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INVESTIGATION ON ONE NONLINEAR PROBLEM OF ISOTROPIC PLATE BY I. VEKUA'S METHOD FOR APPROXIMATION N=1

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Abstract

In the present paper a boundary value problem of a cylindrical body for St Venant-Kirchhoff materials is considered. This problem is reduced to the two-dimensional problem by I. Vekua's method on the midsurface of the plate. The obtained problem is investigated by the implicit function theorem for approximation N=1 .

Key words and phrases: boundary value problem, St Venant Venant-Kirchhoff materials, Implicit function theorem .

AMS subject classification: 35G30, 74B20.

The question of existence of solutions of the nonlinear boundary value problem of three-dimensional elasticity can be approached in two ways:

In one approach, it is assumed that the material is hyperelastic, so that particular solutions are obtained as minimizers of the energy over a set of admissible deformations with appropriate smoothness [1];

Another approach represents applying the implicit function theorem directly to the boundary value problem of three-dimensional elasticity ([2], [3]).

Let $(\mathbf{e}_i)$ denote the basis of the Euclidean space $\mathbb{R}^3$, and let ω be a domain in plane spanned by the vectors $\mathbf{e}_\alpha$. We define the sets

$$\Omega^h := \omega \times]-h,h[, \quad \Gamma := \partial\omega \times]-h,h[, \quad \Gamma_+ := \omega \times \{h\}, \quad \Gamma_- := \omega \times \{-h\},$$

$$\partial_j := \frac{\partial}{\partial x_j}, \quad h = const > 0.$$

Under repeating indexes we mean sumation, the Latin letters taking the values 1,2,3 and the Greek one - 1,2. Ω^h is cilindre, which thickness $2h$.

Let Ω^h consist St Venant-Kirchhoff materials [3]. Consider the three-dimensional boundary value problem with a vector of displacement $\mathbf{u} = (u_1, u_2, u_3)$

$$
\begin{cases}
-\partial_j(\sigma_{ij} + \sigma_{kj}\partial_k u_i) = f_i \quad \text{in} \ \Omega^{\mathrm{h}}, \\
u_i = 0 \quad \text{on} \ \Gamma, \\
\sigma_{i3} + \sigma_{k3}\partial_k u_i = t_{i3}^{+} \quad \text{on} \ \Gamma_{+}, \\
\sigma_{i3} + \sigma_{k3}\partial_k u_i = t_{i3}^{-} \quad \text{on} \ \Gamma_{-},
\end{cases} \tag{1}
$$

where

$$\sigma_{ij} = \lambda E_{pp}(\mathbf{u})\delta_{ij} + 2\mu E_{ij}(\mathbf{u}), \tag{2}$$

$$E_{ij}(\mathbf{u}) = \frac{1}{2}\left(\partial_i u_j + \partial_j u_i + \partial_i u_m \partial_j u_m\right), \tag{3}$$

σ_{ij} - are the components of the second Piola-Kirchhoff stress tensor, E_{ij} - are the components of the Green-St Venant strain tensor, f_i - is the given density per unit volume of the applied body forces, t_{i3}^{+}, t_{i3}^{-} are given functions on upper and lower plane, E_{ij} are the components of the Creen-St Venant strain tensor, $\lambda > 0$ and $\mu > 0$ are the Lame's constants, δ_{ij}-is Kroneker symbol.

Problem (1) may be written with respect to the first Piola-Kirchhoff stress tensor, the components t_{ij} are connected with σ_{ij} by the following form

$$t_{ij} = \sigma_{kj}(\delta_{ik} + \partial_k u_i) = \sigma_{ij} + \sigma_{kj}\partial_k u_i. \tag{4}$$

$$
\begin{cases}
-(\partial_\alpha t_{i\alpha} + \partial_3 t_{i3}) = f_i \quad \text{in} \ \Omega^{\mathrm{h}}, \\
u_i = 0, \quad \text{on} \ \Gamma, \\
t_{i3}(x_1, x_2, \pm h) = t_{i3}^{\pm}, \quad \text{on} \ \Gamma_{+} \ \text{and} \ \Gamma_{-}.
\end{cases} \tag{1$'$}
$$

As well known, the components of the linearized strain have the form

$$e_{ij} = \frac{1}{2}(\partial_i u_j + \partial_j u_i), \tag{5}$$

then the formulas (2) and (3) may be written as follows

$$\sigma_{ij} = a_{ijpq}\left(e_{pq} + \frac{1}{2}\partial_p u_k \partial_q u_k\right), \tag{2$'$}$$

$$E_{ij} = e_{ij} + \partial_i u_k \partial_j u_k, \tag{3$'$}$$

where

$$a_{ijpq} = \lambda\delta_{ij}\delta_{pq} + \mu(\delta_{ip}\delta_{jq} + \delta_{iq}\delta_{jp}).$$

If $(2')$ substitute into (4) we get

$$t_{ij} = a_{ijpq}e_{pq}(\mathbf{u}) + N_{ij}(\mathbf{u}), \tag{6}$$

where $N_{ij}(\mathbf{u})$ is the following nonlinear term

$$N_{ij}(\mathbf{u}) := \frac{1}{2} a_{ijpq} \partial_p u_m \partial_q u_m + a_{kjpq} \partial_p u_q \partial_k u_i + \frac{1}{2} a_{kjpq} \partial_p u_m \partial_q u_m \partial_k u_i. \quad (7)$$

The three-dimensional problem $(1^{'})$ will be reduced to the two-dimensional one by I.Vekua's method on the midsurface of the plate ω ([4], [5], [6], [13]). For this both side of equation $(1^{'})$ multiply on the following functions

$$\frac{2m+1}{2h} P_m \left(\frac{x_3}{h} \right), \quad m = 0, 1, ..., \quad (8)$$

where P_m - are Legandre polinoms of order m and integrate it from $-h$ to h with respect to x_3.

$$- \left\{ \partial_\alpha \overset{(m)}{t}_{i\alpha} + \frac{2m+1}{2h} \int_{-h}^{h} \partial_3 t_{i3} P_m \left(\frac{x_3}{h} \right) dx_3 \right\} = \overset{(m)}{f}_i, \quad m = 0, 1, 2, ..., \quad (9)$$

where

$$\begin{aligned} \overset{(m)}{t}_{ij}(x_1, x_2) &:= \frac{2m+1}{2h} \int_{-h}^{h} t_{ij}(x_1, x_2, x_3) P_m \left(\frac{x_3}{h} \right) dx_3, \\ \overset{(m)}{f}_i(x_1, x_2) &:= \frac{2m+1}{2h} \int_{-h}^{h} f_i(x_1, x_2, x_3) P_m \left(\frac{x_3}{h} \right) dx_3. \end{aligned} \quad (10)$$

Take into account that the functions (8) are complete in $L^2(]-1,1[)$, the infinite system (9) is formal equivalent to $(1^{'})$.

Integration by parts of (9) and using following formula

$$P_m^{'}(x) = (2m-1) P_{m-1}(x) + (2m-5) P_{m-3}(x) + ..., \quad P_m(\pm 1) = (\pm 1)^m,$$
$$m = 0, 1, ...,$$

we obtain

$$\begin{aligned} & - \left\{ \partial_\alpha \overset{(m)}{t}_{i\alpha} - \frac{2m+1}{h} \left(\overset{(m-1)}{t}_{i3} + \overset{(m-3)}{t}_{i3} + \cdots \right) \right\} \\ & = \overset{(m)}{f}_i + \frac{2m+1}{2h} \left(t_{i3}^{+} - (-1)^m t_{i3}^{-} \right), \quad m = 0, 1, ... \end{aligned} \quad (11)$$

From (6) by use (10) we get

$$\overset{(m)}{t}_{ij} = a_{ijpq} \overset{(m)}{e}_{pq}(u) + \overset{(m)}{N}_{ij}, \quad m = 0, 1, ..., \quad (12)$$

where

$$\left(\overset{(m)}{e}_{ij},\ \ \overset{(m)}{N}_{ij}\right) := \frac{2m+1}{2h}\int\limits_{-h}^{h}(e_{ij},\ \ N_{ij})P_m\left(\frac{x_3}{h}\right)dx_3,\ \ m=0,1,\dots .$$

From (5) for the quantite $\overset{(m)}{e}_{ij}$ we get the following formulas

$$\begin{aligned}
\overset{(m)}{e}_{\alpha\beta} &= \frac{1}{2}\left(\partial_\alpha \overset{(m)}{u}_{\beta} + \partial_\beta \overset{(m)}{u}_{\alpha}\right),\\
\overset{(m)}{e}_{\alpha 3} &= \frac{1}{2}\left(\partial_\alpha \overset{(m)}{u}_{3} + \frac{1}{h}\overset{(m)}{u}{}'_{\alpha}\right),\\
\overset{(m)}{e}_{33} &= \frac{1}{h}\overset{(m)}{u}{}'_{3},
\end{aligned} \tag{13}$$

where

$$\begin{aligned}
\overset{(m)}{u}_{j} &= \frac{2m+1}{2h}\int\limits_{-h}^{h} u_j P_m\left(\frac{x_3}{h}\right)dx_3,\ \ m=0,1,...,\\
\overset{(m)}{u}{}'_{j} &= (2m+1)\left(\overset{(m+1)}{u}_{j} + \overset{(m+3)}{u}_{j} + \cdots\right),\ \ \ m=0,1,....
\end{aligned}$$

From (7) for $\overset{(m)}{N}_{ij}$, $\ m=0,1,...,$ we get [13]

$$\begin{aligned}
\overset{(m)}{N}_{ij} \ =\ & \sum_{m_1,m_2=0}^{\infty}\sum_{r_1=0}^{m_1}\alpha_{m_1 m_2 r_1}\left\{\left(\frac{1}{2}a_{ij\alpha\beta}\partial_\alpha \overset{(m_1)}{u}_{l}\,\partial_\beta \overset{(m_2)}{u}_{l} + \frac{1}{h}a_{ij\alpha 3}\partial_\alpha \overset{(m_1)}{u}_{l}\overset{(m_2)}{u}{}'_{l}\right.\right.\\
+\ & \frac{1}{2h^2}a_{ij33}\overset{(m_1)}{u}{}'_{l}\overset{(m_2)}{u}{}'_{l} + a_{\alpha j\beta q}\partial_\beta \overset{(m_1)}{u}_{q}\partial_\alpha \overset{(m_2)}{u}_{i} + \frac{1}{h}a_{\alpha j3q}\overset{(m_1)}{u}_{q}\partial_\alpha \overset{(m_2)}{u}_{i}\\
+\ & \left.\frac{1}{h}a_{3j\beta q}\partial_\beta \overset{(m_1)}{u}_{q}\overset{(m_2)}{u}{}'_{i} + \frac{1}{h^2}a_{3j3q}\overset{(m_1)}{u}{}'_{q}\overset{(m_2)}{u}{}'_{i}\right)\delta_m^{m_1+m_2-2r_1}\\
+\ & \sum_{m_3=0}^{\infty}\sum_{r_2=0}^{m_3}\alpha_{m_3 m r_2}\left(\frac{1}{2}a_{\alpha j\beta\gamma}\partial_\beta \overset{(m_1)}{u}_{l}\,\partial_\gamma \overset{(m_2)}{u}_{l}\partial_\alpha \overset{(m_3)}{u}_{i}\right.\\
+\ & \frac{1}{2h}a_{3j\beta\gamma}\partial_\beta \overset{(m_1)}{u}_{l}\partial_\gamma \overset{(m_2)}{u}_{l}\overset{(m_3)}{u}{}'_{l} + \frac{1}{h^2}a_{3j3\gamma}\overset{(m_1)}{u}{}'_{l}\partial_\gamma \overset{(m_2)}{u}_{l}\overset{(m_3)}{u}{}'_{i}\\
+\ & \frac{1}{h}a_{\alpha j3\gamma}\overset{(m_1)}{u}{}'_{l}\partial_\gamma \overset{(m_2)}{u}_{l}\partial_\alpha \overset{(m_3)}{u}_{i} + \frac{1}{2h^2}a_{\alpha j33}\overset{(m_1)}{u}{}'_{l}\overset{(m_2)}{u}{}'_{l}\partial_\alpha \overset{(m_3)}{u}_{i}\\
+\ & \left.\left.\frac{1}{2h^3}a_{3j33}\overset{(m_1)}{u}{}'_{l}\overset{(m_2)}{u}{}'_{l}\overset{(m_3)}{u}{}'_{i}\right)\frac{(2m+1)\delta_{m+m_3-2r_2}^{m_1+m_2-2r_1}}{2(m+m_3-2r_2)+1}\right\}.
\end{aligned} \tag{14}$$

where

$$\alpha_{mnr} = \frac{A_{m-r}A_rA_{n-r}}{A_{m+n-r}}\frac{2(m+n)-4r+1}{2(m+n)-2r+1}, \quad A_m = \frac{(2m-1)!!}{m!}.$$

By substituting (12) into (11) we obtain

$$- \frac{1}{2m+1}\left\{\partial_\alpha\left[a_{i\alpha pq}\overset{(m)}{e}_{pq} + \overset{(m)}{N}_{i\alpha}\right] - \frac{2m+1}{h}\left[a_{i3pq}\left(\overset{(m-1)}{e}_{pq} + \overset{(m-3)}{e}_{pq} + \cdots\right) + \overset{(m-1)}{N}_{i3} + \overset{(m-3)}{N}_{i3} + \cdots\right]\right\} = \overset{(m)}{\varphi}_i, \tag{15}$$

where

$$\overset{(m)}{\varphi}_i := \frac{1}{2m+1}\overset{(m)}{f}_i + \frac{1}{2h}\left(t_{i3}^+ - (-1)^m t_{i3}^-\right). \tag{16}$$

Assume, that the displacement vector $\mathbf{u}$ is polinomial of order N with respect to coordinate x_3, where N is any non-negativ integer number

$$\mathbf{u}(x_1,x_2,x_3) \approx \sum_{m=0}^{N}\overset{(m)}{\boldsymbol{u}}(x_1,x_2)P_m\left(\frac{x_3}{h}\right).$$

Assume, that $\overset{(k)}{F} = 0$ if $k > N$, or $k < 0$.

Introduce the following notation

$$\overset{1}{\boldsymbol{u}} := \left(\overset{(0)}{u}_1, \overset{(0)}{u}_2, \overset{(0)}{u}_3, \overset{(1)}{u}_1, \overset{(1)}{u}_2, \overset{(1)}{u}_3\right) = \sum_{m=0}^{1}\overset{(m)}{u}_i\overset{m}{\boldsymbol{e}}_i,$$

where $(\overset{m}{\boldsymbol{e}}_i)$, $m = 0,1$ - are components of the basis vector of the 6-dimensional Euclidian space

$$\begin{aligned}
\overset{0}{\boldsymbol{e}}_1 &= (1,0,0,0,0,0),\\
\overset{0}{\boldsymbol{e}}_2 &= (0,1,0,0,0,0),\\
\overset{0}{\boldsymbol{e}}_3 &= (0,0,1,0,0,0),\\
\overset{1}{\boldsymbol{e}}_1 &= (0,0,0,1,0,0),\\
\overset{1}{\boldsymbol{e}}_2 &= (0,0,0,0,1,0),\\
\overset{1}{\boldsymbol{e}}_3 &= (0,0,0,0,0,1).
\end{aligned}$$

Hence we get six equations with six unknowns

$$\begin{cases}
-\sum\limits_{m=0}^{1}\frac{1}{2m+1}\left\{\partial_\alpha\left[a_{i\alpha pq}\overset{(m)}{e}_{pq}(\overset{1}{\boldsymbol{u}}) + \overset{(m)}{N}_{i\alpha}\right] - \frac{2m+1}{h}\right.\\
\left.\times\left[a_{i3pq}\overset{(m-1)}{e}_{pq}(\overset{1}{\boldsymbol{u}}) + \overset{(m-1)}{N}_{i3}\right]\right\}\overset{m}{\boldsymbol{e}}_i = \sum\limits_{m=0}^{1}\overset{(m)}{\varphi}_i\overset{m}{\boldsymbol{e}}_i \quad \text{in } \omega,\\
\overset{1}{\boldsymbol{u}} = \sum\limits_{m=0}^{1}\overset{(m)}{u}_i\overset{m}{\boldsymbol{e}}_i = 0 \quad \text{on } \partial\omega.
\end{cases} \tag{17}$$

Problem (17) we can be written by the following form

$$\begin{cases} -\sum_{m=0}^{1} \frac{1}{2m+1} \left\{ \partial_\alpha \left(\lambda \overset{(m)}{e}_{pp} (\overset{1}{\boldsymbol{u}}) \delta_{i\alpha} + 2\mu \overset{(m)}{e}_{i\alpha} (\overset{1}{\boldsymbol{u}}) + \overset{(m)}{N}_{i\alpha} \right) \right. \\ \left. - \frac{2m+1}{h} \left[\lambda \overset{(m-1)}{e}_{pp} (\overset{1}{\boldsymbol{u}}) \delta_{i3} + 2\mu \overset{(m-1)}{e}_{i3} (\overset{1}{\boldsymbol{u}}) + \overset{(m-1)}{N}_{i3} \right] \right\} \overset{m}{\boldsymbol{e}}_i \\ = \overset{1}{\boldsymbol{\varphi}} := \sum_{m=0}^{1} \overset{(m)}{\varphi}_i \overset{m}{\boldsymbol{e}}_i \quad \text{in } \omega, \\ \overset{1}{\boldsymbol{u}} = \sum_{m=0}^{1} \overset{(m)}{u}_i \overset{m}{\boldsymbol{e}}_i = 0 \quad \text{on } \partial\omega. \end{cases}$$

Introduce the following notation

$$\begin{aligned} \mathbf{A}(\mathbf{v}) := -\sum_{m=0}^{1} \frac{1}{2m+1} \left\{ \partial_\alpha \left[a_{i\alpha pq} \overset{(m)}{e}_{pq} (\mathbf{v}) + \overset{(m)}{N}_{i\alpha} \right] \right. \\ \left. - \frac{2m+1}{h} \left[a_{i3pq} \overset{(m-1)}{e}_{pq} (\mathbf{v}) + \overset{(m-1)}{N}_{i3} \right] \right\} \overset{m}{\boldsymbol{e}}_i . \end{aligned} \tag{18}$$

Lemma 1 . *There exists a neighborhood $\mathbf{V}(\mathbf{0})$ of the origin in the space $\mathbf{W}(\omega) := \left(W^{2,p}(\omega) \cap W_0^{1,p}(\omega)\right)^6$, $p > 2$, such that*

$$\mathbf{v} \in \mathbf{V}(\mathbf{0}) \quad \Rightarrow A(\mathbf{v}) \in \mathbf{F}(\omega)$$

where $\mathbf{F}(\omega)$ is a some neighborhood of the origin in the space $(L^p(\omega))^6$, $p >$ > 2. Futhermore the operator $\mathbf{A}$ is differentiable at $\mathbf{v} = 0$ and the action of the Frechet derivative $\mathbf{A}^{'}(\mathbf{0})$ on an arbritrary element $\mathbf{v} \in \mathbf{W}(\omega)$ is given by the formula

$$\begin{aligned} \mathbf{A}^{'}(\mathbf{0})\mathbf{v} = -\sum_{m=0}^{1} \frac{1}{2m+1} \{ \partial_\alpha \left[a_{i\alpha pq} \overset{(m)}{e}_{pq} (\mathbf{v}) \right] \\ - \frac{2m+1}{h} \left[a_{i3pq} \overset{(m-1)}{e}_{pq} (\mathbf{v}) \right] \right\} \overset{m}{\boldsymbol{e}}_i . \end{aligned} \tag{19}$$

Proof. The $\left(W^{1,p}(\omega)\right)^6$ is a Banach algebra for $p > 2$. [3]. As a consequence, the operator $\mathbf{A}$, maps the any element $\mathbf{v} \in \mathbf{V}(0)$ into the subset of the space $(L^p(\omega))^6$, $p > 2$.

In order to compute $\mathbf{A}^{'}(\mathbf{0})\mathbf{v}$, it suffices, to compute the terms that are linear with respect to $\mathbf{v}$ in the difference $\{\mathbf{A}(\mathbf{v}) - \mathbf{A}(\mathbf{0})\}$ (this follows from the definition of the Frechet derivative)

$$\mathbf{A}(\mathbf{v})-\mathbf{A}(\mathbf{0})=-\sum_{m=0}^{1}\frac{1}{2m+1}\left\{\partial_\alpha\left[a_{i\alpha pq}\overset{(m)}{e}_{pq}(\mathbf{v})\right]\right.$$

$$\left.-\frac{2m+1}{h}a_{i3pq}\overset{(m-1)}{e}_{pq}(\mathbf{v})\right\}\overset{m}{\boldsymbol{e}}_i+o(\|\mathbf{v}\|_{\mathbf{W}(\omega)})\quad\text{in }\ \mathrm{F}(\omega).$$

and the assertion follows. ∎

In order to use the implicit function theorem, consider the following linearized boundary value problem.

Let $\overset{1}{\boldsymbol{\varphi}}=\left(\overset{(0)}{\varphi}_1,\overset{(0)}{\varphi}_2,\overset{(0)}{\varphi}_3,\overset{(1)}{\varphi}_1,\overset{(1)}{\varphi}_2,\overset{(1)}{\varphi}_3\right)\in(L^p(\omega))^6,\quad p>2$ is given function. Find $\overset{1}{\boldsymbol{u}}\in\mathbf{W}(\omega)$ such that

$$\mathbf{A}'(\mathbf{0})\,\overset{1}{\boldsymbol{u}}=\overset{1}{\boldsymbol{\varphi}},\tag{20}$$

or

$$\begin{cases}-\sum_{m=0}^{1}\frac{1}{2m+1}\left\{\partial_\alpha\left[\lambda\overset{(m)}{e}_{pp}(\overset{1}{\boldsymbol{u}})\delta_{i\alpha}+2\mu\overset{(m)}{e}_{i\alpha}(\overset{1}{\boldsymbol{u}})\right]\right.\\ \left.-\frac{2m+1}{h}\left[\lambda\overset{(m-1)}{e}_{pp}(\overset{1}{\boldsymbol{u}})\delta_{i3}+2\mu\overset{(m-1)}{e}_{i3}(\overset{1}{\boldsymbol{u}})\right]\right\}\overset{m}{\boldsymbol{e}}_i=\overset{1}{\boldsymbol{\varphi}}\quad\text{in }\ \omega,\\ \overset{1}{\boldsymbol{u}}=0\quad\text{on }\ \partial\omega.\end{cases}\tag{20'}$$

Lemma 2. *The linear problem (20) is equivalent to finding a solution* $\overset{1}{\boldsymbol{u}}\in\mathbf{W}(\omega)$ *of the following variational problem*

$$\mathbf{B}(\overset{1}{\boldsymbol{u}},\mathbf{v})=\mathbf{L}(\mathbf{v})\ \text{ forall }\ \mathbf{v}\in\mathbf{V},\tag{21}$$

where

$$\mathbf{B}(\overset{1}{\boldsymbol{u}},\mathbf{v}):=\sum_{m=0}^{1}\frac{1}{2m+1}\int_\omega\left\{\lambda\overset{(m)}{e}_{pp}(\overset{1}{\boldsymbol{u}})\overset{(m)}{e}_{qq}(\mathbf{v})+2\mu\overset{(m)}{e}_{ij}(\overset{1}{\boldsymbol{u}})\overset{(m)}{e}_{ij}(\mathbf{v})\right\}dx,$$

$$\mathbf{L}(\mathbf{v}):=\int_\omega\overset{1}{\boldsymbol{\varphi}}\cdot\mathbf{v}dx,\quad(dx=dx_1dx_2).\tag{22}$$

Proof. Both side of the equation $(20')$ multiply on the vector $\mathbf{v} \in \mathbf{W}(\omega)$, to integrate and using the Green's formula we deduce that

$$
\begin{aligned}
- & \sum_{m=0}^{1} \frac{1}{2m+1} \int_{\omega} \Big\{ \lambda \Big[\partial_\alpha \overset{(m)}{e}{}_{pp}(\overset{1}{\boldsymbol{u}})\delta_{i\alpha} - \frac{2m+1}{h} \overset{(m-1)}{e}{}_{pp}(\overset{1}{\boldsymbol{u}})\delta_{i3} \Big] \\
+ & 2\mu \Big[\partial_\alpha \overset{(m)}{e}{}_{i\alpha}(\overset{1}{\boldsymbol{u}}) - \frac{2m+1}{h} \overset{(m-1)}{e}{}_{i3}(\overset{1}{\boldsymbol{u}}) \Big] \Big\} \overset{(m)}{v}{}_i dx \\
= & \sum_{m=0}^{1} \frac{1}{2m+1} \int_{\omega} \Big\{ \lambda \overset{(m)}{e}{}_{pp}(\overset{1}{\boldsymbol{u}}) \Big(\partial_\alpha \overset{(m)}{v}{}_\alpha + \frac{1}{h} \overset{(m)}{v}{}'_3 \Big) \\
+ & 2\mu \Big[\overset{(m)}{e}{}_{\alpha\beta}(\overset{1}{\boldsymbol{u}}) \frac{1}{2} \Big(\partial_\alpha \overset{(m)}{v}{}_\beta + \partial_\beta \overset{(m)}{v}{}_\alpha \Big) \\
+ & \overset{(m)}{e}{}_{\alpha 3}(\overset{1}{\boldsymbol{u}}) \Big(\partial_\alpha \overset{(m)}{v}{}_3 + \frac{1}{h} \overset{(m)}{v}{}'_\alpha \Big) + \overset{(m)}{e}{}_{33}(\overset{1}{\boldsymbol{u}}) \frac{1}{h} \overset{(m)}{v}{}'_3 \Big] \Big\} dx \\
= & \int_{\omega} \overset{N}{\boldsymbol{\varphi}} \cdot \mathbf{v} dx, \qquad \overset{(m)}{v}{}'_k = \begin{cases} \overset{(1)}{v}{}_j, & \text{when } \mathrm{m} = 0, \\ 0, & \text{when } \mathrm{m} = 1. \end{cases}
\end{aligned}
$$

Hence by use the correspondence (13) we get (21).

Conversely, let $\overset{1}{\boldsymbol{u}}$ is a solution of (21). Using the Green's formula again and taking into account the space $\mathbf{W}(\omega)$ is dense in $(L^p(\omega))^6$ we obtain, that $\overset{1}{\boldsymbol{u}}$ is a solution of problem (20). ∎

Lemma 3. *The bilinear form $\mathbf{B}(\mathbf{u}, \mathbf{v})$ given by the formula (22) is continuous in the space $(H^1(\omega))^6$ with respect to norm $\| \cdot \|_{1,\omega}$, i.e. exists a constant β such that*

$$
\mathbf{B}(\mathbf{u}, \mathbf{v}) \leq \beta \|\mathbf{u}\|_{1,\omega} \|\mathbf{v}\|_{1,\omega} \text{ forall } \mathbf{u}, \mathbf{v} \in (\mathrm{H}^1(\omega))^6, \tag{23}
$$

where

$$
\|\mathbf{v}\|_{1,\omega} := \left\{ \sum_{m=0}^{1} \int_{\omega} \left(\overset{(m)}{v}{}_j \overset{(m)}{v}{}_j + \partial_\alpha \overset{(m)}{v}{}_j \partial_\alpha \overset{(m)}{v}{}_j \right) dx \right\}^{\frac{1}{2}}.
$$

Proof. From the definition bilinear form $\mathbf{B}(\mathbf{u}, \mathbf{v})$, to take into consideration the formula (13) we have

$$\mathbf{B}(\mathbf{u},\mathbf{v}) = \sum_{m=0}^{1}\int\limits_{\omega}\frac{1}{2m+1}\left\{\lambda\left[\partial_\gamma \overset{(m)}{u}_\gamma \partial_\eta \overset{(m)}{u}_\eta + \frac{1}{h}\partial_\gamma \overset{(m)}{u}_\gamma \overset{(m)}{v}{}'_3\right.\right.$$

$$\left.+\frac{1}{h}\partial_\eta \overset{(m)}{v}_\eta \overset{(m)}{u}{}'_3\right] + \frac{\mu}{2}\left(\partial_\alpha \overset{(m)}{u}_\beta + \partial_\beta \overset{(m)}{u}_\alpha\right)\left(\partial_\alpha \overset{(m)}{v}_\beta + \partial_\beta \overset{(m)}{v}_\alpha\right)$$

$$\left.+\mu\left(\partial_\alpha \overset{(m)}{u}_3 + \frac{1}{h}\overset{(m)}{u}{}'_\alpha\right)\left(\partial_\alpha \overset{(m)}{v}_3 + \frac{1}{h}\overset{(m)}{v}{}'_\alpha\right) + \frac{\lambda+2\mu}{h^2}\overset{(m)}{u}{}'_3 \overset{(m)}{v}{}'_3\right\} dx.$$

Now using Cauchy-Schwarz inequality, we obtain exists a constant $k > 0$ such that

$$\begin{aligned}\mathbf{B}(\mathbf{u},\mathbf{v}) \;&\leq\; k\sum_{m=0}^{1}\sum_{i=1}^{3}\left[\left(\int\limits_{\omega}\overset{(m)}{u}{}_i^2 dx\right)^{\frac{1}{2}} + \sum_{\alpha=1}^{2}\left(\int\limits_{\omega}(\partial_\alpha \overset{(m)}{u}_i)^2 dx\right)^{\frac{1}{2}}\right]\\ &\times\;\sum_{l=0}^{1}\sum_{j=1}^{3}\left[\left(\int\limits_{\omega}\overset{(l)}{v}{}_j^2 dx\right)^{\frac{1}{2}} + \sum_{\beta=1}^{2}\left(\int\limits_{\omega}(\partial_\beta \overset{(l)}{v}_j)^2 dx\right)^{\frac{1}{2}}\right]\end{aligned}$$

for all $\mathbf{u},\ \mathbf{v} \in (H^1(\omega))^6$, it follows the inequality (23). ∎

We show that holds a first generalized Korn's inequality ([7], [8]), i.e.

Lemma 4. *There exists a constant $c > 0$ such that*

$$\|\mathbf{v}\|_{1,\omega} \leq c\left\{\sum_{m=0}^{1}\int\limits_{\omega}\overset{(m)}{e}_{ij}\overset{(m)}{e}_{ij}dx\right\}^{\frac{1}{2}}, \qquad (24)$$

for all $\mathbf{v} = (\overset{(0)}{v}_1, \overset{(0)}{v}_2, \overset{(0)}{v}_3, \overset{(1)}{v}_1, \overset{(1)}{v}_2, \overset{(1)}{v}_3) \in (H^1_0(\omega))^6$

Proof. For fixed $m = 0, 1$ are used following notations

$$\begin{aligned}\mathbf{B}_m(\mathbf{v},\mathbf{v}) \;&:=\; \int\limits_{\omega}\overset{(m)}{e}_{ij}(\mathbf{v})\overset{(m)}{e}_{ij}(\mathbf{v})dx = \int\limits_{\omega}\overset{(m)}{e}_{\alpha\beta}(\mathbf{v})\overset{(m)}{e}_{\alpha\beta}(\mathbf{v})dx\\ &+\;\; 2\int\limits_{\omega}\overset{(m)}{e}_{\alpha 3}(\mathbf{v})\overset{(m)}{e}_{\alpha 3}(\mathbf{v})dx + \int\limits_{\omega}\overset{(m)}{e}_{33}(\mathbf{v})\overset{(m)}{e}_{33}(\mathbf{v})dx,\\ \mathbf{B}^1_m(\mathbf{v},\mathbf{v}) \;&:=\; \int\limits_{\omega}\overset{(m)}{e}_{\alpha\beta}(\mathbf{v})\overset{(m)}{e}_{\alpha\beta}(\mathbf{v})dx,\\ \mathbf{B}^2_m(\mathbf{v},\mathbf{v}) \;&:=\; 2\int\limits_{\omega}\overset{(m)}{e}_{\alpha 3}(\mathbf{v})\overset{(m)}{e}_{\alpha 3}(\mathbf{v})dx,\end{aligned}$$

$$\mathbf{B}^3_m(\mathbf{v},\mathbf{v}) := \int\limits_\omega \overset{(m)}{e}_{33}(\mathbf{v})\overset{(m)}{e}_{33}(\mathbf{v})dx,$$

$$\mathbf{B}_m(\mathbf{v},\mathbf{v}) = \mathbf{B}^1_m(\mathbf{v},\mathbf{v})+\mathbf{B}^2_m(\mathbf{v},\mathbf{v})+\mathbf{B}^3_m(\mathbf{v},\mathbf{v}).$$

By Korn's inequality ($\overset{(m)}{v}_\gamma \in H^1_0(\omega)$), exists the number $k_1 > 0$ such that

$$\mathbf{B}^1_m(\mathbf{u},\mathbf{v}) \geq k_1 \int\limits_\omega \left(\overset{(m)}{v}_\gamma \overset{(m)}{v}_\gamma + \partial_\gamma \overset{(m)}{v}_\alpha \partial_\gamma \overset{(m)}{v}_\alpha\right) dx$$

for all $\overset{(m)}{v}_\beta \in H^1_0(\omega)$.

$$\begin{aligned}\mathbf{B}^2_m(\mathbf{v},\mathbf{v}) &= \frac{1}{2}\int \left(\partial_\alpha \overset{(m)}{v}_3 + \frac{1}{h}\overset{(m)}{v}{}'_\alpha\right)\left(\partial_\alpha \overset{(m)}{v}_3 + \frac{1}{h}\overset{(m)}{v}{}'_\alpha\right) dx \\ &\geq \frac{1-\varepsilon}{2}\int\limits_\omega \partial_\alpha \overset{(m)}{v}_3 \partial_\alpha \overset{(m)}{v}_3 dx - \frac{1-\varepsilon}{2\varepsilon h^2}\int\limits_\omega \overset{(m)}{v}{}'_\alpha \overset{(m)}{v}{}'_\alpha dx\end{aligned}$$

for any $\varepsilon > 0$ (here using the inequality $(a+b)^2 \geq (1-\varepsilon)a^2 - \dfrac{1-\varepsilon}{\varepsilon}b^2$).

$$\int\limits_\omega \overset{(m)}{v}{}'_\alpha \overset{(m)}{v}{}'_\alpha dx \leq \sum_{l=0}^{1}\int\limits_\omega \overset{(l)}{v}_\gamma \overset{(l)}{v}_\gamma dx, \quad m=0,1.$$

Hence

$$\mathbf{B}^2_m(\mathbf{v},\mathbf{v}) \geq \frac{1-\varepsilon}{2}\int\limits_\omega \partial_\alpha \overset{(m)}{v}_3 \partial_\alpha \overset{(m)}{v}_3 dx - \frac{1-\varepsilon}{2\varepsilon h^2}\sum_{l=0}^{1}\int\limits_\omega \overset{(l)}{v}_\alpha \overset{(l)}{v}_\alpha dx.$$

It follows

$$\begin{aligned}\mathbf{B}_m(\mathbf{v},\mathbf{v}) \geq k_1 \int\limits_\omega (\overset{(m)}{v}_\alpha \overset{(m)}{v}_\alpha + \partial_\gamma \overset{(m)}{v}_\alpha \partial_\gamma \overset{(m)}{v}_\alpha)dx + \frac{1-\varepsilon}{2}\int\limits_\omega \partial_\alpha \overset{(m)}{v}_3 \partial_\alpha \overset{(m)}{v}_3 dx \\ -\frac{1-\varepsilon}{2\varepsilon h^2}\sum_{l=0}^{1}\int\limits_\omega \overset{(l)}{v}_\alpha \overset{(l)}{v}_\alpha dx.\end{aligned}$$

Let $0<\varepsilon<1$. From the last inequality we have

$$\begin{aligned}\mathbf{B}_0(\mathbf{v},\mathbf{v})+\mathbf{B}_1(\mathbf{v},\mathbf{v}) \geq \sum_{m=0}^{1}\Bigg\{k_1\int\limits_\omega \partial_\gamma \overset{(m)}{v}_\alpha \partial_\gamma \overset{(m)}{v}_\alpha dx + \frac{1-\varepsilon}{2}k_1 \\ \times \int\limits_\omega \partial_\alpha \overset{(m)}{v}_3\, \partial_\alpha \overset{(m)}{v}_3 dx + \left(k_1 - \frac{1-\varepsilon}{2\varepsilon h^2}\right)\int\limits_\omega \overset{(m)}{v}_\alpha \overset{(m)}{v}_\alpha dx\Bigg\}.\end{aligned} \tag{25}$$

$$k_1 - \frac{1-\varepsilon}{2\varepsilon h^2} > 0.$$

since $\overset{(m)}{v}_3 \in H_0^1(\omega)$, hence holds the Fridrix inequality.

$$\int\limits_{\omega} (\overset{(m)}{v}_3)^2 dx \le k_2 \int\limits_{\omega} \partial_\alpha \overset{(m)}{v}_3 \partial_\alpha \overset{(m)}{v}_3 dx.$$

From (25) taking into account the last two inequalites we get (24). ∎

From (22) and (24) we have

$$\mathbf{B}(\mathbf{v},\mathbf{v}) \ge \frac{2\mu}{3} \sum_{m=0}^{1} \int\limits_{\omega} \overset{(m)}{e}_{ij}(\mathbf{v}) \overset{(m)}{e}_{ij}(\mathbf{v}) dx \ge \frac{2\mu}{3^2} \|\mathbf{v}\|_{1,\omega} \tag{26}$$

for all $\mathbf{v} \in (H_0^1(\omega))^6$.

Theorem 1. (existence of a weak solution). *Let the* $\overset{1}{\boldsymbol{\varphi}} \in (L^p(\omega))^6$, $p > 1$ *be a given function. Then there is one and only function* $\overset{1}{\boldsymbol{u}} \in$ $\in (H_0^1(\omega))^6$, *that satisfies*

$$\mathbf{B}(\overset{N}{\boldsymbol{u}}, \mathbf{v}) = \mathbf{L}(\mathbf{v}) \ \text{ forall } \ \mathbf{v} \in (\mathrm{H}_0^1(\omega))^6,$$

also

$$\| \overset{1}{\boldsymbol{u}} \|_{1,\omega} \le C \| \overset{1}{\boldsymbol{\varphi}} \|_{0,p,\omega},$$

where $C > 0$ *is independent from* $\overset{1}{\boldsymbol{\varphi}}$ $\overset{1}{\boldsymbol{u}}$.
In addition

$$\mathbf{J}(\overset{1}{\boldsymbol{u}}) = \inf_{\mathbf{v} \in (H_0^1(\omega))^6} \mathbf{J}(\mathbf{v}), \ \text{ where } \ \mathbf{J}(\mathbf{v}) = \frac{1}{2}\mathbf{B}(\mathbf{v},\mathbf{v}) - \mathbf{L}(\mathbf{v}).$$

Proof. The Sobolev imbedding theorem imply that the linear form $\mathbf{L}$ is cuntinuous on the space $(H_0^1(\omega))^6$ if $\overset{1}{\boldsymbol{\varphi}} \in (L^p(\omega))^6$, $p > 1$. The symmetric form $\mathbf{B}$ is continuous and $(H_0^1(\omega))^6$-elliptic by inequalites (23) and (26). Hence the conclusion follows by the Riesz representation theorem [9]. ∎

The weak solution possesses additional regularity if the boundary $\partial\omega$ and the right-hand side $\overset{1}{\boldsymbol{\varphi}}$ also possess additional regularity.

Theorem 2. *Let* ω *be a domain* $\mathbb{R}^2$ *with a boundary* $\partial\omega$ *of class* $\mathcal{C}^2$, *let* $\overset{1}{\boldsymbol{\varphi}} \in (L^p(\omega))^6$, $p > 1$. *Then the weak solution* $\overset{1}{\boldsymbol{u}} \in (H_0^1(\omega))^6$ *of the linearized pure displacement problem is in the space* $(W^{2,p}(\omega))^6$ *and it satisfies*

$$-\sum_{m=0}^{1} \frac{1}{2m+1} \left\{ \partial_\alpha \left[a_{i\alpha pq} \overset{(m)}{e}_{pq}(\overset{1}{\boldsymbol{u}}) \right] - \frac{2m+1}{h} a_{i3pq} \overset{(m-1)}{e}_{pq}(\overset{1}{\boldsymbol{u}}) \right\} \overset{m}{\boldsymbol{e}}_i$$
$$= \overset{1}{\boldsymbol{\varphi}} \ \text{ in } \ (\mathrm{L}^{\mathrm{p}}(\omega))^6.$$

Proof. Because the linear operator $\mathbf{A}^{'}(\mathbf{0})$ is strongly elliptic, the implication

$$\overset{1}{\boldsymbol{\varphi}} \in (L^p(\omega))^6 \Rightarrow \overset{1}{\boldsymbol{u}} \in (H^2(\omega))^6 \cap (H_0^1(\omega))^6,$$

holds it the boundary $\partial\omega$ is of class $\mathcal{C}^2$ [10]. Hence the announced regularity holds for $p = 2$.

Because the linearized problem is uniformly elliptic and satisfies the comlpementing conditions ([11], [12]). That the mapping

$$\mathbf{A}^{'}(\mathbf{0}): \quad \mathbf{v} \in \mathbf{V}^p := \left\{ \mathbf{v} \in (W^{2,p}(\omega))^6; \;\; \mathbf{v} = \mathbf{0} \;\text{ on }\; \partial\omega \right\} \to$$

$$-\sum_{m=0}^{1} \frac{1}{2m+1} \left\{ \partial_\alpha \left[a_{i\alpha pq} \overset{(m)}{e}_{pq}(\overset{1}{\boldsymbol{u}}) \right] - \frac{2m+1}{h} a_{i3pq} \overset{(m-1)}{e}_{pq}(\overset{1}{\boldsymbol{u}}) \right\} \overset{m}{\boldsymbol{e}}_i$$

$$\in (L^p(\omega))^6$$

has an index $ind\mathbf{A}^{'}(\mathbf{0})$ that is independent of $p \in]1, \infty[$. In our case, we know that $ind\mathbf{A}^{'}(\mathbf{0}) = 0$ for $p = 2$, since $\mathbf{A}^{'}(\mathbf{0})$ is a bijection in this case.

Since $\mathbf{V^p}(\omega)$ is continuously imbedded in $H_0^1(\omega))^6$, i.e. $\mathbf{V^p}(\omega) \hookrightarrow (H_0^1(\omega))^6$ for $p \geq 1$ [9], $\mathbf{A}^{'}(\mathbf{0}): \;\; \mathbf{V^p}(\omega) \to (L^p(\omega))^6$ of p; hence $dimKer\mathbf{A}^{'}(\mathbf{0}) = \mathbf{0}$. Since $ind\mathbf{A}^{'}(\mathbf{0}) = 0$ on the other hand, the mapping $\mathbf{A}^{'}(\mathbf{0})$, is also surjective in this case. Hence the regularity result holds for $p > 1$.

The weak solution $\overset{1}{\boldsymbol{u}} \in (W^{2,p}(\omega))^6 \cap (H_0^1(\omega))^6$ satisfies

$$\int_\omega \sum_{m=0}^{1} \frac{1}{2m+1} \left\{ \lambda \overset{(m)}{e}_{pp}(\overset{1}{\boldsymbol{u}}) \overset{(m)}{e}_{qq}(\mathbf{v}) + 2\mu \overset{(m)}{e}_{ij}(\overset{1}{\boldsymbol{u}}) \overset{(m)}{e}_{ij}(\mathbf{v}) \right\} dx = \int_\omega \overset{1}{\boldsymbol{\varphi}} \cdot \mathbf{v} dx$$

for all $\mathbf{v} \in (D(\omega))^6$, ($D(\omega)$ denote the space of functions whose support is a compact subset of ω. Hence we can apply the Green formula to the left-hand sides; this gives

$$\int_\omega \sum_{m=0}^{1} \frac{1}{2m+1} \left\{ \lambda \overset{(m)}{e}_{pp}(\overset{1}{\boldsymbol{u}}) \overset{(m)}{e}_{qq}(\mathbf{v}) + 2\mu \overset{(m)}{e}_{ij}(\overset{1}{\boldsymbol{u}}) \overset{(m)}{e}_{ij}(\mathbf{v}) \right\} dx$$

$$= \; -\int_\omega \sum_{m=0}^{1} \frac{1}{2m+1} \left\{ \partial_\alpha \left[\lambda \overset{(m)}{e}_{pp}(\overset{1}{\boldsymbol{u}}) \delta_{i\alpha} + 2\mu \overset{(m)}{e}_{i\alpha}(\overset{1}{\boldsymbol{u}}) \right] \right.$$

$$\left. - \frac{2m+1}{h} \left[\lambda \overset{(m-1)}{e}_{pp}(\overset{1}{\boldsymbol{u}}) \delta_{i3} + 2\mu \overset{(m-1)}{e}_{i3}(\overset{1}{\boldsymbol{u}}) \right] \right\} \overset{(m)}{v}_i dx.$$

and the conclusion follows since $\{D(\omega)\}^{-} = L^p(\omega)$. ∎

The above nonlinear problem can be converted into a problem: Find a vector field $\overset{1}{\boldsymbol{u}}: \;\; \overline{\omega} \to \mathbb{R}^6$, that satisfies

$$\left\{ \begin{array}{l} \mathbf{A}(\overset{1}{\boldsymbol{u}}) = \overset{1}{\boldsymbol{\varphi}} \;\text{ in }\; \omega, \\ \overset{1}{\boldsymbol{u}} = \mathbf{0} \;\text{ on }\; \partial\omega, \end{array} \right. \tag{27}$$

where operator $\mathbf{A}$ is defined by the formula (18).

Theorem 3. *Let ω be a domain $\mathbb{R}^2$ with a boundary $\partial\omega$ of class $\mathcal{C}^2$, and assume*

$$\overset{(m)}{t}_{ij}(\overset{1}{\boldsymbol{u}}) = a_{ijpq}\overset{(m)}{e}_{pq}(\overset{1}{\boldsymbol{u}}) + \overset{(m)}{N}_{ij}(\overset{1}{\boldsymbol{u}}), \quad m = 0, 1.$$

Then for each number $p > 2$ there exists a neighborhood $\mathbf{F}^p$ of the origin in the space $(L^p(\omega))^6$ and a neighborhood $\mathbf{U}^p$ of the origin in the space

$$\mathbf{V}(\omega) := \left\{\mathbf{v} \in (W^{2,p}(\omega))^6, \ \ \mathbf{v} = \mathbf{0} \ \text{ on } \ \partial\omega\right\},$$

such that, for each $\overset{1}{\boldsymbol{\varphi}} \in \mathbf{F}^p$ the boundary value problem (25) has exactly one solution $\overset{1}{\boldsymbol{u}} \in \mathbf{U}^p$.

Proof. The Sobolev space $(W^{1,p}(\omega))^6$ is a Banach algebra for $p > 3$. As a consequence, the nonlinear operator $\mathbf{A}$ maps the space $(W^{2,p}(\omega))^6$ into the space $(L^p(\omega))^6$ and it is infinitely differentiable between these two spaces, since it is a sum of continuous linear, bilinear and trilinear mappings (hence all it's derivatives of order ≥ 4 vanish).

Since $\overset{1}{\boldsymbol{u}} = \mathbf{0}$ is clearly a solution of problem (25) corresponding to $\overset{1}{\boldsymbol{\varphi}} = 0$. In order to use the implicit function theorem, we must verify that the derivative $\mathbf{A}'(\mathbf{0})$ is an isomorphism between the spaces $\mathbf{V}^p(\omega)$ $(L^p(\omega))^6$. But the problem: Find $\overset{1}{\boldsymbol{u}}$ such that

$$\mathbf{A}'(\mathbf{0})\,\overset{1}{\boldsymbol{u}} = \overset{1}{\boldsymbol{\varphi}}$$

is the linearized pure displacement problem:

$$\begin{cases} -\displaystyle\sum_{m=0}^{1} \frac{1}{2m+1}\left\{\partial_\alpha a_{i\alpha pq}\overset{(m)}{e}_{pq}(\overset{1}{\boldsymbol{u}}) - \frac{2m+1}{h} a_{i3pq}\overset{(m-1)}{e}_{pq}(\overset{1}{\boldsymbol{u}})\right\}\mathbf{e}_i = \overset{1}{\boldsymbol{\varphi}} \ \text{ in } \ \omega. \\ \\ \overset{1}{\boldsymbol{u}} = \mathbf{0} \ \text{ on } \ \partial\omega, \end{cases}$$

which have one and only one solution.

Hence the continuous linear operator $\mathbf{A}'(\mathbf{0}) : \mathbf{V}^p(\omega) \to (L^p(\omega))^6$ is bijective. Since, by the closed graph theorem [9] its inverse is also continuous.

REFERENCES

1. Ball J. M.// Arch. Rational Mech. Anal. 1977. 63. P. 337-403.
2. Ciarlet P., Rabier P. Uravnenia Karmana. M., 1983.
3. Ciarlet P. Matematicheskaia teoria uprugosti. M., 1992.
4. Vekua I.N.// Nekotorye obshie metody postroenia razlichnyx variantov teorii obolochek. M., 1982.

5. Vekua I.N. Ob odnom metode raschota prizmaticheskikh obolochek. Tbilisi, 1955.

6. Meunargia T.V.// Trudy instituta prikl. mat. im. I.N. Vekua TGU , 1990. N38. 5-43.

7. Gordeziani D.G.//Dokl. AN SSSR. 1974. T. 215. N6. C. 1289-1292. 8. Khoma I.U. Obobshennaia teoria anizotropnyx obolochek. Kiev. 1986.

9. Sobolev S.L. Nekotorye primenenia funkcional'nogo analiza v mat. fizike. M., 1988.

10. Necas J. Les Methodes Directes en Theorie des Equations Elliptiques. Paris. 1967.

11. Agmon S., Douglis A., Nirenberg L.// Comm. Pure Appl. Math. 1964. 17. P. 35-92.

12. Geymonat G.// Ann. Mat. Pura Appl. 1965. 69. P. 207-284.

13. Meunargia T.V.// Proceeding of A. Razmadze Math. Institute 119. 1999. P.131-154.