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A GENERAL STOCHASTIC MODEL OF THE PRIORITY SERVICE SYSTEM WITH TWO TYPES OF OPERATIONS

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Abstract. The standby technical many-element system with renewal and replacement operations is considered. A stochastic model of this system is constructed taking into account the duration and priority of replacement. The question of economic efficiency is considered and the corresponding estimates are obtained.
The results are obtained for any number of replacement and renewal elements.

Keywords and phrases: Priority system of queues, standby, replacement, renewal, economic efficiency.

AMS subject classification: 60K25; 90B22.

Introduction

In 1992 Recommendation E.862 "Dependability Planning of Telecommunication Networks" [1] developed by Swedish specialists by the order of International Society of Electric Communication was finally approved. There a particular attention is paid to models and methods of planning, design, functioning and technical maintenance dependability of telecommunication networks and to the problems of application of these methods to different services in international network. Out of two possible approaches to dependable planning, intuitive and analytical, Recommendation E.862 gives absolute priority to analytical one. This Recommendation became a stimulus for the activity of many scientists.

Due to the investigations carried out by the group of scientists of the Georgian Technical University the study of the problems of complex multi-element stand-by systems has been considerably advanced in the recent years. In particular, with consideration of the duration of replacement, different models have been considered and partially investigated [2–7].

Among the most important trends of dependable planning the problem of definition of optimum number of standby elements and of technical maintenance means is highlighted for the cases when system elements fail, lose workability and need replacement. Until recently neither in mathematical reliability theory nor in the theory of mass service, is the theoretical basis of reliability theory, are the models of multi-component standby systems with renewable elements developed taking into account replacement duration of the failed element of the system. As a rule, in the models of dependability and technical maintenance the duration of replacement was not considered as a separate independent operation. This value was supposed to be zero (i.e. there happened instant replacement), or replacement time was supposed to be so small that it might be neglected.

In complex many–element standby systems the supposition of instant replacement in a number of cases is quite unjustified. In many systems, particularly in radio electronic ones, the duration of replacement has the order not lower that the duration of renewal (repair) of the failed element of the system.

Model Formulation

The standby technical system considered in the given work consists of m main and n standby elements. All elements are identical. For normal functioning of the system all m main elements are necessary to be maintained in operating condition. The system continues to function in case when main elements number is reduced, but the effectiveness of system functioning drops down, as a result it becomes necessary to turn to standby elements and to replace the failed main element with the standby one.

The main elements fail with intensity α and standby ones - with intensity β. The failed main element is replaced with an operating standby element at the first possibility, the replacement being done with intensity λ. If in the system there is no necessity to make replacement (all m main elements are in operating state) or if there is an organ that is not replaced (the number of replacement and renewal organs exceeds the number of failed main elements), the failed element is renewed. Renewal is done with intensity μ. The number of organs executing replacement or renewal is equal to r. In the process of replacement or renewal of one element only one organ takes place.

Note, that in [4–6] the above described standby system is investigated for the case $r = 1$. The case when the number of replacement and renewal organs are more than one are of practical interest. The presented article considers a general case of the standby system, one of the particular case being the case with a single organ of replacement and renewal.

For construction of a mathematical model of the considered system, introduce the notion of system state. We say that system is in state $s_{i,j}$ $(i = \overline{0,m},\ j = \overline{0,n+m})$, if the number of deficit main elements is i, and the number of non-operating elements (main and standby) is j.

Before constructing the model we assume that number of replacement and renewal organs is $r = m + n$.

The analysis of the possible states of the considered system shows that the states $s_{i,j}$ $(i = \overline{0,m},\ j = \overline{0,n+i})$ are essential for it.

The process of the system functioning is described by the functions $p(i,j,t)$, which implies the probability that at the moment of time t the system is in the state $s_{i,j}$.

Let us assume that t has a small increment h and find the probability $p(i,j,t+h)$ that during the time h the system will pass to the state $s_{i,j}$. Using the total probability formula, our reasoning will be as follows. At the moment of time $t+h$ the system may be in the state $s_{i,j}$ if: 1) at the moment of time t it was in the same state $s_{i,j}$ and during the time h it still remained in this state, i.e. there occurred no failure either of a working element or of standby elements and there was no replacement or renewal; 2) at the moment of time t the system was in the state $s_{i-1,j-1}$ and during the time h there occurred failure of one of the working elements, but there was no failure of a standby element and there was no replacement or renewal; 3) the system was in the state $s_{i,j-1}$ and during the time h there occurred failure of a standby element, but there was no failure of the main element and there was no renewal or replacement; 4) being at the moment of time t in the state $s_{i+1,j}$, the replacement took place, but the element did not go out of service and there was no renewal; 5) at the moment of time t the system is in the state $s_{i,j+1}$ and during the time h the renewal took place, but there was no failure or replacement of an element. Any other state of the system by the moment of time t never ensures its state $s_{i,j}$ by the time $t+h$. It should be noted that the probability of the presence of two operations during the time h is a value of order $o(h)$ which can be neglected in our model. (Under one operation we understand one of the following events: failure of a working element, failure of a standby element, replacement or renewal of an element.)

Note that in the case of the state $s_{i,j}$ the failed main elements are replaced by i replacement organs. However in that case it is necessary to have serviceable standby elements in a quantity not less than i.

Proceeding from the above reasoning and having made some small transformations, we write the equation for the probability $p(i,j,t+h)$:

$$\begin{aligned}p(i,j,t+h) &= p(i,j,t) + p(i,j,t)(-\alpha(m-i) - \beta(n-(j-i)) - \mu j - \xi_{i,j})h \\ &\quad + p(i-1,j-1,t)\alpha(m-i+1)h + p(i,j-1,t)\beta(n-j+1)h \\ &\quad + p(i,j+1,t)\mu(j+1)h + p(i+1,j,t)\lambda(n-j+i+1)h + o(h^2),\end{aligned}$$

where $\xi_{i,j} = \lambda(n-j+i)$, if $i \neq 0$ and $\xi_{i,j} = 0$, if $i = 0$; $i = \overline{0,m}$, $j = \overline{0,n+i}$. After simple transformations, assuming that h tends to zero we come to the differential equations (Kolmogorov's equations)

$$\begin{aligned}\frac{d}{dt}\,p(i,j,t) &= -\big(\alpha(m-i) + \beta(n-(j-i)) + \mu j + \xi_{i,j}\big)\,p(i,j,t) \\ &\quad + \alpha(m-i+1)\,p(i-1,j-1,t) + \beta(n-j+i+1)\,p(i,j-1,t) \\ &\quad + \mu(j+1)\,p(i,j+1,t) + \lambda(n-j+i+1)\,p(i+1,j,t), \quad i = \overline{0,m}, \\ &\qquad j = \overline{0,n+i}.\end{aligned}$$

The above differential equations are written for each state $s_{i,j}$ $(i=\overline{0,m}$, $j=\overline{0,n+i})$. The number of equations coincides with the number of states $s_{i,j}$ and, accordingly, with the number of probabilities $p(i,j,t)$ we want to find.

Let us consider the question of final probabilities, i.e., probabilities $p_{i,j} = \lim\limits_{t\to\infty} p(i,j,t)$. The number of states of the considered system is finite and, after a finite number of steps, the system may pass from any of its states to any other state. As is known, this is sufficient for the existence of final probabilities. Since we also have $\frac{d}{dt}\,p(i,j,t) = 0$, for defining $p_{i,j}$ (final probabilities no longer depending on t) we obtain the following system of linear algebraic equations:

$$\begin{aligned}\big(\alpha(m-i)+\beta(n-(j-i))+\mu j+\xi_{i,j}\big)\,p_{i,j} &= \alpha(m-i+1)\,p_{i-1,j-1}\\ &+\beta(n-j+i+1)\,p_{i,j-1}+\mu(j+1)\,p_{i,j+1}\\ &+\lambda(n-j+i+1)\,p_{i+1,j}, \qquad (1)\\ & i=\overline{0,m}, \quad j=\overline{0,n+i}.\end{aligned}$$

As we see, we obtain a homogeneous system. For its solution it is necessary to use standard condition $\sum\limits_{i=0}^{m}\sum\limits_{j=0}^{n+i} p_{i,j} = 1$ and it is connected with pure calculation difficulties.

When analyzing one-dimensional random processes with discrete space of states, it is convenient to use graphical scheme, the so-called state graph. As to complex technical systems for the description of which it is necessary to consider two-dimensional random processes, such approach is supposed to be quite complicated, and the result is not always evident.

We present the construction of graphical scheme, "map of system states" which enables to evidently visualize two-dimensional random processes. Besides evident representation of the possible states of the system and transition from one state into another, the proposed "map" allows (without using Kolmogorov's equations) direct recording of linear algebraic equations for final probabilities of system states $p_{i,j}$ - the probability of system being in state $s_{i,j}$.

As it was noted states $s_{i,j}$ $(i=\overline{0,m},\ j=\overline{0,n+i})$ are essential for the considered system. Without limiting the generality, suppose $m>n$.

Then, on coordinate plane iOj the system states may be presented schematically in the form of trapezoid with vertexes $s_{0,0}, s_{0,n}, s_{m,m+n}, s_{m,0}$ (Fig.1).

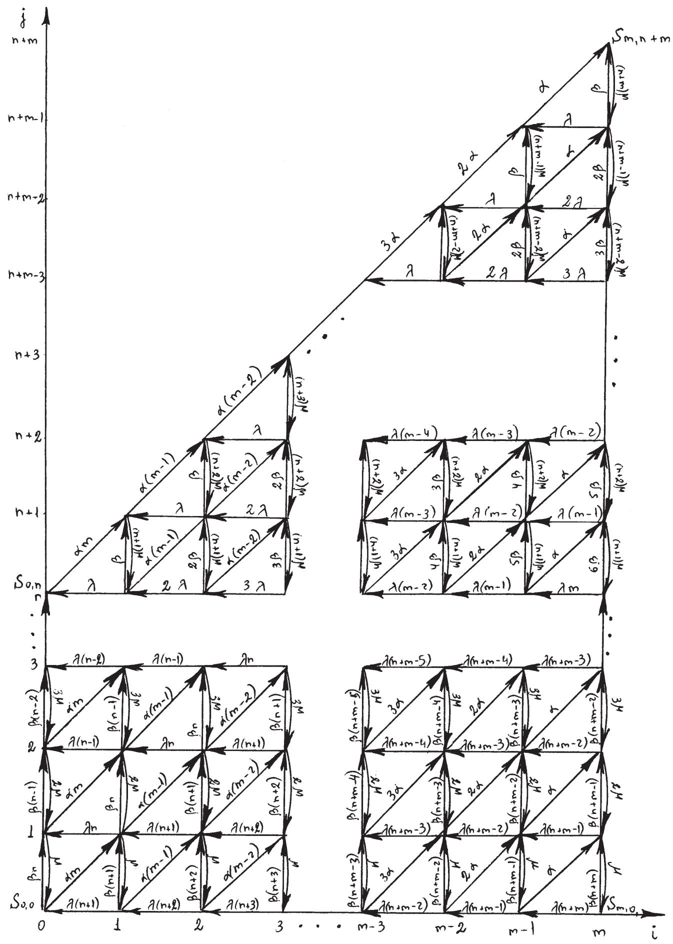


Fig. 1

Each node of state map (point (i, j)) symbolizes the state of system $s_{i,j}$. In trapezoid region the arrows are shown on which the intensities of that event flow are marked which transfers the system into a new state according to the given arrow.

The map itself is drawn as follows: In state $s_{i,j}$ ($i = \overline{1, m-1}$, $j = \overline{1, n}$), the system may occur: (a) from state $s_{i-1,j-1}$ as a result of the failure of one of $(m - (i-1))$ operating main elements; (b) from state $s_{i,j-1}$ as a result of the failure of one of $(n - (j - 1 - i))$ operating standby elements; (c) from state $s_{i+1,j}$ as a result of replacement of one of $(i + 1)$ failed main elements; (d) from state $s_{i,j+1}$ as a result of renewal of one of $(j + 1)$ non-operating elements. Note, that if $j = \overline{n+1, m+n-1}$, in case (a) the intensity of replacement is $\lambda\ (i + 1 - (j - n))$ and not $\lambda\ (i + 1)$. This is caused by the number of operating standby elements existing in the given moment. The analogous reasoning enable to draw the whole map of system states.

After the mentioned constructions the linear algebraic system of equations for determination of probabilities $p_{i,j}$ is obtained directly, in a quite simple way. Namely, for each node, on the plane iOj, the equation is composed as follows: in the left part we write the probability of system being in the considered state multiplied by sum intensity of all flows removing the system out of the given state (out coming arrows), in the right part of the equation we write the sum of products of the probabilities of all states from which the arrows go to the node symbolizing the given state, on the intensity of the corresponding flows of events. As a result we obtain the system of linear algebraic equations (1).

On drawing up the above presented map of system states and the corresponding model we did not limit the number of replacement and renewal organs. In other words, the model is constructed for case $r = m + n$.

Present the algorithm of transformation of system state map in case $0 < r < m + n$ (when $r = 0$, many essential states of the system consist of one element $s_{m,n+m}$ and are of no practical importance).

For the convenience of reasoning denote the coefficients at intensities λ and μ through k and l, respectively. Then, on the map of states the intensities shown on arrows are transformed in the following way:

1) if $k + l > r$, that is if the sum of coefficients at λ and μ is more than r, we conduct action (2). Otherwise the values of intensities are $k\lambda$ and $l\mu$.

2) $l = l - 1$.

3) if $l < 1$, system transition into a new state by the renewal of the failed element (transition by arrow $l\mu$) does not happen. Pass to point (5).

4) if $l \geq 1$, we pass to point (1).

5) if $k > r$, conduct action (6). Otherwise the value of intensity is $k\lambda$.

6) $k = k - 1$.

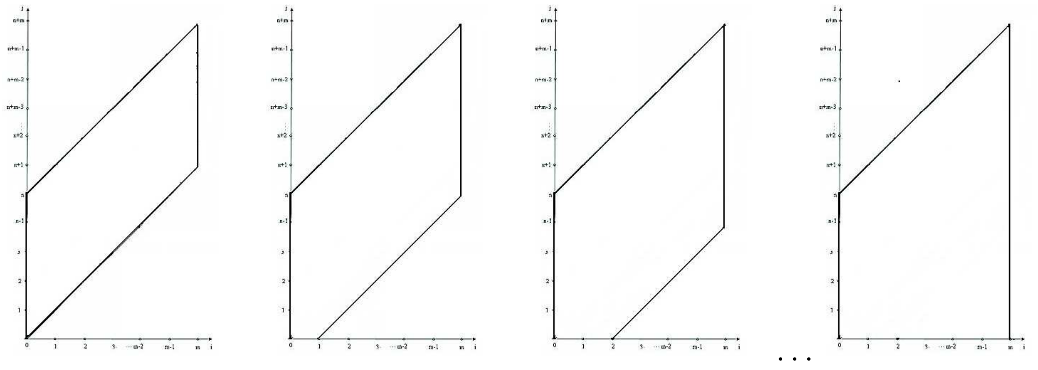
7) if $k = 1$, intensity is λ. Otherwise we pass to point (5).

The essence of the given algorithm is as follows: In the first place all replacements of failed elements are done (as the number of operating standby elements and the number of organs r allow) and only then becomes possible the renewal of the failed elements. In the case of a sufficient number of replacement and renewal organs (r is more than the number of failing main elements) or if it is found impossible to make replacement (in the case of insufficient number of operating standby elements) parallel replacement and renewal becomes possible.

With the change of the number of replacement and renewal organs r changes the area of essential states (trapezoid with vertices $s_{0,0}, s_{0,n}, s_{m.n+m}$, $s_{m,0}$ when $r = m+n$), the system of linear algebraic equations (1) is respectively changed. However, in accordance with the above given algorithm, having drawn the map of system state for concrete r, we easily get the corresponding system of linear algebraic equations for determination of final probabilities of system state $p_{i,j}$.

In [4, 5] the particular case of the considered system is studied and map of states is drawn for $r = 1$.

Specifically, for different $r = \overline{1, m+n}$, the area of essential states of the system in the map of states respectively takes the form (Fig. 2):



$r = 1$ $\quad$ $r = 2$ $\quad$ $r = 3$ $\quad \cdots \quad$ $r = m + n$

Fig. 2

It should be noted that the models of analysis developed by us are suitable for investigation of many other systems and thus are, in a sense, universal. This universality is based on isomorphism of processes going on in different nature systems, abstract processes in their mathematical models.

Calculation of Economical Efficiency

Construction and study of the models for determination of probability characteristics $p_{i,j}$ are of no original interest. These results are just the prerequisite for the methods of determination of dependability parameters and economic effectiveness of functioning of various systems, such, for example, as computer and telecommunication networks.

Consider the problem of economical effectiveness of system functioning. In the case when the number of replacement and renewal organs equals one ($r = 1$), the given problem is considered in [5].

Introduce the index of economical effectiveness as an effectiveness function:

$$F(m, n) = (r_1 - c_1)E_1 + (r_2 - c_2)E_2 - \sum_{k=3}^{6} c_k E_k. \tag{2}$$

Here: r_1 is the income in time unit from one operating main element; r_2 is the income in time unit from one operating standby element; c_1 is expenses in time unit for one operating main element; c_2 is expenses in time unit for one operating standby element; c_3 is expenses in time unit for one non-operating standby element; c_4 is expenses in time unit for one operating renewal organ; c_5 is expenses in time unit for one operating replacement organ; c_6 is expenses in time unit for one non-operating organ of replacement and renewal. Denote $E_i = E_i(m, n)$, where E_1 is a mean number of operating main elements; E_2 is a mean number of operating standby elements; E_3 is a mean number of non-operating standby elements; E_4 is a mean number of replacement and renewal organs occupied with renewal; E_5 is a mean number of replacement and renewal organs occupied with replacement; E_6 is a mean number of non-operating replacement and renewal organs.

Using the map of system state (Fig.1) for clearness, we get the formulas for calculation of mean values E_i, $i = 1, \dots, 6$.

In order to determine the mean quantity of main elements E_1, consider a random variable, the number of operating main elements.

Note that on the map of system state the nodes arranged along straight lines parallel to Oj axis correspond to each concrete value of the considered random variable. The distribution of a random variable is as follows:

0	1	2	$\cdots$	$m-2$	$m-1$	m
$\sum_{j=0}^{n+m} p_{m,j}$	$\sum_{j=0}^{n+m-1} p_{m-1,j}$	$\sum_{j=0}^{n+m-2} p_{m-2,j}$	$\cdots$	$\sum_{j=0}^{n+2} p_{2,j}$	$\sum_{j=0}^{n+1} p_{1,j}$	$\sum_{j=0}^{n} p_{0,j}$

Hence we determine $E_1 = \sum_{l=0}^{m} l \sum_{k=0}^{n+m-l} p_{m-l,k}$.

Considering the absolute priority of replacement the random variable, the number of operating standby elements is equal to $n-(j-i)$. On the map of system state the nodes arranged along the axes parallel to straight $j = i$ correspond to each concrete value of a random variable. The distribution of a random variable has the form:

0	1	$\cdots$	n	$n+1$	$\cdots$	$m+n-1$	$m+n$
$\sum_{i=0}^{m} p_{i,i+n}$	$\sum_{i=0}^{m} p_{i,i+(n-1)}$	$\cdots$	$\sum_{i=0}^{m} p_{i,i}$	$\sum_{i=1}^{m} p_{i,i-1}$	$\cdots$	$\sum_{i=m-1}^{m} p_{i,i-m+1}$	$p_{m,0}$

Then $E_2 = \sum_{l=0}^{n} l \sum_{k=0}^{m} p_{k,k+n-l} + \sum_{l=n+1}^{m+n} l \sum_{k=l-n}^{m} p_{k,k+n-l}$.

The considered standby system consists of main, operating standby and non-operating standby elements. Therefore, for the mean number of non-operating standby elements we can write $E_3 = n + m - (E_1 + E_2)$.

The renewal of the failed elements happens only if all replacement and renewal organs are not occupied with replacement. In accordance to the map of system states, write distribution of a random variable, the number of replacement and renewal organs occupied with renewal:

0	1	$\cdots$	n	$n+1$	$\cdots$	$m+n-1$	$m+n$
$\sum_{i=0}^{m} p_{i,0}$	$\sum_{i=0}^{m} p_{i,1}$	$\cdots$	$\sum_{i=0}^{m} p_{i,n}$	$\sum_{i=1}^{m} p_{i,n+1}$	$\cdots$	$\sum_{i=m-1}^{m} p_{i,n+m-1}$	$p_{m,m+n}$

Then $E_4 = \sum_{l=0}^{n} l \sum_{k=0}^{m} p_{k,l} + \sum_{l=n+1}^{m+n} l \sum_{k=l-n}^{m} p_{k,n+k}$.

In accordance to the map of states the replacement is done at any state of $s_{i,j}$j with the exception of states arranged on axes $(s_{0,0}, s_{0,n})$ and $(s_{0,n}, s_{m,m+n})$. Thus, we can write the distribution of a random variable, the number of replacement and renewal organs occupied with replacement:

0	1	2	$\cdots$	$n-1$	n	$n+1$	$n+2$	$\cdots$	$m+n$
$\sum_{j=0}^{n} p_{0,j} + \sum_{i=1}^{m} p_{i,n+i}$	$\sum_{i=1}^{m} p_{i,n+i-1}$	$\sum_{i=1}^{m} p_{i,n+i-2}$	$\cdots$	$\sum_{i=1}^{m} p_{i,i+1}$	$\sum_{i=1}^{m} p_{i,n+i}$	$\sum_{i=1}^{m} p_{i,i-1}$	$\sum_{i=2}^{m} p_{i,i-2}$	$\cdots$	$p_{m,0}$

Then the mean number of replacement and renewal organs occupied with replacement is equal to $E_5 = \sum_{l=1}^{n} l \sum_{k=1}^{m} p_{k,n+k-l} + \sum_{l=1}^{m} (n+l) \sum_{k=l}^{m} p_{k,k-l}$.

The mean number of non-operating replacement and renewal organs is $E_6 = r - (E_4 + E_5)$.

Within the theory of queuing and dependability, the initial economical characteristics r_i $(i = 1, 2)$ and c_i $(i = 1, \ldots, 6)$ are considered preset. Solving the system of linear algebraic equations (1), define probability characteristics $p_{i,j}$ and further, according to the above given formulas calculate the values of averages E_i $(i = 1, \ldots, 6)$. In these conditions the problem of mathematical programming (optimization problem) is formulated.

Namely, in case of fixed m function F, given by formula (2), depends on one argument n. Therefore, the problem is stated: for the given value m the value n is to be found which gives function F the maximum value. In other words, at the given definite number of main elements of standby system, we define optimum number of the necessary standby elements, i.e. such number when economic effect of system functioning will be maximum.

Function F is defined for natural values of argument n which in practical cases, as a rule, has no great values (not more than ten). Thus, optimization problem can be solved by simple search of all possible values of the number of standby elements n.

We have considered the problem of system functioning by the number of standby elements m, although it is easy to notice that the problem of optimization can be solved with other parameters of (2), as well.

As a conclusion we want to note that for $r < m + n$, in accordance of the above presented algorithm, the map of system states will change and accordingly it is easy to get formulas defining mean values of E_i.

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