\frac{\partial^m e^{a\theta}\rho^{-b}}{\partial y^m}\right|_{\xi=x+yt} = O\left(t|^{-\mathrm{Re}b-m}\right), \quad |t| \to +\infty, \quad a \neq 0; \tag{29}$$

$$\left.\frac{\partial^m \rho^{-b}}{\partial y^m}\right|_{\xi=x+yt} = O\left(|t|^{-\mathrm{Re}b-2\left(m-\left[\frac{m}{2}\right]\right)}\right), \quad |t| \to +\infty; \tag{30}$$

$$\left.\frac{\partial^m \theta^j}{\partial y^m}\right|_{\xi=x+yt} = O\left(|t|^{-m}\right), \quad |t| \to +\infty, \quad j \in \mathbb{N}. \tag{31}$$

In view of (26)-(31), in the above mentioned domain from (25) we get

$$\frac{\partial^m \theta^j e^{a\theta}\rho^{-b}}{\partial y^m}$$

$$= \begin{cases} O\left(|x-\xi|^{-\mathrm{Re}b-m}\right) = O\left(|t|^{-\mathrm{Re}b-m}\right), \quad |\xi|, |t| \to +\infty, \\ \textit{when either } a \neq 0,\ m \in \mathbb{N}^0, \ \textit{ or } a=0,\ j\neq 0,\ m\in\mathbb{N}^0, \\ \textit{or } a=j=m=0, \ \textit{or } a=j=0,\ b\neq 0,-2,\ldots,-2\left(m-\left[\dfrac{m}{2}\right]-1\right), \\ \qquad m \in \mathbb{N}_2; \\ O\left(|x-\xi|^{-\mathrm{Re}b-m-1}\right) = O\left(|t|^{-\mathrm{Re}b-m-1}\right), \quad |\xi|, |t| \to +\infty, \\ \textit{when } a=j=0,\ b\neq 0,-2,\ldots,-2\left(m-\left[\dfrac{m}{2}\right]-1\right),\ m\in\mathbb{N}_1; \\ 0, \ \textit{when } a=j=0,\ b\in\left\{0,-2,\ldots,-2\left(m-\left[\dfrac{m}{2}\right]-1\right)\right\},\ m\in\mathbb{N}. \end{cases} \tag{32}$$

Indeed, the cases $a \neq 0$ and $a = j = m = 0$ are obvious. In the cases $a = j = 0$ $m \in \mathbb{N}$ and $a = 0$, $j \neq 0$, $m \in \mathbb{N}_0$ we have to take into account

$$2\left(m-\left[\frac{m}{2}\right]\right)=\begin{cases} 2\left(m-\dfrac{m}{2}\right)=m, & m\in\mathbb{N}_2;\\ 2\left(m-\dfrac{m-1}{2}\right)=m+1, & m\in\mathbb{N}_1,\end{cases}$$

and

$$\left|\frac{\partial^m\theta^j\rho^{-b}}{\partial y^m}\right|=\left|\sum_{\kappa=0}^{m}\begin{pmatrix} m\\ \kappa\end{pmatrix}\frac{\partial^\kappa\theta^j}{\partial y^\kappa}\cdot\frac{\partial^{m-\kappa}\rho^{-b}}{\partial y^{m-\kappa}}\right|\leq\sum_{\kappa=0}^{m}C_\kappa|x-\xi|^{-\mathrm{Re}b-m-\left(m-\kappa-2\left[\frac{m-\kappa}{2}\right]\right)}$$

$$=\sum_{\kappa=0}^{m}C_\kappa\begin{cases}|x-\xi|^{-\mathrm{Re}b-m} & for\ \ m-\kappa\in\mathbb{N}_2;\\ |x-\xi|^{-\mathrm{Re}b-m-1} & for\ \ m-\kappa\in\mathbb{N}_1,\end{cases}$$

$$\leq C|x-\xi|^{-\mathrm{Re}b-m},\quad |\xi|\to+\infty,\quad C,C_\kappa=\mathrm{const},$$

respectively.

If

$$a=j=0,\ \ b\in\left\{0,-2,\dots,-2\left(m-\left[\frac{m}{2}\right]-1\right)\right\}\quad m\in\mathbb{N},$$

then

$$\frac{\partial^m\theta^j e^{a\theta}\rho^{-b}}{\partial y^m}=\frac{\partial^m\rho^{-b}}{\partial y^m}=0, \tag{33}$$

because of

$$m>0,2,\dots,2\left(m-\left[\frac{m}{2}\right]-1\right),$$

since

$$2\left(m-\left[\frac{m}{2}\right]-1\right)=\begin{cases} m-2, & m\in\mathbb{N}_2;\\ m-1, & m\in\mathbb{N}_1.\end{cases} \tag{34}$$

From (34) it is easy to see, that

$$b\in\left\{0,-2,\dots,-2\left(m-\left[\frac{m}{2}\right]-1\right)\right\},\ m\in\mathbb{N},$$

can be rewritten as

$$b=-2n,\quad n=\begin{cases} 0,1,\dots,\dfrac{m-2}{2}, & m\in\mathbb{N}_2,\\ 0,1,\dots,\dfrac{m-1}{2}, & m\in\mathbb{N}_1.\end{cases}$$

From (27) there follows (4).

By virtue of (5)-(7) and (23), (24), we have

$$\left.\frac{\partial^m e^{a\theta}\rho^{-b}}{\partial y^m}\right|_{\xi=x+yt}$$
$$=y^{-b-m}\sum_{\kappa=1}^{\left[\frac{m}{2}\right]+1}\tilde{B}_\kappa(b,m;at)e^{a\cdot arc\ cot(-t)}(1+t^2)^{-\frac{b}{2}-m+\kappa-1}, \tag{35}$$

where

$$\tilde{B}_1(b,m;at) = (-1)^m \prod_{l=1}^{m} [at+b+2[l-1)],$$

$$\tilde{B}_\kappa(b,m;at) = (-1)^{m-2\kappa+2} \sum_{\alpha_{\kappa-1}=2\kappa-3}^{m-1} \left(\prod_{j=1}^{\kappa-1} \sum_{\alpha_j=2j-1}^{\alpha_{j+1}-2} \right) \left\{ \prod_{k=1}^{\kappa-1} [b+2(\alpha_k-k)] \right.$$

$$\left. \times (m-\alpha_k) \prod_{\substack{l=1 \\ l\neq\alpha_i-i+1 \\ i=1,2,\ldots,\kappa-1}}^{m-\kappa+1} [at+b+2(l-1)] \right\}, \quad \kappa = 2,3,\ldots,\left[\frac{m}{2}\right]+1;$$

and

$$\left.\frac{\partial^m \theta^j}{\partial y^m}\right|_{\xi=x+yt}$$

$$= \begin{cases} (-1)^{m-j}\,[\mathrm{sign}(-t)]^{-m}\, y^{-m}\,(1+t^2)^{-\frac{m}{2}} \sum\limits_{\kappa_j=0}^{m} \left(\prod\limits_{k=1}^{j-2} \sum\limits_{\kappa_{k+1}=0}^{\kappa_k+2} \right) \left\{ \begin{pmatrix} m \\ \kappa_j \end{pmatrix} \right. \\ \times \prod\limits_{k=1}^{j-2} \begin{pmatrix} \kappa_{k+2} \\ \kappa_{k+1} \end{pmatrix} (\kappa_2-1)!\,(m-\kappa_j-1)! \\ \times \prod\limits_{k=1}^{j-2} (\kappa_{k+2}-\kappa_{k+1}-1)!\sin[\kappa_2 \arctan(-t)] \\ \times \sin[(m-\kappa_j)\arctan(-t)] \\ \left. \times \prod\limits_{k=1}^{j-2} \sin[(\kappa_{k+2}-\kappa_{k+1})\arctan(-t)] \right\}, \quad j \geq 2; \\ (-1)^{m-1}(m-1)!y^{-m}\,(1+t^2)^{-\frac{m}{2}}\,[\mathrm{sign}(-t)]^{-m} \\ \times \sin[m\arctan(-t)], \quad j=1, \end{cases} \tag{36}$$

respectively.

If (3) is fulfilled and $m,k \in N_1$, then

$$M_k(0,b,0,m) = y^{b+m-k-1} \int_{-\infty}^{+\infty} (\xi-x)^k \frac{\partial^m \rho^{-b}}{\partial y^m} d\xi = y^{b+m} \int_{-\infty}^{+\infty} t^k \left.\frac{\partial^m \rho^{-b}}{\partial y^m}\right|_{\xi=x+yt} dt = 0,$$

since the integrand because of (35) is an odd function with respect to t while the integral, in view of (3), is convergent. So, (4) is proved.

After substitution $\xi = x+yt$ the expression (1) will get the following form

$$M_k(a,b,j,m) = y^{b+m} \int_{-\infty}^{+\infty} t^k \left.\frac{\partial^m \theta^j e^{a\theta} \rho^{-b}}{\partial y^m}\right|_{\xi=x+yt} dt.$$

Hence, by virtue of (25), (35), (36) it is evident that the right-hand side of the last equality and therefore, the function $M_k(a,b,j,m)$ is independent of x, y.

Remark 2. In view of (32), if condition (2) is fulfilled, the function $M_k(a,b,j,m)$ is defined. When

$$a=j=0,\ b\neq 0,-2,\dots,-2\left(m-\left[\frac{m}{2}\right]-1\right),\ m\in\mathbb{N}_1 \tag{37}$$

it is defined under the weaker restriction (3).

3. Particular Classes of Special Functions

Let

$$M_k(a,b,m):=M_k(a,b,0,m),\quad M(a,b,m):=M_0(a,b,m), \tag{38}$$

$$\Lambda_k(a,b):=M_k(a,2-b,0),\quad \Lambda(a,b):=\Lambda_0(a,b), \tag{39}$$

$$\overset{*}{\Lambda}(a,b):=M_0(a,2-b,1,0), \tag{40}$$

$$(\alpha,m):=\alpha(\alpha+1)\cdots(\alpha+m-1),\quad m>1;\quad (\alpha,0)\equiv 1.$$

Theorem 3. *Under restrictions of the theorem 1*

$$M_k(a,b,j,m+1)=(k-b-m+1)M_k(a,b,j,m). \tag{41}$$

The following equalities are valid:

$$M_k(a,b,j,m)=(-1)^m(b-k-1,m)M_k(a,b,j,0) \tag{42}$$

and

$$M_k(a,b,m)=(-1)^m(b-k-1,m)\Lambda_k(a,2-b) \tag{43}$$

for $\mathrm{Re}b>1+k,\ k,m\in\mathbb{N}^0$*;*

$$\Lambda(a,2-b-2m)=\frac{a^2+(b+2m)^2}{(b+2m)(b+2m+1)}\Lambda(a,-b-2m), \tag{44}$$

for $\mathrm{Re}b>1-2m$*;*

$$M(a,b,m)=\frac{(-1)^m\prod\limits_{\kappa=1}^{m}\left\{a^2+[b+2(\kappa-1)]^2\right\}}{(b+m-1,m)}\Lambda(a,2-b-2m), \tag{45}$$

when either $\mathrm{Re}b>1-m$, $m\in\mathbb{N}$ *or if* $a=0$, *when* $\mathrm{Re}b>m$, $m\in\mathbb{N}_1$ *(in the last case in (45)* $b=1-m$ *is allowed if the right-hand side we consider as a corresponding limit which will be equal to zero);*

$$M(a,b,m)=(-1)^{m-1}(b,m-1)M(a,b,1)\ \text{ for }\ \mathrm{Re}b>0,\quad m\in\mathbb{N}. \tag{46}$$

When $a = 0$, $m \in \mathbb{N}_1$ *(46) is valid also for* $\mathrm{Re}b > -1$.

Proof. Equality (41) can be obtained as follows

$$\begin{aligned}
M_k(a,b,j,m+1) &= y^{b+m-k}\int_{-\infty}^{+\infty}(\xi-x)^k\frac{\partial^{m+1}\theta^j e^{a\theta}\rho^{-b}}{\partial y^{m+1}}d\xi \\
&= y^{b+m-k}\frac{\partial}{\partial y}\int_{-\infty}^{+\infty}(\xi-x)^k\frac{\partial^{m}\theta^j e^{a\theta}\rho^{-b}}{\partial y^{m}}d\xi \\
&= \frac{\partial}{\partial y}\left[y^{b+m-k}\int_{-\infty}^{+\infty}(\xi-x)^k\frac{\partial^{m}\theta^j e^{a\theta}\rho^{-b}}{\partial y^{m}}d\xi\right] \\
&\quad -(b+m-k)y^{b+m-k-1}\int_{-\infty}^{+\infty}(\xi-x)^k\frac{\partial^{m}\theta^j e^{a\theta}\rho^{-b}}{\partial y^{m}}d\xi \\
&= \frac{\partial}{\partial y}\left[yM_k(a,b,j,m)\right]-(b+m-k)M_k(a,b,j,m) \\
&= (1+k-b-m)M_k(a,b,j,m).
\end{aligned}$$

Using l-times formula (41), when $\mathrm{Re}b+m-l-k-1>0$ (when $a=0$, $j=0$, $m-l\in\mathbb{N}_1$, we can take $\mathrm{Re}b+m-l-k>0$), we get

$$M_k(a,b,j,m) = (2+k-b-m,l)M_k(a,b,j,m-l).$$

Therefore, in particular, for $j=k=0$, $l=m-1$, we obtain (46), because of

$$(2-b-m,m-1) = (-1)^{m-1}(b,m-1),$$

while for $l=m$, we have

$$M_k(a,b,j,m) = (2+k-b-m,m)M_k(a,b,j,0) = (-1)^m(b-k-1,m)M_k(a,b,j,0),$$

i.e., (42). Hence, we get (43) since

$$M_k(a,b,0,0) = M_k(a,b,0) = \Lambda_k(a,2-b).$$

For $\mathrm{Re}b>1$, $a\neq 0$, using twice integration by parts, we get

$$\Lambda(a,-b) = \int_0^{\pi} e^{a\theta}\sin^b\theta d\theta = \frac{b(b-1)}{a^2}\Lambda(a,2-b) - \frac{b^2}{a^2}\Lambda(a,2-b).$$

Thus,

$$\Lambda(a,2-b) = \frac{a^2+b^2}{b(b-1)}\Lambda(a-b).$$

The last remains valid also in the case $a = 0$, what immediately follows from the following equalities

$$\Lambda(0,2-b) = -\int_0^\pi \sin^b\theta d\cot\theta = b\int_0^\pi \sin^{b-2}\theta\cos^2\theta d\theta = b\int_0^\pi \sin^{b-2}\theta d\theta - b\int_0^\pi \sin^b\theta d\theta.$$

If we replace b by $b+2m$ we get (44).

When $m = 1$, because of

$$\Lambda_1(a,b) = -\frac{a}{b}\Lambda(a,b) \ \text{ for } \ \mathrm{Re}b < 0, \tag{47}$$

evidently,

$$\begin{aligned} M(a,b,1) &= y^b \int_{-\infty}^{+\infty} \frac{\partial e^{a\theta}\rho^{-b}}{\partial y} d\xi = -\int_{-\infty}^{+\infty} (at+b) e^{a\cdot \mathrm{arccot}(-t)} \left(1+t^2\right)^{-\frac{b}{2}-1} dt \\ &= -a\Lambda_1(a,-b) - b\Lambda(a,-b) = -\frac{a^2+b^2}{b}\Lambda(a,-b), \ \ \mathrm{Re}b > 0. \end{aligned}$$

In the case $a = 0$ we can assume $\mathrm{Re}b > -1$. So, formula (45) is true for $m = 1$. Now, we assume its validity for $m = n$ and consider $M(a,b,n+1)$. By virtue of (41), when $j = k = 0$ and either $\mathrm{Re}b > 1-n$, $n \in \mathbb{N}$ or $a = 0$ and $\mathrm{Re}b > -n$, $n \in \mathbb{N}_1$, we have

$$\begin{aligned} &M(a,b,n+1) = (1-b-n)M(a,b,n) \\ &= (1-b-n)\frac{(-1)^n \prod\limits_{\kappa=1}^{n} \left\{a^2 + [b+2(\kappa-1)]^2\right\}}{(b+n-1,n)}\Lambda(a,2-b-2n) \\ &= \frac{(-1)^{n+1} \prod\limits_{\kappa=1}^{n} \left\{a^2 + [b+2(\kappa-1)]^2\right\}}{(b+n,n-1)}\Lambda(a,2-b-2n). \end{aligned}$$

Whence, taking into account (44) for $m = n$, we get

$$M(a,b,n+1) = (-1)^{n+1} \frac{\prod\limits_{\kappa=1}^{n} \left\{a^2 + [b+2(\kappa-1)]^2\right\}}{(b+n,n+1)}\Lambda(a,-b-2n).$$

But, both the sides of this equality are analytic functions with respect to b when either $\mathrm{Re}b > -n$ or $a = 0$, $\mathrm{Re}b > -n-1$, $n+1 \in \mathbb{N}_1$ (in the last case points $b = -n$, $n \in \mathbb{N}_2$ are removable points of singularity for the right-hand side), which coincide either for $\mathrm{Re}b > 1-n$, $n \in \mathbb{N}$ or in case $a = 0$ for $\mathrm{Re}b > -n$, $n \in \mathbb{N}_1$. Then, according to the uniqueness theorem of analytic function both the sides coincide in the whole domain of their analyticity. □

It is well known that (s., e.g., [4], pp. 491 and 386):

$$\Lambda_k(a,b) = (-1)^k \frac{2^b \pi e^{\pi\left(\frac{a}{2}-i\frac{k}{2}\right)} \sum_{n=0}^{k} \frac{(-1)^n(-k,n)\left(\frac{a}{2i}+\frac{b}{2},n\right)}{\left(1+\frac{a}{2i}-\frac{b}{2}-k,n\right)(1,n)}}{(1-b-k)\tilde{B}\left(1-\frac{a}{2i}-\frac{b}{2},1+\frac{a}{2i}-\frac{b}{2}-k\right)}, \quad \mathrm{Re} b < 1-k, \tag{48}$$

$$A(c,b) := \int_0^\pi \cos(c\theta)\sin^{-b}\theta d\theta = \frac{2^b\pi\cos\frac{c\pi}{2}}{(1-b)\tilde{B}\left(\frac{2+c-b}{2},\frac{2-c-b}{2}\right)}, \quad \mathrm{Re} b < 1, \quad c \in \mathbb{R}^1; \tag{49}$$

$$B(c,b) := \int_0^\pi \sin(c\theta)\sin^{-b}\theta d\theta = \frac{2^b\pi\sin\frac{c\pi}{2}}{(1-b)\tilde{B}\left(\frac{2+c-b}{2},\frac{2-c-b}{2}\right)}, \quad \mathrm{Re} b < 1, \quad c \in \mathbb{R}^1; \tag{50}$$

where

$$\tilde{B}(x,y) := \int_0^1 t^{x-1}(1-t)^{y-1}dt, \quad \mathrm{Re}\, x > 0, \quad \mathrm{Re}\, y > 0,$$

is the Euler Beta function (see [4], pp. 962-964). Evidently, when b and c are real numbers

$$\tilde{B}\left(\frac{2+c-b}{2},\frac{2-c-b}{2}\right) > 0 \ \text{ for } \ b < 2 \pm c.$$

Taking into account the last one, from (49) and (50) we conclude that

$$A(2k+1,b) = 0, \quad A(c,b) \neq 0, \quad c \neq 2k+1, \quad k = 0, \pm 1, \pm 2, \ldots;$$

$$B(2k,b) = 0, \quad B(c,b) \neq 0, \quad c \neq 2k, \quad k = 0, \pm 1, \pm 2, \ldots;$$

$$A^2(c,b) + B^2(c,b) = 2^{2b}(1-b)^{-2}\pi^2\tilde{B}^{-2}\left(\frac{2+c-b}{2},\frac{2-c-b}{2}\right) > 0.$$

Theorem 4. *For complex numbers a, b, and $k \in \mathbb{N}^0$, $\mathrm{Re}\, b < 1-k$, the inequality*

$$\Lambda_k(a,b) \neq 0$$

is valid if and only if when

$$b - ia, \quad b + ia + 2k \bar{\in} \mathbb{N}_2, \tag{51}$$

$$\sum_{n=0}^{k} \frac{(-1)^n(-k,n)\left(\frac{a}{2i}+\frac{b}{2},n\right)}{(1,n)\left(1+\frac{a}{2i}-\frac{b}{2}-k,n\right)} \neq 0, +\infty. \tag{52}$$

For $a, b \in \mathbb{R}^1$, $b < 1-k$ and $k \in \mathbb{N}^0$ we have

$$\Lambda_k(a,b) \begin{cases} > 0, & \text{when either } k \in \mathbb{N}_2^0, \text{ or } a > 0, \ k \in \mathbb{N}_1; \\ < 0, & \text{when } a < 0, \ k \in \mathbb{N}_1; \\ = 0, & \text{when } a = 0, \ k \in \mathbb{N}_1, \end{cases} \tag{53}$$

while, for $k \in \mathbb{N}$ *we have*

$$\begin{aligned}&(k+1)\Lambda_k(a,-k-1)\\ &+a\Lambda_{k+1}(a,-k-1)\begin{cases} >0, & \text{when either } k\in\mathbb{N}_2, \text{ or } a>0,\ k\in\mathbb{N}_1;\\ <0, & \text{when } a<0,\ k\in\mathbb{N}_1;\\ =0, & \text{when } a=0,\ k\in\mathbb{N}_1,\end{cases}\end{aligned} \tag{54}$$

Proof. Substituting

$$\tilde{B}\left(1-\frac{a}{2i}-\frac{b}{2}, 1+\frac{a}{2i}-\frac{b}{2}-k\right)=\frac{\Gamma\left(1-\frac{a}{2i}-\frac{b}{2}\right)\Gamma\left(1+\frac{a}{2i}-\frac{b}{2}-k\right)}{\Gamma(2-b-k)}$$

in (48) and taking into account that the Euler function

$$\Gamma(z)=\frac{1}{z}\prod_{n=1}^{\infty}\left\{\left(1+\frac{1}{n}\right)^z\left(1+\frac{z}{n}\right)^{-1}\right\}$$

[see below (62)] is not equal to zero in the complex plane but the points $z=0,-1,-2,\dots,$ where it has poles (see [5], p. 16), the first part of the theorem becomes clear.

From

$$\begin{aligned}\Lambda_k(a,b)&=\int_{-\infty}^{+\infty} t^k e^{a\cdot\operatorname{arccot}(-t)}\left(1+t^2\right)^{\frac{b}{2}-1}dt\\ &=\int_0^{+\infty}\left[e^{a\cdot\operatorname{arccot}(-t)}+(-1)^k e^{a\cdot\operatorname{arccot}t}\right]t^k\left(1+t^2\right)^{\frac{b}{2}-1}dt\end{aligned} \tag{55}$$

it is obvious that if $k\in\mathbb{N}_2^0$, then $\Lambda_k(a,b)>0$; if $a=0$, $k\in\mathbb{N}_1$, then $\Lambda_k(0,b)=0$; if $a\neq 0$, then (53) is valid because of inequalities

$$\operatorname{arccot}(-t)>\operatorname{arccot}t \text{ for } t\in]0,+\infty[; \tag{56}$$

$$e^{a\cdot\operatorname{arccot}(-t)}-e^{a\cdot\operatorname{arccot}t}\begin{cases}>0, & a>0;\\ <0, & a<0,\end{cases}\quad t\in]0,+\infty[. \tag{57}$$

In view of (55), we have

$$\begin{aligned}&(k+1)\Lambda_k(a,-k-1)+a\Lambda_{k+1}(a,-k-1)\\ &=(k+1)\int_0^{+\infty}\left[e^{a\cdot\operatorname{arccot}(-t)}+(-1)^k e^{a\cdot\operatorname{arccot}t}\right]t^k\left(1+t^2\right)^{\frac{k+1}{2}-1}dt\\ &+a\int_0^{+\infty}\left[e^{a\cdot\operatorname{arccot}(-t)}+(-1)^{k+1}e^{a\cdot\operatorname{arccot}t}\right]t^k\left(1+t^2\right)^{\frac{k+1}{2}-1}dt.\end{aligned}$$

Whence, using inequalities (56) and (57) and separately considering the cases: a is arbitrary, $k \in \mathbb{N}_2$; $a > 0$, $k \in \mathbb{N}_1$; $a < 0$, $k \in \mathbb{N}_1$; $a = 0$, $k \in \mathbb{N}_1$ it is easy to see that (54) is valid. □

Corollary 5. For complex numbers a and b the inequality

$$M(a, b, m) \neq 0$$

is valid if and only if when $2 - b - 2m \pm ia \bar{\in} N_2$ and either Re$b > 1 - m$, $m \in \mathbb{N}^0$, $a^2 + [b + 2(\kappa - 1)]^2 \neq 0$, $\kappa = 1, ..., m$; or if $a = 0$, when Re$b > -m$, $m \in \mathbb{N}_1$, $b \neq 0, -2, ..., -2\Big(m - \left[\frac{m}{2}\right] - 1\Big)$.

If $a, b \in \mathbb{R}^1$ then

$$M(a, b, m) \neq 0$$

when either $a \neq 0$, $b > 1 - m$, $m \in \mathbb{N}^0$; or $a = 0$, $b \neq 0, -2, ..., -2\Big(m - \left[\frac{m}{2}\right] - 1\Big)$ and either $b > 1 - m$, $m \in \mathbb{N}_2$ or $b > -m$, $m \in \mathbb{N}_1$.

Proof. According to formula (45) and Theorem 4 it is not difficult to prove the corollary 5 . It should be only mentioned that for $a, b \in \mathbb{R}^1$ the numbers $2 - b - 2m \pm ia$ can not be even positive ones, since when $a \neq 0$ they are pure complex numbers, while when $a = 0$ we have $1 - m \leq 0$ and either $1 - b - m < 0$, or $1 - b - m < 1$, i.e, in both the cases $2 - b - 2m < 1$.□

Theorem 6. *For* Re$b < 1 - k$ *and* $k \geq 2$ *we have*

$$\Lambda_k(a, b) = -\frac{\Lambda_0(a, b)}{(b, k)} \begin{cases} \sum\limits_{j=0}^{\frac{k-2}{2}} \overset{(k)}{c}_{2j} a^{2j} - a^k, \quad k \in \mathbb{N}_2; \\ \sum\limits_{j=0}^{\frac{k-3}{2}} \overset{(k)}{c}_{2j+1} a^{2j+1} + a^k, \quad k \in \mathbb{N}_1 \backslash \{1\}, \end{cases} \tag{58}$$

where $\overset{(k)}{c}_j$ *are independent of* a.

Proof. Let us prove in advance that

$$\Lambda_k(a, b) = \frac{-1}{b + k - 1} \left[(k - 1)\Lambda_{k-2}(a, b) + a\Lambda_{k-1}(a, b) \right], \tag{59}$$

$$\text{Re}b < 1 - k, \; k \in \mathbb{N}_1 \backslash \{1\}.$$

Indeed,

$$\Lambda_k(a,b) = \int\limits_{-\infty}^{+\infty} t^{k-1} e^{a\cdot \operatorname{arccot}(-t)} d\frac{(1+t^2)^{\frac{b}{2}}}{b}$$
$$= -\frac{1}{b}\int\limits_{-\infty}^{+\infty} (k-1)t^{k-2} e^{a\cdot \operatorname{arccot}(-t)} \left(1+t^2\right)^{\frac{b}{2}} dt$$
$$-\frac{a}{b}\int\limits_{-\infty}^{+\infty} t^{k-1} e^{a\cdot \operatorname{arccot}(-t)} \left(1+t^2\right)^{\frac{b}{2}-1} dt$$
$$= -\frac{k-1}{b}\Lambda_{k-2}(a,b+2) - \frac{a}{b}\Lambda_{k-1}(a,b)$$
$$-\frac{k-1}{b}\Lambda_{k-2}(a,b) - \frac{k-1}{b}\Lambda_k(a,b) - \frac{a}{b}\Lambda_{k-1}(a,b), \qquad (60)$$

since, as it is easy to see,

$$\Lambda_{k-2}(a,b+2) = \int\limits_{-\infty}^{+\infty} t^{k-2} e^{a\cdot \operatorname{arccot}(-t)} \left(1+t^2\right)^{\frac{b}{2}} dt$$
$$= \int\limits_{-\infty}^{+\infty} t^{k-2} e^{a\cdot \operatorname{arccot}(-t)} \left(1+t^2\right)^{\frac{b}{2}-1} dt$$
$$+ \int\limits_{-\infty}^{+\infty} t^{k} e^{a\cdot \operatorname{arccot}(-t)} \left(1+t^2\right)^{\frac{b}{2}-1} dt = \Lambda_{k-2}(a,b) + \Lambda_k(a,b).$$

(59) immediately follows from (60).

Because of (59) we have

$$\Lambda_2(a,b) = -\frac{1}{b+1}\left[\Lambda_0(a,b) + a\Lambda_1(a,b)\right]$$
$$= -\frac{1}{b+1}\left[\Lambda_0(a,b) - \frac{a^2}{b}\Lambda_0(a,b)\right] = -\frac{\Lambda_0(a,b)}{b(b+1)}\left(b-a^2\right);$$
$$\Lambda_3(a,b) = -\frac{1}{b+1}\left[2\Lambda_1(a,b) + a\Lambda_2(a,b)\right]$$
$$= -\frac{\Lambda_0(a,b)}{b(b+1)(b+2)}\left[-a(3b+2)+a^3\right],$$

i.e., formula (58) is true for $k = 2, 3$. Assuming that it is valid for $2, 3, ..., k$ let us prove its validity for $k+1$.

By virtue of (58) and (59), for Re$b < -k$ we have

$$\Lambda_{k+1}(a,b) = -\frac{1}{b+1}\left[k\Lambda_{k-1}(a,b) + a\Lambda_k(a,b)\right]$$

$$= -\frac{\Lambda_0(a,b)}{b+k}\left[-\frac{k}{(b,k-1)}\begin{cases}\sum\limits_{j=0}^{\frac{k-3}{2}} \overset{(k-1)}{c_{2j}}\ a^{2j} - a^{k-1}, & k-1 \in \mathbb{N}_2;\\ \sum\limits_{j=0}^{\frac{k-4}{2}} \overset{(k-1)}{c_{2j+1}}\ a^{2j+1} + a^{k-1}, & k-1 \in \mathbb{N}_1,\end{cases}\right.$$

$$\left.-\frac{a}{(b,k)}\begin{cases}\sum\limits_{j=0}^{\frac{k-3}{2}} \overset{(k)}{c_{2j+1}}\ a^{2j+1} + a^{k}, & k \in \mathbb{N}_1;\\ \sum\limits_{j=0}^{\frac{k-2}{2}} \overset{(k)}{c_{2j}}\ a^{2j} - a^{k}, & k \in \mathbb{N}_2,\end{cases}\right]$$

$$= -\frac{\Lambda_0(a,b)}{(b,k+1)}\begin{cases}\sum\limits_{j=0}^{\frac{k-1}{2}} \overset{(k+1)}{c_{2j}}\ a^{2j} - a^{k+1}, & k \in \mathbb{N}_1, \text{ i.e., } k+1 \in \mathbb{N}_2;\\ \sum\limits_{j=0}^{\frac{k-2}{2}} \overset{(k-1)}{c_{2j+1}}\ a^{2j+1} + a^{k+1}, & k \in \mathbb{N}_2, \text{ i.e., } k+1 \in \mathbb{N}_1,\end{cases}$$

i.e., formula (59) is valid for $k+1$. □

Remark 7. It is well known that (see [4], p. 460),

$$\overset{*}{\Lambda}(0,b) = 2^{b-1}\pi^2\Gamma(1-b)\Gamma^{-2}\left(1-\frac{b}{2}\right), \tag{61}$$

where

$$\Gamma(z) := \int\limits_0^\infty e^{-t}t^{z-1}dt, \quad \text{Re}z > 0, \tag{62}$$

is the Euler Gamma function (see [4], pp. 947-951). From (48) we get

$$\begin{aligned}\Lambda(0,b) &= \frac{2^b\pi}{1-b}\tilde{B}^{-1}\left(1-\frac{b}{2}, 1-\frac{b}{2}\right) = \frac{2^b\pi\Gamma(2-b)}{(1-b)\Gamma^2\left(1-\frac{b}{2}\right)}\\ &= 2^b\pi\Gamma(1-b)\Gamma^{-2}\left(1-\frac{b}{2}\right),\end{aligned} \tag{63}$$

since (see [4], pp. 951 and 964)

$$\Gamma(2-b) = (1-b)\Gamma(1-b) \quad \text{and} \quad \tilde{B}(a,b) = \frac{\Gamma(a)\Gamma(B)}{\Gamma(a+b)} = \tilde{B}(b,a).$$

From (61) and (63) it is easy to conclude that

$$\overset{*}{\Lambda}(0,b) = \frac{\pi}{2}\Lambda(0,b), \quad b < 1. \tag{64}$$

R E F E R E N C E S

1. Jaiani G. Euler-Poisson-Darboux Equation, *Tbilisi University Press, Tbilisi*, 1984 (in Russian, Georgian and English summaries).

2. Jaiani G. Initial and boundary value problems for singular differential equations and applications to the theory of cusped bars and plates, Complex Methods for Partial Differential Equations (ISAAC Serials, **6** (1999), 113-149), Eds.: H. Begehr, O. Celebi, W. Tutschke. *Kluwer, Dordrecht.*

3. Fichtenholz G.M. A Course of Differential and Integral Calculus, Vol. I, *Fiz.-Mat.-Lit., Moscow,* 1962 (in Russian).

4. Gradstein I.S., Ryshik I.M. Tables, Integrals, Summs, Series, and Products, *Fiz.-Mat.-Lit., Moscow,* 1963 (in Russian).

5. Whittaker E.T., Watson G.N. A Course of Modern Analysis, vol. 2, *Fiz.-Mat.-Lit., Moscow,* 1963 (translation into Russian).

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