Proceedings of I. Vekua Institute
of Applied Mathematics
Vol. 56-57, 2006-2007

EXPLICIT ESTIMATES FOR ERROR OF AVERAGED DECOMPOSITION SCHEME

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Abstract. In this work D. Gordeziani averaged differential and difference decomposition schemes for approximate solution of evolution equation are considered. Explicit a priori estimates for error of approximate solution have been obtained on the basis of semigroup approximation.

Keywords and phrases: Decomposition method, semigroup, Cauchy's abstract problem.

AMS subject classification (2000): 65M12; 65M15; 65M55.

1. Differential Schemes

Let us consider Cauchy problem in Banach space X:

$$u'(t) + Au(t) = f(t), \quad u(0) = \varphi, \quad t > 0, \tag{1.1}$$

where A is linear densely defined closed operator in X represented in the following form: $A = A_1 + A_2$; A_1 and A_2 are also densely defined closed operators in X.

We consider approximate solution of the problem (1.1) by D.Gordeziani averaged decomposition scheme. Our aim is to obtain explicit estimates for error of approximate solution. Under the explicit estimates we imply such a priori estimates for solution approximation, where constants on the right-hand side do not depend on the solution of initial continuous problem, i.e. they are absolute constants.

Different types of decomposition schemes are examined in G.Marchuk's well-known book (see [1] and extensive bibliography added to it).

D. Gordeziani averaged decomposition differential scheme for approximate solution of problem (1.1) have the form (see [2]):

$$\frac{dv_k^{(1)}(t)}{dt} + A_1 v_k^{(1)}(t) = \sigma_0 f(t), \ v_k^{(1)}(t_{k-1}) = u_{k-1}(t_{k-1}), \ u_0(0) = \varphi,$$

$$\frac{dv_k^{(2)}(t)}{dt} + A_2 v_k^{(2)}(t) = (1-\sigma_0) f(t), \qquad v_k^{(2)}(t_{k-1}) = v_k^{(1)}(t_k), \qquad (1.2)$$

$$\frac{dw_k^{(1)}(t)}{dt} + A_2 w_k^{(1)}(t) = \sigma_1 f(t),\ w_k^{(1)}(t_{k-1}) = u_{k-1}(t_{k-1}),\ u_0(0) = \varphi,$$

$$\frac{dw_k^{(2)}(t)}{dt} + A_1 w_k^{(2)}(t) = (1-\sigma_1) f(t), \qquad w_k^{(2)}(t_{k-1}) = w_k^{(1)}(t_k), \qquad (1.3)$$

$$t \in [t_{k-1}, t_k], \quad u_k(t_k) = \frac{1}{2}(v_k^{(2)}(t_k) + w_k^{(2)}(t_k)), \qquad (1.4)$$

where $k = 1, 2, \dots, t_k = k \cdot \tau, \tau > 0$ is step of time.

The following theorem takes place.

Theorem 1.1. *Assume the following conditions are fulfilled:*

(a) There exists such $\omega_0 > 0$, that for any $\lambda > \omega_0$, operator $A + \lambda I$ is invertible and the estimate is valid:

$$\|(A + \lambda I)^{-k}\| \le \frac{M}{(\lambda - \omega_0)^k}, \quad M = const > 0, \quad k = 1, 2, \cdots .$$

(b) There exists such $\omega_1 > 0$, that for any $\xi > \omega_1$, operators $A_i + \xi I,\ i = 1, 2$ are invertible and the following estimates are valid:

$$\|(A_i + \xi I)^{-1}\| \le \frac{1}{\xi - \omega_1}.$$

(c) $D(A^m) \subset D(A_i^m)$, $m = 1, 2, 3$ $(i = 1, 2)$, operators A_i map $D(A^m)$, $m = 2, 3$, in $D(A^{m-1})$ $(A_i : D(A^m) \to D(A^{m-1}))$ and the following inequalities are valid:

$$\|A_i^2 u\| + \|A_i A_{3-i} u\| \le c\|A_0^2 u\|, \quad u \in D(A^2),$$

$$\|A_i^3 u\| + \|A_i^2 A_{3-i} u\| + \|A_1 A_2 A_1 u\| \le c\|A_0^3 u\|, \quad u \in D(A^3),$$

where $A_0 = A - \lambda_0 I$, λ_0 is regular point operator of A, $c = const > 0$.

(d) $f(t)$ is continuously differentiable function and $f'(t)$ satisfies Lipschitz condition; for each fixed t from $[0; +\infty[$, $f(t) \in D(A^3), f'(t) \in D(A)$ and $\varphi \in D(A^3)$.

Then, if $\sigma_1 = 1 - \sigma_0$, for error of scheme (1.2)-(1.4) the following estimate is valid:

$$\|u(t_k) - u_k(t_k)\| \le c\tau^2 \big[e^{\omega t_k} \big(t_k \|A_0^3 \varphi\| + \int_0^{t_k} \|A_0 f'(t)\| dt$$

$$+ \tau \sum_{i=1}^{k} \big(\|A_0^2 f(t_{i-\frac{1}{2}})\| + \|A_0 f(t_{i-\frac{1}{2}})\| \big) + t_k \big) + \int_0^{t_k} (t_k - s) e^{\omega(t_k - s)} \|A_0^3 f(s)\| ds \big], \qquad (1.5)$$

where $\omega = max(\omega_0, 2\omega_1)$, $c = const > 0$.

To prove Theorem 1.1, we need two lemmas.

Lemma 1.2. *If operators A_1, A_2 and A satisfy the conditions of Theorem 1.1, then for any natural number n the following estimate is valid:*

$$\left\| \left[U(t) - \left(V\left(\frac{t}{n}\right)\right)^n \right]\varphi \right\| \le \frac{ct^3}{n^2} e^{\omega t} \|A^3\varphi\|, \quad \varphi \in D(A^3), \tag{1.6}$$

$$V(t) = \frac{1}{2}\left[U_1(t)U_2(t) + U_2(t)U_1(t)\right],$$

where $U(t) = exp(-tA)$ and $U_i(t) = exp(-tA_i)$ are strongly continuous semigroups generated by operators A and A_i $(i = 1, 2)$ respectively.

Proof. It is easy to prove that for the semigroup $U(t)$ the following expansion is valid:

$$U(t) = \sum_{k=0}^{n} (-1)^k \frac{t^k}{k!} A^k + R^{(n+1)}(t), \tag{1.7}$$

where

$$R^{(n+1)}(t) = (-A)^{n+1} \int_0^t \int_0^{s_1} \int_0^{s_2} \cdots \int_0^{s_n} U(s) ds ds_n ds_{n-1} \cdots ds_1.$$

Note that in this case, in accordance with formula (1.7), we have

$$V(\tau) = \frac{1}{2}\left[U_1(\tau)U_2(\tau) + U_2(\tau)U_1(\tau)\right]$$

$$= \frac{1}{2}\left[U_1(\tau)\left(I - \tau A_2 + \frac{\tau^2}{2}A_2^2 + R_2^{(3)}(\tau)\right) + U_2(\tau)\left(I - \tau A_1 + \frac{\tau^2}{2}A_1^2 + R_1^{(3)}(\tau)\right)\right]$$

$$= \frac{1}{2}\Big[U_1(\tau) - \tau U_1(\tau)A_2 + \frac{\tau^2}{2}U_1(\tau)A_2^2 + U_1(\tau)R_2^{(3)}(\tau)$$

$$+U_2(\tau) - \tau U_2(\tau)A_1 + \frac{\tau^2}{2}U_2(\tau)A_1^2 + U_2(\tau)R_1^{(3)}(\tau)\Big]$$

$$= \frac{1}{2}\Big[I - \tau A_1 + \frac{\tau^2}{2}A_1^2 + R_1^{(3)}(\tau) - \tau\left(I - \tau A_1 + R_1^{(2)}(\tau)\right)A_2$$

$$+\frac{\tau^2}{2}\left(I + R_1^{(1)}(\tau)\right)A_2^2 + U_1(\tau)R_2^{(3)}(\tau)$$

$$+I - \tau A_2 + \frac{\tau^2}{2}A_2^2 + R_2^{(3)}(\tau) - \tau\left(I - \tau A_2 + R_2^{(2)}(\tau)\right)A_1$$

$$+\frac{\tau^2}{2}\left(I + R_2^{(1)}(\tau)\right)A_1^2 + U_2(\tau)R_1^{(3)}(\tau)\Big]$$

$$= I - \tau(A_1 + A_2) + \frac{\tau^2}{2}(A_1^2 + A_1A_2 + A_2A_1 + A_2^2) + R_3(\tau),$$

where the remainder term $R_3(\tau)$ is $O(\tau^3)$.

In view of the last formula and (1.7) we obtain (1.6) (see in [3] proof of the analogous lemma).

Lemma 1.3. *(see[3])Assume A operator and $f(t)$ function satisfy the conditions of Theorem 1.1. Then the following estimate is valid:*

$$\|\int\limits_{t_{i-1}}^{t_i} U(t_i-s)f(s)ds-\tau U(\frac{\tau}{2})f(t_{i-\frac{1}{2}})\|$$

$$\leq c\tau^2 e^{\omega_0\tau}\big(\int\limits_{t_{i-1}}^{t_i}\|Af'(t)\|dt+\tau(\|A^2f(t_{i-\frac{1}{2}})\|+1)\big),\quad c=const>0, \quad (1.8)$$

where $U(t)=exp(-tA)$) is strongly continuous semigroup generated by operator A, $\tau=t_i-t_{i-1}$ ($t_i\geq t_{i-1}\geq 0$).

Proof of Theorem 1.1. As it is known, solution of problem (1.1), by means of semigroup $U(t)=exp(-tA)$, is expressed by the following formula (see [4],[5]):

$$u(t)=U(t)\varphi+\int\limits_0^t U(t-s)f(s)ds. \quad (1.9)$$

From the system (1.2), according to formula (1.9) we obtain

$$v_k^{(1)}(t_k)=U_1(\tau)u_{k-1}(t_{k-1})+\sigma_0\int\limits_{t_{k-1}}^{t_k}U_1(t_k-s)f(s)ds,$$

$$v_k^{(2)}(t_k)=U_2(\tau)v_k^{(1)}(t_k)+(1-\sigma_0)\int\limits_{t_{k-1}}^{t_k}U_2(t_k-s)f(s)ds.$$

From these equations it follows that

$$v_k^{(2)}(t_k)=U_2(\tau)U_1(\tau)u_{k-1}(t_{k-1})$$

$$+\sigma_0\int\limits_{t_{k-1}}^{t_k}U_2(\tau)U_1(t_k-s)f(s)ds,(1-\sigma_0)\int\limits_{t_{k-1}}^{t_k}U_2(t_k-s)f(s)ds. \quad (1.10)$$

From the system (1.3) similarly follows that

$$w_k^{(2)}(t_k)=U_1(\tau)U_2(\tau)u_{k-1}(t_{k-1})$$

$$+\sigma_1\int\limits_{t_{k-1}}^{t_k}U_1(\tau)U_2(t_k-s)f(s)ds+(1-\sigma_1)\int\limits_{t_{k-1}}^{t_k}U_1(t_k-s)f(s)ds. \quad (1.11)$$

It is clear, that according to the formulas (1.10) and (1.11), we have (see (1.4)):

$$u_k(t_k) = \frac{1}{2}\big(U_1(\tau)U_2(\tau) + U_2(\tau)U_1(\tau)\big)u_{k-1}(t_{k-1})$$

$$+\frac{1}{2}\big[\sigma_0 \int\limits_{t_{k-1}}^{t_k} U_2(\tau)U_1(t_k - s)f(s)ds + \sigma_1 \int\limits_{t_{k-1}}^{t_k} U_1(\tau)U_2(t_k - s)f(s)ds$$

$$+(1-\sigma_0) \int\limits_{t_{k-1}}^{t_k} U_2(t_k - s)f(s)ds + (1-\sigma_1) \int\limits_{t_{k-1}}^{t_k} U_1(t_k - s)f(s)ds\big]. \quad (1.12)$$

If we introduce natation:

$$V_0(\tau, t) = \frac{1}{2}\big(\sigma_0 U_2(\tau)U_1(t) + \sigma_1 U_1(\tau)U_2(t) + (1-\sigma_0)U_2(t) + (1-\sigma_1)U_1(t)\big),$$

then (1.12) will be expressed as

$$u_k(t_k) = V(\tau)u_{k-1}(t_{k-1}) + \int\limits_{t_{k-1}}^{t_k} V_0(\tau, t_k - s)f(s)ds.$$

We obtain

$$u_k(t_k) = (V(\tau))^k\varphi + \sum_{i=1}^{k} \int\limits_{t_{i-1}}^{t_i} (V(\tau))^{k-i} V_0(\tau, t_i - s)f(s)ds. \quad (1.13)$$

From the (1.9) equations it follows that

$$u(t_k) = (U(\tau))^k\varphi + \sum_{i=1}^{k} \int\limits_{t_{i-1}}^{t_i} U(t_k - s))f(s)ds. \quad (1.14)$$

It is clear that according to the formulas (1.13) and (1.14) we have

$$u(t_k) - u_k(t_k) = [(U(\tau))^k - (V(\tau))^k]\varphi$$

$$+\sum_{i=1}^{k} \int\limits_{t_{i-1}}^{t_i} [(U(\tau))^{k-i} - (V(\tau))^{k-i}]U(t_i - s)f(s)ds$$

$$+\sum_{i=1}^{k} (V(\tau))^{k-i} \int\limits_{t_{i-1}}^{t_i} [U(t_i - s) - V_0(\tau, t_i - s)]f(s)ds. \quad (1.15)$$

According to Lemma 1.2 the following estimates are obtained

$$\|[(U(\tau))^k - (V(\tau))^k]\varphi\| \le c\tau^2 t_k e^{\omega t_k}\|A_0^3\varphi\|, \quad \varphi \in D(A^3), \quad (1.16)$$

$$\sum_{i=1}^{k}\int\limits_{t_{i-1}}^{t_i}\|[(U(\tau))^{k-i}-(V(\tau))^{k-i}]U(t_i-s)f(s)\|ds$$
$$\leq c\tau^2\int\limits_{0}^{t_k}(t_k-s)e^{\omega(t_k-s)}\|A^3f(s)\|ds. \qquad (1.17)$$

Let us express the integral in second sum in (1.15) as follows:

$$\int\limits_{t_{i-1}}^{t_i}[U(t_i-s)-V_0(\tau,t_i-s)]f(s)ds=J_1+J_2+J_3,$$

where

$$J_1=\int\limits_{t_{i-1}}^{t_i}U(t_i-s)f(s)ds-\tau U(t_i-t_{i-\frac{1}{2}})f(t_{i-\frac{1}{2}}),$$
$$J_2=\tau[U(t_i-t_{i-\frac{1}{2}})-V_0(\tau,t_i-t_{i-\frac{1}{2}})]f(t_{i-\frac{1}{2}}),$$
$$J_3=\tau V_0(\tau,t_i-t_{i-\frac{1}{2}})f(t_{i-\frac{1}{2}})-\int\limits_{t_{i-1}}^{t_i}V_0(\tau,t_i-s)f(s)ds.$$

It is clear that for J_3 we have the following expression:

$$J_3=-\frac{1}{2}\big[\sigma_0J_{3,1}+\sigma_1J_{3,2}+(1-\sigma_0)J_{3,3}+(1-\sigma_1)J_{3,4}\big],$$

where

$$J_{3,1}=U_2(\tau)\Big[\int\limits_{t_{i-1}}^{t_i}U_1(t_i-s)f(s)ds-\tau U_1(t_i-t_{i-\frac{1}{2}})f(t_{i-\frac{1}{2}})\Big],$$

$$J_{3,2}=U_1(\tau)\Big[\int\limits_{t_{i-1}}^{t_i}U_2(t_i-s)f(s)ds-\tau U_2(t_i-t_{i-\frac{1}{2}})f(t_{i-\frac{1}{2}})\Big],$$

$$J_{3,3}=\int\limits_{t_{i-1}}^{t_i}U_2(t_i-s)f(s)ds-\tau U_2(t_i-t_{i-\frac{1}{2}})f(t_{i-\frac{1}{2}}),$$

$$J_{3,4}=\int\limits_{t_{i-1}}^{t_i}U_1(t_i-s)f(s)ds-\tau U_1(t_i-t_{i-\frac{1}{2}})f(t_{i-\frac{1}{2}}).$$

In accordance with Lemma1.3 the following estimates are valid:

$$\|J_{3,1}\|\leq ce^{2\omega_1\tau}\tau^2\Big[\int\limits_{t_{i-1}}^{t_i}\|A_1f'(t)\|dt+\tau(\|A_1^2f(t_{i-\frac{1}{2}})\|+1)\Big],$$

$$\|J_{3,2}\| \leq ce^{2\omega_1\tau}\tau^2\big[\int\limits_{t_{i-1}}^{t_i} \|A_2f'(t)\|dt + \tau(\|A_2^2f(t_{i-\frac{1}{2}})\| + 1)\big],$$

$$\|J_{3,3}\| \leq ce^{\omega_1\tau}\tau^2\big[\int\limits_{t_{i-1}}^{t_i} \|A_2f'(t)\|dt + \tau(\|A_2^2f(t_{i-\frac{1}{2}})\| + 1)\big],$$

$$\|J_{3,3}\| \leq ce^{\omega_1\tau}\tau^2\big[\int\limits_{t_{i-1}}^{t_i} \|A_1f'(t)\|dt + \tau(\|A_1^2f(t_{i-\frac{1}{2}})\| + 1)\big],$$

$$\|J_1\| \leq ce^{\omega_0\tau}\tau^2\big[\int\limits_{t_{i-1}}^{t_i} \|Af'(t)\|dt + \tau(\|A^2f(t_{i-\frac{1}{2}})\| + 1)\big].$$

These inequalities, according to condition (c) of Theorem 1.1 provide the following inequalities:

$$\|J_1\| \leq ce^{\omega\tau}\tau^2\big[\int\limits_{t_{i-1}}^{t_i} \|A_0f'(t)\|dt + \tau(\|A_0^2f(t_{i-\frac{1}{2}})\| + 1)\big], \tag{1.18}$$

$$\|J_3\| \leq ce^{\omega\tau}\tau^2\big[\int\limits_{t_{i-1}}^{t_i} \|A_0f'(t)\|dt + \tau(\|A_0^2f(t_{i-\frac{1}{2}})\| + 1)\big]. \tag{1.19}$$

Let's evaluate norm of J_2. According to formula (1.7) we have

$$V_0(\tau,\frac{\tau}{2}) = \frac{1}{2}\big[\sigma_0U_2(\tau)U_1(\frac{\tau}{2})+\sigma_1U_1(\tau)U_2(\frac{\tau}{2})+(1-\sigma_0)U_2(\frac{\tau}{2})+(1-\sigma_1)U_1(\frac{\tau}{2})\big]$$

$$= \frac{1}{2}\big[\sigma_0U_2(\tau)\big(I - \frac{\tau}{2}A_1 + R_1^{(2)}(\frac{\tau}{2})\big) + \sigma_1U_1(\tau)\big(I - \frac{\tau}{2}A_2 + R_2^{(2)}(\frac{\tau}{2})\big)$$

$$+(1-\sigma_0)U_2(\frac{\tau}{2}) + (1-\sigma_1)U_1(\frac{\tau}{2})\big]$$

$$= \frac{1}{2}\big[\sigma_0\big(U_2(\tau)-\frac{\tau}{2}U_2(\tau)A_1+U_2(\tau)R_1^{(2)}(\frac{\tau}{2})\big)+\sigma_1\big(U_1(\tau)-\frac{\tau}{2}U_1(\tau)A_2+U_1(\tau)R_2^{(2)}(\frac{\tau}{2})\big)$$

$$+(1-\sigma_0)U_2(\frac{\tau}{2}) + (1-\sigma_1)U_1(\frac{\tau}{2})\big]$$

$$= \frac{1}{2}\big[\sigma_0\big(I - \tau A_2 + R_2^{(2)}(\tau) - \frac{\tau}{2}(I + R_2^{(1)}(\tau))A_1 + U_2(\tau)R_1^{(2)}(\frac{\tau}{2})\big)$$

$$+\sigma_1\big(I - \tau A_1 + R_1^{(2)}(\tau) - \frac{\tau}{2}(I + R_1^{(1)}(\tau))A_2 + U_1(\tau)R_2^{(2)}(\frac{\tau}{2})\big)$$

$$+(1-\sigma_0)\big(I + \frac{\tau}{2}A_2 + R_2^{(2)}(\frac{\tau}{2})\big) + (1-\sigma_1)\big(I + \frac{\tau}{2}A_1 + R_1^{(2)}(\frac{\tau}{2})\big)\big]$$

$$= I - \frac{\tau}{2}\big(\frac{1}{2}\sigma_0 - \sigma_1 + \frac{1}{2}(1-\sigma_1)\big)A_1 - \frac{\tau}{2}\big(\sigma_0 - \frac{1}{2}\sigma_1 + \frac{1}{2}(1-\sigma_0)\big)A_2 + R_2(\tau),$$

where

$$R_2(\tau)=\sum_{j=1}^{6}R_{2,j}(\tau), \qquad (1.20)$$

$$R_{2,1}(\tau)=\frac{1}{2}(\sigma_0 U_2(\tau)+(1-\sigma_1)I)R_1^{(2)}(\frac{\tau}{2}),$$

$$R_{2,2}(\tau)=\frac{1}{2}(\sigma_1 U_1(\tau)+(1-\sigma_0)I)R_2^{(2)}(\frac{\tau}{2}),$$

$$R_{2,3}(\tau)=-\frac{\tau}{4}\sigma_0 R_2^{(1)}(\tau)A_1,\quad R_{2,4}(\tau)=-\frac{\tau}{4}{}_1R_1^{(1)}(\tau)A_2,$$

$$R_{2,5}(\tau)=\frac{1}{2}\sigma_0 R_2^{(2)}(\tau),\quad R_{2,6}(\tau)=\frac{1}{2}\sigma_1 R_1^{(2)}(\tau).$$

Thus, we obtain that

$$V_0(\tau,\frac{\tau}{2})=I-\frac{\tau}{2}\big(\frac{1}{2}\sigma_0-\sigma_1+\frac{1}{2}(1-\sigma_1)\big)A_1$$

$$-\frac{\tau}{2}\big(\sigma_0-\frac{1}{2}\sigma_1+\frac{1}{2}(1-\sigma_0)\big)A_2+R_2(\tau). \qquad (1.21)$$

For $U(\frac{\tau}{2})$, in accordance with formula (1.7), we have:

$$U(\frac{\tau}{2})=I-\frac{\tau}{2}(A_1+A_2)+R^{(2)}(\frac{\tau}{2}). \qquad (1.22)$$

On the basis of expressions (1.21) and (1.22) we make conclusion: if parameters σ_0 and σ_1 satisfy the following system

$$\frac{1}{2}\sigma_0+\sigma_1+\frac{1}{2}(1-\sigma_1)=1$$

$$\sigma_0+\frac{1}{2}\sigma_1+\frac{1}{2}(1-\sigma_0)=1,$$

then difference $U(\frac{\tau}{2})-V_0(\tau,\frac{\tau}{2})$ will be of the same order as $O(\tau^2)$. Hence $\sigma_1=1-\sigma_0$.

Thus, when , $\sigma_1=1-\sigma_0$. we have:

$$U(\frac{\tau}{2})-V_0(\tau,\frac{\tau}{2})=R_2(\tau)-R^{(2)}(\frac{\tau}{2}),$$

where $R_2(\tau)$ and $R^{(2)}(\frac{\tau}{2})$, respectively, are calculated by formulas (1.20) and (1.7).

It is clear that

$$\|(V_0(\tau,\frac{\tau}{2})-U(\frac{\tau}{2}))\varphi\|\le\sum_{j=1}^{6}\|R_{2,j}(\tau)\varphi\|+\|R^{(2)}(\frac{\tau}{2})\varphi\|. \qquad (1.23)$$

According to conditions (a) and (b) of Theorem 1.1 we have:

$$\|U(t)\|\le Me^{\omega_0 t}, \qquad (1.24)$$

$$\|U_i(t)\| \le e^{\omega_1 t}. \tag{1.25}$$

According to estimates (1.24) and (1.25) and condition (c) of Theorem 1.1 we have:

$$\|R_{2,j}(\tau)\varphi\| \le c\tau^2 e^{2\omega_1\tau}\|A_0^2\varphi\|, \ \varphi \in D(A^3), \ j = 1, \cdots, 6, \tag{1.26}$$

$$\|R^{(2)}(\frac{\tau}{2})\varphi\| \le \|A^2 \int_0^{\frac{\tau}{2}}\int_0^{s_1} U(s)\varphi ds ds_1\| \le c\tau^2 e^{\omega_0\frac{\tau}{2}}\|A_0^2\varphi\|, \ \varphi \in D(A^2). \tag{1.27}$$

From (1.23), taking into account estimates (1.26) and (1.27), we obtain

$$\|(V_0(\tau, \frac{\tau}{2}) - U(\frac{\tau}{2}))\varphi\| \le c\tau^2 e^{\omega\tau}\|A_0^2\varphi\|, \quad \varphi \in D(A^2). \tag{1.28}$$

From (1.15), taking into account estimates (1.25), (1.16), (1.17), (1.18), (1.19) and (1.28), estimates (1.5) is obtained. □

2. Difference Analogue

To find approximate solution of problem (1.1) we apply difference analogue of differential decomposition scheme (1.2) – (1.4) :

$$\frac{v_k^{(1)} - v_{k-1}^{(1)}}{\tau} + A_1\frac{v_k^{(1)} + v_{k-1}^{(1)}}{2} = \sigma_0 f(t_{k-\frac{1}{2}}), \ v_{k-1}^{(1)} = u_{k-1}, \quad u_0 = \varphi,$$

$$\frac{v_k^{(2)} - v_{k-1}^{(2)}}{\tau} + A_2\frac{v_k^{(2)} + v_{k-1}^{(2)}}{2} = (1-\sigma_0) f(t_{k-\frac{1}{2}}) \qquad v_{k-1}^{(2)} = v_k^{(1)}, \tag{2.1}$$

$$\frac{w_k^{(1)} - w_{k-1}^{(1)}}{\tau} + A_2\frac{w_k^{(1)} + w_{k-1}^{(1)}}{2} = \sigma_1 f(t_{k-\frac{1}{2}}), \ w_{k-1}^{(1)} = u_{k-1}, \quad u_0 = \varphi,$$

$$\frac{w_k^{(2)} - w_{k-1}^{(2)}}{\tau} + A_1\frac{w_k^{(2)} + w_{k-1}^{(2)}}{2} = (1-\sigma_1) f(t_{k-\frac{1}{2}}) \qquad w_{k-1}^{(2)} = w_k^{(1)}, \tag{2.2}$$

$$u_k = \frac{1}{2}\big(v_k^{(2)} + w_k^{(2)}\big). \tag{2.3}$$

The following theorem takes place:

Theorem 2.1. *Let us assume that the conditions (a),(c),(d) of Theorem 1.1 and additionally the following condition are fulfilled:*

For any $\tau > 0$, operators $I + \tau A_i$, $i = 1, 2$, are invertible and the following inequalities are valid:

$$\|(I - \tau A_i)(I + \tau A_i)^{-1}\| \le e^{\omega_1\tau}, \quad \omega_1 = const > 0,$$

$$\|(I + \tau A_i)^{-1}\| \le ce^{\omega_1\tau}, \quad c = const > 0; \tag{2.4}$$

then, if $\sigma_1 = 1 - \sigma_0$*, for error of scheme* (2.1)-(2.3) *the following estimate is valid:*

$$\|u(t_k) - u_k\| \le c\tau^2 \big[e^{\omega t_k}\big(t_k\|A_0^3\varphi\| + \int_0^{t_k} \|A_0 f'(t)\|dt +$$

$$\tau\sum_{i=1}^{k}\big(\|A_0^2 f(t_{i-\frac{1}{2}})\| + \|A_0 f(t_{i-\frac{1}{2}})\|\big) + t_k\big) + \int_0^{t_k}(t_k - s)e^{\omega(t_k-s)}\|A_0^3 f(s)\|ds\big], \tag{2.5}$$

where $\omega = max(\omega_0, 2\omega_1)$*,* $c = const > 0$*.*

Proof. From (2.1) and (2.2), respectively, the following is obtained:

$$v_k^{(2)} = S_2(\tau)S_1(\tau)v_{k-1}^{(2)} + \tau\sigma_0 S_2(\tau)L_1(\tau)f(t_{k-\frac{1}{2}}) + \tau(1-\sigma_0)L_2(\tau)f(t_{k-\frac{1}{2}}),$$

$$w_k^{(2)} = S_1(\tau)S_2(\tau)w_{k-1}^{(2)} + \tau\sigma_1 S_1(\tau)L_2(\tau)f(t_{k-\frac{1}{2}}) + \tau(1-\sigma_1)L_1(\tau)f(t_{k-\frac{1}{2}}),$$

where

$$S_i(\tau) = (I - \frac{\tau}{2}A_i)(I + \frac{\tau}{2}A_i)^{-1}, \quad L_i(\tau) = (I + \frac{\tau}{2}A_i)^{-1}, \quad i = 1, 2.$$

Consequently, according to (2.3), we have:

$$u_k = V(\tau)u_{k-1} + \tau L(\tau)f(t_{k-\frac{1}{2}}). \tag{2.6}$$

where

$$V(\tau) = \frac{1}{2}\big(S_2(\tau)S_1(\tau) + S_1(\tau)S_2(\tau)\big),$$

and

$$L(\tau) = \frac{1}{2}\big(\sigma_0 S_2(\tau)L_1(\tau) + \sigma_1 S_1(\tau)L_2(\tau) + (1-\sigma_0)L_2(\tau) + (1-\sigma_1)L_1(\tau)\big).$$

From (2.6) by induction we have:

$$u_k = (V(\tau))^k\varphi + \tau\sum_{i=1}^{k}(V(\tau))^{k-i}L(\tau)f(t_{i-\frac{1}{2}}). \tag{2.7}$$

Taking into consideration the identity

$$U(t_k - s) = U(t_{k-i})U(t_i - s) = (U(\tau))^{k-i}U(t_i - s),$$

then (1.14) could be written as:

$$u(t_k) = (U(\tau))^k\varphi + \sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}(U(\tau))^{k-i}U(t_i - s))f(s)ds. \tag{2.8}$$

According to the formulas (2.7) and (2.8) the following is obtained:

$$u(t_k) - u_k = [(U(\tau))^k - (V(\tau))^k]\varphi$$

$$+\sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} (U(\tau))^{k-i} U(t_i - s)) f(s) ds - \sum_{i=1}^{k} (V(\tau))^{k-i} L(\tau) f(t_{i-\frac{1}{2}}) ds.$$

Let us rewrite the right-hand side as:

$$u(t_k)-u_k = [(U(\tau))^k-(V(\tau))^k]\varphi+\sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} [(U(\tau))^{k-i}-(V(\tau))^{k-i}]U(t_i-s)f(s)ds$$

$$-\sum_{i=1}^{k} (V(\tau))^{k-i}\big[\tau(L(\tau) - U(\frac{\tau}{2}))f(t_{i-\frac{1}{2}})$$

$$+(\tau U(\frac{\tau}{2})f(t_{i-\frac{1}{2}}) - \int_{t_{i-1}}^{t_i} U(t_i - s)f(s)ds)\big]. \qquad (2.9)$$

Let us estimate the difference $L(\tau) - U(\frac{\tau}{2})$.
The following formulas are valid:

$$S_i(\tau) = I + R_i^{(1)}(\tau), \quad R_i^{(1)}(\tau) = -\frac{\tau}{2}A_i(I + S_i(\tau)); \qquad (2.10)$$

$$S_i(\tau) = I - \tau A_i + R_i^{(2)}(\tau), \quad R_i^{(2)}(\tau) = \frac{\tau^2}{4}A_i^2(I + S_i(\tau)); \qquad (2.11)$$

$$S_i(\tau) = I - \tau A_i + \frac{\tau^2}{4}A_i^2 + R_i^{(3)}(\tau), \; R_i^{(3)}(\tau) = -\frac{\tau^3}{8}A_i^3(I + S_i(\tau)); \qquad (2.12)$$

$$L_i(\tau) = I - \frac{\tau}{2}A_i + \frac{\tau^2}{4}A_i^2 L_i(\tau). \qquad (2.13)$$

According to the formulas (2.13) and (2.10) - (2.12), we have:

$$L(\tau) = \frac{1}{2}\big[\sigma_0 S_2(\tau)L_1(\tau) + \sigma_1 S_1(\tau)L_2(\tau) + (1 - \sigma_0)L_2(\tau) + (1 - \sigma_1)L_1(\tau)\big]$$

$$= \frac{1}{2}\big[\sigma_0 S_2(\tau)\big(I - \frac{\tau}{2}A_1 + \frac{\tau^2}{4}A_1^2 L_1(\tau))\big) + \sigma_1 S_1(\tau)\big(I - \frac{\tau}{2}A_2 + \frac{\tau^2}{4}A_2^2 L_2(\tau))\big)$$

$$+(1 - \sigma_0)\big(I - \frac{\tau}{2}A_2 + \frac{\tau^2}{4}A_2^2 L_2(\tau))\big) + (1 - \sigma_1)\big(I - \frac{\tau}{2}A_1 + \frac{\tau^2}{4}A_1^2 L_1(\tau))\big)\big]$$

$$= \frac{1}{2}\big[\sigma_0\big(S_2(\tau) - \frac{\tau}{2}S_2(\tau)A_1 + \frac{\tau^2}{4}S_2(\tau)A_1^2 L_1(\tau)\big)$$

$$+\sigma_1\big(S_1(\tau) - \frac{\tau}{2}S_1(\tau)A_2 + \frac{\tau^2}{4}S_1(\tau)A_2^2 L_2(\tau)\big)$$

$$+(1-\sigma_0)\big(I-\frac{\tau}{2}A_2+\frac{\tau^2}{4}A_2^2L_2(\tau))\big)+(1-\sigma_1)\big(I-\frac{\tau}{2}A_1+\frac{\tau^2}{4}A_1^2L_1(\tau))\big)\big]$$

$$=\frac{1}{2}\big[\sigma_0\big(I-\tau A_2+R_2^{(2)}(\tau)-\frac{\tau}{2}(I+R_2^{(1)}(\tau))A_1+\frac{\tau^2}{4}S_2(\tau)A_1^2L_1(\tau)\big)$$

$$+\sigma_1\big(I-\tau A_1+R_1^{(2)}(\tau)-\frac{\tau}{2}(I+R_1^{(1)}(\tau))A_2+\frac{\tau^2}{4}S_1(\tau)A_2^2L_2(\tau)\big)$$

$$+(1-\sigma_0)\big(I-\frac{\tau}{2}A_2+\frac{\tau^2}{4}A_2^2L_2(\tau))\big)+(1-\sigma_1)\big(I-\frac{\tau}{2}A_1+\frac{\tau^2}{4}A_1^2L_1(\tau))\big)\big]$$

$$=I-\frac{\tau}{2}\big(\frac{1}{2}\sigma_0-\sigma_1+\frac{1}{2}(1-\sigma_1)\big)A_1-\frac{\tau}{2}\big(\sigma_0-\frac{1}{2}\sigma_1+\frac{1}{2}(1-\sigma_0)\big)A_2+R_2(\tau),$$

where

$$R_2(\tau)=\sum_{j=1}^{6}R_{2,j}(\tau), \tag{2.14}$$

$$R_{2,1}(\tau)=\frac{\tau^2}{8}(\sigma_0S_2(\tau)L_1(\tau)+(1-\sigma_1)L_1(\tau))A_1^2,$$

$$R_{2,2}(\tau)=\frac{\tau^2}{8}(\sigma_1S_1(\tau)L_2(\tau)+(1-\sigma_0)L_2(\tau))A_2^2,$$

$$R_{2,3}(\tau)=-\frac{\tau}{4}\sigma_0R_2^{(1)}(\tau)A_1,\quad R_{2,4}(\tau)=-\frac{\tau}{4}\sigma_1R_1^{(1)}(\tau)A_2,$$

$$R_{2,5}(\tau)=\frac{1}{2}\sigma_0R_2^{(2)}(\tau),\quad R_{2,6}(\tau)=\frac{1}{2}\sigma_1R_1^{(2)}(\tau).$$

Hence

$$L(\tau)=I-\frac{\tau}{2}\big(\frac{1}{2}\sigma_0-\sigma_1+\frac{1}{2}(1-\sigma_1)\big)A_1$$

$$-\frac{\tau}{2}\big(\sigma_0-\frac{1}{2}\sigma_1+\frac{1}{2}(1-\sigma_0)\big)A_2+R_2(\tau), \tag{2.15}$$

For $U(\frac{\tau}{2})$, according to formula (1.7) we have:

$$U(\frac{\tau}{2})=I-\frac{\tau}{2}(A_1+A_2)+R^{(2)}(\frac{\tau}{2}). \tag{2.16}$$

On the basis of expressions (2.15) and (2.16) conclude that if parameters σ_0 and σ_1 satisfy the following system

$$\frac{1}{2}\sigma_0+\sigma_1+\frac{1}{2}(1-\sigma_1)=1$$

$$\sigma_0+\frac{1}{2}\sigma_1+\frac{1}{2}(1-\sigma_0)=1,$$

then difference $U(\frac{\tau}{2})-L(\frac{\tau}{2})$ will be of the same order as $O(\tau^2)$. Hence $\sigma_1=1-\sigma_0$.

Thus, when $\sigma_1=1-\sigma_0$, we have:

$$U(\frac{\tau}{2})-L(\tau)=R_2(\tau)-R^{(2)}(\frac{\tau}{2}),$$

where $R_2(\tau)$ and $R^{(2)}(\frac{\tau}{2})$, respectively, are calculated by formulas (2.14) and (1.7).

It is clear that inequality:

$$\|(L(\tau) - U(\frac{\tau}{2}))\varphi\| \leq \sum_{j=1}^{6} \|R_{2,j}(\tau)\varphi\| + \|R^{(2)}(\frac{\tau}{2})\varphi\|. \qquad (2.17)$$

From (2.17), taking into account evaluations (1.26) and (1.27), the following is obtained:

$$\|(L(\tau) - U(\frac{\tau}{2}))\varphi\| \leq c\tau^2 e^{\omega\tau} \|A_0^2\varphi\|, \quad \varphi \in D(A^2). \qquad (2.18)$$

According to (2.4)inequality, we have:

$$\|V(\tau)\| = \frac{1}{2}\big\|S_2(\tau)S_1(\tau) + S_1(\tau)S_2(\tau)\big\| \leq e^{2\omega_1\tau}. \qquad (2.19)$$

Analogously to (1.16) we obtain

$$\|[(U(\tau))^k - (V(\tau))^k]\varphi\| \leq c\tau^2 t_k e^{\omega t_k} \|A_0^3\varphi\|, \quad \varphi \in D(A^3). \qquad (2.20)$$

From (2.9), taking into account estimates (1.18), (1.24), (2.18), (2.19) and (2.20), estimate (2.5) is obtained. □

The estimates, given in Theorems 1.1 and 1.2, are obtained earlier and appeared in [6].

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Received 18.06.2007; revised 17.09.2007; accepted 5.12.2007.