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LONG TIME BEHAVIOR OF SOLUTIONS TO NONLINEAR INTEGRO-DIFFERENTIAL EQUATION

Jangveladze T., Kiguradze Z.

*Iv. Javakhishvili Tbilisi State University,
I. Vekua Institute of Applied Mathametics,
Tbilisi State University,
2 University Str., 0186 Tbilisi, Georgia
e.mail: tjangv@.yahoo.com*
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Abstract

The large time asymptotic behavior of solutions to the nonlinear integro-differential equation associated with the penetration of a magnetic field into a substance is studied. The rates of convergence are given too.

Key words and phrases: nonlinear integro-differential equation, magnetic field, asymptotic behavior.
AMS subject classification: 35K55, 45K05

1. Introduction

Process of diffusion of the magnetic field into a substance is modelled by Maxwell's system of partial differential equations [1]. As it is shown in [2] if the coefficient of thermal heat capacity and electroconductivity of the substance depend on temperature, then in the simple case the Maxwell's system can be rewritten in the following form

$$\frac{\partial U}{\partial t} = \frac{\partial}{\partial x}\left[a\left(\int_0^t \left(\frac{\partial U}{\partial x}\right)^2 d\tau\right)\frac{\partial U}{\partial x}\right], \tag{1.1}$$

where function $a = a(S)$ is defined for $S \in [0, \infty)$.

Note that the integro-differential equation (1.1) is complex and still yields to the investigation only for particular cases [2-8].

The existence, uniqueness and asymptotic behavior of the solutions of the initial-boundary value problems for the equations of type (1.1) in suitable classes are studied in the works [2-12] and in a number of other works as well. The existence theorems, that are proved in [2-4], [8] are based on a priori estimates, Galerkin's method and compactness arguments as in [11], [12] for nonlinear parabolic equations. Investigations for multidimensional space cases at first are carried out in the work [4].

In the work [7] is proposed some generalization of the equations of type (1.1)

$$\frac{\partial U}{\partial t} = a\left(\int_0^t \int_0^1 \left(\frac{\partial U}{\partial x}\right)^2 dx d\tau\right) \frac{\partial^2 U}{\partial x^2}. \tag{1.2}$$

The purpose of this note is to study the asymptotic behavior of solutions of the equation (1.2). Our object is to give large time asymptotic behavior as $t \to \infty$ of the solutions of the first boundary value problems for the equation (1.2). The attention is paid on the case $a(S) = (1+S)^p$, $p > 0$. The solvability, uniqueness and asymptotic behavior to the solutions of (1.2) type models are studied in [8-10]. The rest of the paper is organized as follows. In the second section we discuss the initial-boundary value problem with zero lateral boundary data. Section three is devoted to the study of the problem with non zero boundary data in part of lateral boundary.

Mathematical results, that are given below, show difference between stabilization rates of solutions with homogeneous and nonhomogeneous boundary conditions.

2. The problem with zero boundary conditions

In the domain $Q = (0,1) \times (0,\infty)$ let us consider the following initial-boundary value problem:

$$\frac{\partial U}{\partial t} = a(S)\frac{\partial^2 U}{\partial x^2}, \quad (x,t) \in Q, \tag{2.1}$$

$$U(0,t) = U(1,t) = 0, \quad t \ge 0, \tag{2.2}$$

$$U(x,0) = U_0(x), \quad x \in [0,1], \tag{2.3}$$

where

$$S(t) = \int_0^t \int_0^1 \left(\frac{\partial U}{\partial x}\right)^2 dx d\tau,$$

$a(S) = (1+S)^p$, $p > 0$; $U_0 = U_0(x)$ is given function.

The existence and uniqueness of the solution of such problems in suitable classes have been proved in [8].

Now we are going to estimate the solution of the problem (2.1)-(2.3). Let us introduce usual L_2-inner product and norm:

$$(u,v) = \int_0^1 u(x)v(x)dx, \quad \|u\| = (u,u)^{1/2}.$$

Denote by $W_2^k(0,1)$ and $\overset{o}{W}{}_2^k(0,1)$ the usual Sobolev spaces of real functions on (0,1) which first k derivatives are square integrable.

Theorem 2.1. *If* $U_0 \in \overset{o}{W}{}_2^1(0,1)$, *then for the solution of the problem* (2.1)-(2.3) *the following estimate is true*

$$\|U\| + \left\| \frac{\partial U}{\partial x} \right\| \le C \exp\left(-\frac{t}{2} \right).$$

Here and below C denotes positive constant independent from t.

Proof. Let us multiply the equation (2.1) by U and integrate over $(0,1)$. After integrating by parts using the boundary conditions (2.2) we get

$$\frac{1}{2}\frac{d}{dt}\|U\|^2 + \int\limits_0^1 (1+S)^p \left(\frac{\partial U}{\partial x} \right)^2 dx = 0.$$

From this, taking into account the relation $(1+S)^p \ge 1$ and Poincare's inequality, we obtain

$$\frac{1}{2}\frac{d}{dt}\|U\|^2 + \left\| \frac{\partial U}{\partial x} \right\|^2 \le 0, \quad \frac{1}{2}\frac{d}{dt}\|U\|^2 + \|U\|^2 \le 0. \tag{2.4}$$

Let us multiply the equation (2.1) scalarly by $\partial^2 U/\partial x^2$. Using again formula of integrating by parts and boundary conditions (2.2) we get

$$\left. \frac{\partial U}{\partial t}\frac{\partial U}{\partial x} \right|_0^1 - \int\limits_0^1 \frac{\partial^2 U}{\partial x \partial t}\frac{\partial U}{\partial x} dx = \int\limits_0^1 (1+S)^p \left(\frac{\partial^2 U}{\partial x^2} \right)^2 dx,$$

$$\frac{1}{2}\frac{d}{dt}\left\| \frac{\partial U}{\partial x} \right\|^2 + \left\| \frac{\partial^2 U}{\partial x^2} \right\|^2 \le 0,$$

or

$$\frac{d}{dt}\left\| \frac{\partial U}{\partial x} \right\|^2 \le 0. \tag{2.5}$$

From (2.4) and (2.5) we receive

$$\frac{d}{dt}\left[\exp(t) \left(\|U\|^2 + \left\| \frac{\partial U}{\partial x} \right\|^2 \right) \right] \le 0.$$

This inequality immediately proves Theorem 2.1.

Note that Theorem 2.1 gives exponential stabilization of the solution of the problem (2.1)-(2.3) in the norm of the space $W_2^1(0,1)$. Let us show that the stabilization is also achieved in the norm of the space $C^1(0,1)$. In particular, let us show that the following statement takes place.

Theorem 2.2. *If* $U_0 \in W_2^2(0,1) \cap \overset{o}{W}{}_2^1(0,1)$, *then for the solution of the problem* (2.1)-(2.3) *the following relation holds*

$$\left| \frac{\partial U(x,t)}{\partial x} \right| \le C \exp\left(-\frac{t}{2} \right).$$

At first let us prove an auxiliary statement.

Lemma 2.1. *For the solution of the problem* (2.1)-(2.3) *the following estimate is true*

$$\left\| \frac{\partial U(x,t)}{\partial t} \right\| \le C \exp\left(-\frac{t}{2}\right).$$

Proof. Let us differentiate equation (2.1) with respect to t

$$\frac{\partial^2 U}{\partial t^2} = (1+S)^p \frac{\partial^3 U}{\partial x^2 \partial t} + p(1+S)^{p-1} \int_0^1 \left(\frac{\partial U}{\partial x}\right)^2 dx \frac{\partial^2 U}{\partial x^2}$$

and multiply scalarly by $\partial U/\partial t$. We deduce

$$\frac{1}{2}\frac{d}{dt}\int_0^1 \left(\frac{\partial U}{\partial t}\right)^2 dx + (1+S)^p \int_0^1 \left(\frac{\partial^2 U}{\partial x \partial t}\right)^2 dx +$$

$$+p(1+S)^{p-1} \int_0^1 \left(\frac{\partial U}{\partial x}\right)^2 dx \int_0^1 \frac{\partial U}{\partial x}\frac{\partial^2 U}{\partial x \partial t} dx = 0.$$

After simple transformations we have

$$\frac{d}{dt}\int_0^1 \left(\frac{\partial U}{\partial t}\right)^2 dx + (1+S)^p \int_0^1 \left(\frac{\partial U}{\partial t}\right)^2 dx \le p^2 (1+S)^{p-2} \left[\int_0^1 \left(\frac{\partial U}{\partial x}\right)^2 dx\right]^3. \tag{2.6}$$

Note that Theorem 2.1 helps us to estimate function S

$$S(t) = \int_0^t \int_0^1 \left(\frac{\partial U}{\partial x}\right)^2 dx d\tau = \int_0^t \left\|\frac{\partial U}{\partial x}\right\|^2 d\tau \le C \int_0^t \exp(-\tau) d\tau \le C.$$

So, we have

$$1 \le 1 + S(t) \le C. \tag{2.7}$$

Combining Theorem 2.1 and relations (2.6), (2.7) we arrive at

$$\frac{d}{dt}\left(\exp(t) \left\|\frac{\partial U}{\partial t}\right\|^2\right) \le C \exp(-2t). \tag{2.8}$$

Let us integrate (2.8) from 0 to t

$$\exp(t) \left\|\frac{\partial U}{\partial t}\right\|^2 \le C \int_0^t \exp(-2\tau) d\tau.$$

Therefore, Lemma 2.1 is proved.

Now, let us estimate $\partial^2 U/\partial x^2$ in the space $L_1(0,1)$. From (2.1) we have

$$\frac{\partial^2 U}{\partial x^2} = (1+S)^{-p}\frac{\partial U}{\partial t}.$$

So, applying Lemma 2.1 and (2.7) we derive

$$\int_0^1 \left|\frac{\partial^2 U}{\partial x^2}\right| dx \le \left[\int_0^1 (1+S)^{-2p} dx\right]^{1/2} \left[\int_0^1 \left|\frac{\partial U}{\partial t}\right|^2 dx\right]^{1/2} \le C\exp\left(-\frac{t}{2}\right).$$

From this, taking into account the relation

$$\frac{\partial U(x,t)}{\partial x} = \int_0^1 \frac{\partial U(y,t)}{\partial y} dy + \int_0^1 \int_y^x \frac{\partial^2 U(\xi,t)}{\partial \xi^2} d\xi dy$$

and the boundary conditions (2.2), it follows that

$$\left|\frac{\partial U(x,t)}{\partial x}\right| = \left|\int_0^1 \int_y^x \frac{\partial^2 U(\xi,t)}{\partial \xi^2} d\xi dy\right| \le \int_0^1 \left|\frac{\partial^2 U(y,t)}{\partial y^2}\right| dy \le C\exp\left(-\frac{t}{2}\right).$$

So, Theorem 2.2 is proved.

3. The problem with non zero data on one side of lateral boundary

In the domain Q let us consider the following initial-boundary value problem:

$$\frac{\partial U}{\partial t} = a(S)\frac{\partial^2 U}{\partial x^2}, \quad (x,t)\in Q, \tag{3.1}$$

$$U(0,t) = 0, \quad U(1,t) = \psi, \quad t \ge 0, \tag{3.2}$$

$$U(x,0) = U_0(x), \quad x\in[0,1], \tag{3.3}$$

where

$$S(t) = \int_0^t \int_0^1 \left(\frac{\partial U}{\partial x}\right)^2 dx d\tau, \tag{3.4}$$

$a(S) = (1+S)^p$, $p>0$, $\psi = Const > 0$; $U_0 = U_0(x)$ is given function.

The main purpose of this section is to prove the following statement.

Theorem 3.1. *If $U_0 \in W_2^2(0,1)$, $U_0(0) = 0$, $U_0(1) = \psi$, then for the solution of the problem* (3.1)-(3.4) *the following estimate is true*

$$\left|\frac{\partial U(x,t)}{\partial x} - \psi\right| \le Ct^{-1-p}, \quad t\ge 1.$$

Before we proceed to the proof of the Theorem 3.1, we establish some auxiliary lemmas.

Lemma 3.1. *The following estimates are true:*

$$\varphi^{\frac{1}{1+2p}}(t) \le 1 + S(t) \le C\varphi(t)^{\frac{1}{1+2p}}, \quad t \ge 0,$$

where

$$\varphi(t) = 1 + \int_0^t \int_0^1 (1+S)^{2p} \left(\frac{\partial U}{\partial x}\right)^2 dx d\tau. \tag{3.5}$$

Here and below C denotes again positive constants independent from t.

Proof. From (3.4) it follows that

$$\frac{dS}{dt} = \int_0^1 \left(\frac{\partial U}{\partial x}\right)^2 dx, \quad S(0) = 0. \tag{3.6}$$

Let us multiply equation (3.6) on $(1+S)^{2p}$ and introduce following notation

$$\sigma = (1+S)^p \frac{\partial U}{\partial x}.$$

We have

$$\frac{1}{1+2p} \frac{dS^{1+2p}}{dt} = \int_0^1 \sigma^2 dx.$$

Integrating this equation on $(0,t)$ we arrive at

$$\frac{1}{1+2p}(1+S)^{1+2p} = \int_0^t \int_0^1 \sigma^2 dx d\tau + \frac{1}{1+2p}.$$

Note that $0 < \dfrac{1}{1+2p} < 1$. So, we get

$$\varphi^{\frac{1}{1+2p}}(t) \le 1 + S(t) \le \left[(1+2p)\varphi(t)\right]^{\frac{1}{1+2p}}.$$

Lemma 3.1 is proved.

Lemma 3.2. *The following estimates are true*

$$\psi^2 \varphi^{\frac{2p}{1+2p}}(t) \le \int_0^1 \sigma^2 dx \le C\varphi^{\frac{2p}{1+2p}}(t), \ \ t \ge 0.$$

Proof. Taking into account Lemma 3.1 we get

$$\int_0^1 \sigma^2 dx = \int_0^1 (1+S)^{2p} \left(\frac{\partial U}{\partial x}\right)^2 dx \ge \varphi^{\frac{2p}{1+2p}}(t) \int_0^1 \left(\frac{\partial U}{\partial x}\right)^2 dx \ge$$

$$\geq \varphi^{\frac{2p}{1+2p}}(t)\left[\int_0^1 \frac{\partial U}{\partial x}dx\right]^2 = \psi^2\varphi^{\frac{2p}{1+2p}}(t),$$

or

$$\int_0^1 \sigma^2 dx \geq \psi^2\varphi^{\frac{2p}{1+2p}}(t). \tag{3.7}$$

Let's multiply equation (3.1) scalarly by $(1+S)^{-p}\partial U/\partial t$. Using formula of integrating by parts and boundary conditions (3.2) we have

$$\int_0^1 (1+S)^{-p}\left(\frac{\partial U}{\partial t}\right)^2 dx + \frac{1}{2}\frac{d}{dt}\int_0^1 \left(\frac{\partial U}{\partial x}\right)^2 dx = 0.$$

After integrating from 0 to t we arrive at

$$\int_0^t\int_0^1 (1+S)^{-p}\left(\frac{\partial U}{\partial t}\right)^2 dx d\tau + \frac{1}{2}\int_0^1 \left(\frac{\partial U}{\partial x}\right)^2 dx = C.$$

From this we get

$$\int_0^1 \left(\frac{\partial U}{\partial x}\right)^2 dx \leq C. \tag{3.8}$$

Using (3.8) and Lemma 3.1 we conclude

$$\int_0^1 \sigma^2 dx = (1+S)^{2p}\int_0^1 \left(\frac{\partial U}{\partial x}\right)^2 dx \leq C\varphi^{\frac{2p}{1+2p}}(t).$$

Now taking into account (3.7) from the last inequality the prove of the Lemma 3.2 is over.

From Lemma 3.2 and relation (3.5) we have following estimates:

$$\psi^2\varphi^{\frac{2p}{1+2p}}(t) \leq \frac{d\varphi(t)}{dt} \leq C\varphi^{\frac{2p}{1+2p}}(t). \tag{3.9}$$

Lemma 3.3. *$\partial U/\partial t$ satisfy the inequality*

$$\int_0^1 \left(\frac{\partial U}{\partial t}\right)^2 dx \leq C\varphi^{-\frac{2}{1+2p}}(t), \quad t \geq 0.$$

Proof. From (2.6), using Lemma 3.1 and relation (3.8), we get

$$\frac{d}{dt}\int_0^1 \left(\frac{\partial U}{\partial t}\right)^2 dx + \varphi^{\frac{p}{1+2p}}(t)\int_0^1 \left(\frac{\partial U}{\partial t}\right)^2 dx \leq C\varphi^{\frac{p-2}{1+2p}}(t).$$

Using Gronwall's inequality we have

$$\int_0^1 \left(\frac{\partial U}{\partial t}\right)^2 dx \le \exp\left(-\int_0^t \varphi^{\frac{p}{1+2p}}(\tau)d\tau\right)\left[\int_0^1 \left(\frac{\partial U}{\partial t}\right)^2 dx\bigg|_{t=0} + \right.$$
$$\left. + C\int_0^t \exp\left(\int_0^\tau \varphi^{\frac{p}{1+2p}}(\xi)d\xi\right)\varphi^{\frac{p-2}{1+2p}}(\tau)d\tau\right]. \tag{3.10}$$

Noting that $\varphi(t) \ge 1$, applying L'Hopital's rule and estimate (3.9), we have

$$\lim_{t\to\infty} \frac{\int_0^t \exp\left(\int_0^\tau \varphi^{\frac{p}{1+2p}}(\xi)d\xi\right)\varphi^{\frac{p-2}{1+2p}}(\tau)d\tau}{\exp\left(\int_0^t \varphi^{\frac{p}{1+2p}}(\tau)d\tau\right)\varphi^{-\frac{2}{1+2p}}(t)} =$$

$$= \lim_{t\to\infty} \frac{\exp\left(\int_0^t \varphi^{\frac{p}{1+2p}}(\tau)d\tau\right)\varphi^{\frac{p-2}{1+2p}}(t)}{\exp\left(\int_0^t \varphi^{\frac{p}{1+2p}}(\tau)d\tau\right)\left(\varphi^{\frac{p-2}{1+2p}}(t) - \frac{2}{1+2p}\varphi^{\frac{-3-2p}{1+2p}}(t)\frac{d\varphi}{dt}\right)} \le \tag{3.11}$$
$$\le \lim_{t\to\infty} \frac{1}{1 - \frac{C}{1+2p}\varphi^{-\frac{p+1}{1+2p}}(t)} \le C.$$

Therefore, Lemma 3.2 follows from (3.10) and (3.11).

Now according to the method applying in the section 2, taking into account Lemmas 3.1 and 3.3, we derive

$$\left|\frac{\partial U(x,t)}{\partial x} - \psi\right| = \left|\int_0^1 \int_y^x \frac{\partial^2 U(\xi,t)}{\partial \xi^2} d\xi dy\right| \le \int_0^1 \left|\frac{\partial^2 U(x,t)}{\partial x^2}\right| dx \le$$

$$\le \int_0^1 \left|\frac{\partial U}{\partial t}(1+S)^{-p}\right| dx \le \left[\int_0^1 (1+S)^{-2p} dx\right]^{1/2} \left[\int_0^1 \left|\frac{\partial U}{\partial t}\right|^2 dx\right]^{1/2} \le$$
$$\le C\varphi^{-\frac{p}{1+2p}}(t)\varphi^{-\frac{1}{1+2p}}(t) = C\varphi^{-\frac{p+1}{1+2p}}(t).$$

Hence, we have

$$\left|\frac{\partial U(x,t)}{\partial x} - \psi\right| \le C\varphi^{-\frac{p+1}{1+2p}}(t). \tag{3.12}$$

After integrating from (3.9) it is easy to show that

$$\psi^2 t^{1+2p} \le \varphi(t) \le Ct^{1+2p}, \quad t \ge 1.$$

From this taking into account estimate (3.12) we receive validity of the Theorem 3.1.

For the adiabatic shearing of incompressible fluids with temperature-dependent viscosity the results similar to the Theorem 3.1 are obtained in [13].

The existence of a globally defined solutions of the problems (2.1)-(2.3) and (3.1)-(3.3) can now be reobtained by a routine procedure. One first establishes the existence of the local solutions on a maximal time interval and then inferring from the estimates that this solution cannot escape in a finite time.

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