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# SOME BOUNDARY VALUE PROBLEMS OF ELASTICITY FOR SEMI-ELLIPSES

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**Summary**: Using the method of separation of variables analytical solutions for semi-elliptical domains are found when the displacement vector or stresses are given on the elliptic part of the boundary while on the linear part (the major axis of the ellipse) either the normal component of the displacement vector and tangential component of the exterior stress or the normal component of the exterior stress and tangential component of the displacement vector are given.

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Boundary value problems of elasticity are considered for the domains bounded by coordinates lines of the elliptic system of coordinates $\theta$, $\alpha$, $(0 \leq \theta < \infty, -\pi < \alpha \leq \pi)$ [1], or to be more precise, for a semi-elliptic domain $(0 \leq \theta < \theta_1,\ 0 < \alpha < \pi)$.

Prior to the statement of the problem we shall introduce the following notation. We denote by $u$ and $v$ the components of the vector $\vec{U}$ along the tangents to the coordinate lines $\alpha$ and $\theta$, respectively; $N_\theta$ and $N_\alpha$, respectively, denote normal stresses while $S$ denotes tangential stress; $h$, $\sigma$, $E$ and $c$ , respectively, denote Lame's coefficient, Poisson's ratio, Young's modulus and scale factor; $\overline{u} = \dfrac{2hu}{c^2}$, $\overline{v} = \dfrac{2hv}{c^2}$, $h = \dfrac{c}{\sqrt{2}} h_0 = \dfrac{c}{\sqrt{2}} \sqrt{\cosh(2\theta) - \cos(2\alpha)}$, $æ = 4(1-\sigma)$, $\mu = \dfrac{E}{2(1+\sigma)}$.

**Problem statement**.

In the domain $\Omega = \{0 \leq \theta < \theta_1, 0 < \alpha < \pi)$ find functions $K(\theta, \alpha)$,

$B(\theta,\alpha)$,$u(\theta,\alpha)$, $v(\theta,\alpha)$ which satisfy the system of equilibrium equations [2]

$$\begin{aligned}
&\frac{\partial K}{\partial \theta}-\frac{\partial B}{\partial \alpha}=0,\\
&\frac{\partial K}{\partial \alpha}+\frac{\partial B}{\partial \theta}=0,\\
&\frac{\partial \bar{u}}{\partial \theta}+\frac{\partial \bar{v}}{\partial \alpha}=\frac{æ-2}{æ\mu}h_0^2K,\\
&\frac{\partial \bar{v}}{\partial \theta}-\frac{\partial \bar{u}}{\partial \alpha}=\frac{1}{\mu}h_0^2B
\end{aligned}\tag{1}$$

under the following boundary conditions

$$\overline{v}\Big|_{\substack{\alpha=0\\ \alpha=\pi}}=0,\ \frac{h_0^2}{\mu}S\Big|_{\substack{\alpha=0\\ \alpha=\pi}}=0,\ \overline{u}|_{\theta=0}=f_{01}(\alpha),\ \frac{h_0^2}{\mu}S\Big|_{\theta=0}=F_{02}(\alpha) \tag{2}$$

or

$$\frac{h_0^2}{\mu}N_\theta\Big|_{\substack{\alpha=0\\ \alpha=\pi}}=0,\ \overline{u}\Big|_{\substack{\alpha=0\\ \alpha=\pi}}=0,\ \frac{h_0^2}{\mu}N_\theta\Big|_{\theta=0}=F_{01}(\alpha),\ \overline{v}|_{\theta=0}=f_{02}(\alpha); \tag{3}$$

$$\begin{aligned}
&a)\ \ \frac{h_0^2}{\mu}N_\theta\Big|_{\theta=\theta_1}=F_{11}(\alpha),\ \frac{h_0^2}{\mu}S\Big|_{\theta=\theta_1}=F_{12}(\alpha),\quad \text{or}\\
&b)\ \ \overline{u}|_{\theta=\theta_1}=f_{11}(\alpha),\ \overline{v}|_{\theta=\theta_1}=f_{12}(\alpha)
\end{aligned}\tag{4}$$

By the superposition of problems (1), $(2)^0$ , (4) and (1), $(3)^0$, (4) we obtain the solutions of problems (1), (4) where $(2)^0$ and $(3)^0$ denote formulas (2) and (3) while $f_{01}(\alpha)=0$, $F_{02}(\alpha)=0$ and $F_{01}(\alpha)=0$, $f_{02}(\alpha)=0$.

**Solution representation.**

The technique used for the solution of the problems will be illustrated by problems (1), (2), (4$a$) the solution of which will be represented as the sum of the solutions of the two following problems.

1) In the domain $\Omega^*=\{0<\theta<\infty, 0<\alpha<\pi)$ find functions $K^{(1)}(\theta,\alpha)$, $B^{(1)}(\theta,\alpha)$, $u^{(1)}(\theta,\alpha)$, $v^{(1)}(\theta,\alpha)$ which satisfy system (1) of equilibrium equations and the following boundary conditions:

$$\begin{gathered}
\overline{v}^{(1)}\Big|_{\substack{\alpha=0\\ \alpha=\pi}}=0,\ \frac{h_0^2}{\mu}S^{(1)}\Big|_{\substack{\alpha=0\\ \alpha=\pi}}=0,\ \overline{u}^{(1)}\Big|_{\theta=0}=\\
=f_{01}(\alpha),\ \frac{h_0^2}{\mu}S^{(1)}\Big|_{\theta=0}=F_{02}(\alpha).
\end{gathered}\tag{5}$$

2) In the domain $\Omega = \{0 \le \theta < \theta_1,\ 0 < \alpha < \pi)$ find functions $K^{(2)}(\theta, \alpha)$, $B^{(2)}(\theta, \alpha)$, $u^{(2)}(\theta, \alpha)$, $v^{(2)}(\theta, \alpha)$ which satisfy system (1) of equilibrium equations and the following boundary conditions

$$\overline{v}^{(2)}\Big|_{\substack{\alpha=0\\ \alpha=\pi}} = 0,\ \ \frac{h_0^2}{\mu} S^{(2)}\Bigg|_{\substack{\alpha=0\\ \alpha=\pi}} = 0,\ \ \overline{u}^{(2)}\Big|_{\theta=0} = 0,\ \ \frac{h_0^2}{\mu} S^{(2)}\Bigg|_{\theta=0} = 0,$$
$$\frac{h_0^2}{\mu} N_\theta^{(2)}\Bigg|_{\theta=\theta_1} = F_{11}(\alpha) - \widetilde{F}_{01}(\alpha),\quad \frac{h_0^2}{\mu} S^{(2)}\Bigg|_{\theta=\theta_1} = F_{12}(\alpha) - \widetilde{F}_{02}(\alpha), \tag{6}$$

where $\overline{u}^{(1)}$, $\overline{v}^{(1)}$, $N_\theta^{(1)}$ $N_\alpha^{(1)}$, $S^{(1)}$ and $\overline{u}^{(2)}$, $\overline{v}^{(2)}$, $N_\theta^{(2)}$ $N_\alpha^{(2)}$, $S^{(2)}$, respectively, are components of the displacement vector and stress tensor obtained after problems 1) and 2) have been solved, $\widetilde{F}_{01}(\alpha) = \dfrac{h_0^2}{\mu} N_\theta^{(1)}\Bigg|_{\theta=\theta_1}$, $\widetilde{F}_{02}(\alpha) = \dfrac{h_0^2}{\mu} S^{(1)}\Bigg|_{\theta=\theta_1}$.

Before solving problems 1) and 2) introduce the following notation:

$$L_{1s}^{(j)}(\varphi_1, \varphi_2) = s\frac{\partial \varphi_1^{(j)}}{\partial \theta} + (-1)^j (s-2)\frac{\partial \varphi_2^{(j)}}{\partial \alpha},\ L_{2s}^{(j)}(\varphi_1, \varphi_2) =$$
$$= (-1)^{j+1}(s-2)\frac{\partial \varphi_1^{(j)}}{\partial \alpha} + s\frac{\partial \varphi_2^{(j)}}{\partial \theta},$$
$$j = 1, 2;\ \ s = 1, k.$$

**Solution of Problem 1).**

The solution of system 1) for the first problem (on the domain $\Omega^*$) using the harmonics $\varphi_1^{(1)}$ and $\varphi_2^{(1)}$ will be expressed as follows:

$$K^{(1)} = \frac{æ\mu}{h_0^2}\left[\cosh\theta \sin\alpha\, L_{21}^{(1)}(\varphi_1, \varphi_2) + \sinh\theta\cos\alpha\, L_{11}^{(1)}(\varphi_1, \varphi_2)\right],$$

$$B^{(1)} = \frac{æ\mu}{h_0^2}\left[\sinh\theta \cos\alpha\, L_{21}^{(1)}(\varphi_1, \varphi_2) - \cosh\theta\sin\alpha\, L_{11}^{(1)}(\varphi_1, \varphi_2)\right];$$

$$\overline{u}^{(1)} = \sinh\theta\sin\alpha\, L_{21}^{(1)}(\varphi_1, \varphi_2) - (æ-1)\varphi_1^{(1)}\sinh\theta\cos\alpha -$$
$$-(æ-1)\varphi_2^{(1)}\cosh\theta\sin\alpha,$$

$$\overline{v}^{(1)} = \sinh\theta\sin\alpha\, L_{11}^{(1)}(\varphi_1, \varphi_2) + (æ-1)\varphi_1^{(1)}\cosh\theta\sin\alpha -$$
$$(æ-1)\varphi_2^{(1)}\sinh\theta\cos\alpha, \tag{7}$$

while the components of the stress tensor have the following form:

$$\frac{h_0^2}{\mu} N_\theta^{(1)} = -\sinh\theta\cos\alpha\, L_{1k}^{(1)}(\varphi_1, \varphi_2) - \cosh\theta\sin\alpha\, L_{2k}^{(1)}(\varphi_1, \varphi_2) -$$

$$-2\sinh\theta\sin\alpha\frac{\partial L_{11}^{(1)}(\varphi_1,\varphi_2)}{\partial\alpha}-$$

$$-\frac{4\sinh^2\theta}{h_0^2}\left[\sinh\theta\cos\alpha\, L_{11}^{(1)}(\varphi_1,\varphi_2)+\cosh\theta\sin\alpha\, L_{21}^{(1)}(\varphi_1,\varphi_2)\right],$$

$$\frac{h_0^2}{\mu}S^{(1)}=\cosh\theta\sin\alpha\, L_{1k}^{(1)}(\varphi_1,\varphi_2)-\sinh\theta\cos\alpha\, L_{2k}^{(1)}(\varphi_1,\varphi_2)+$$

$$+2\sinh\theta\sin\alpha\frac{\partial L_{21}^{(1)}(\varphi_1,\varphi_2)}{\partial\alpha}-$$

$$-\frac{4\sinh^2\theta}{h_0^2}\left[\cosh\theta\sin\alpha\, L_{11}^{(1)}(\varphi_1,\varphi_2)-\sinh\theta\cos\alpha\, L_{21}^{(1)}(\varphi_1,\varphi_2)\right],$$

$$N_\alpha^{(1)}=\frac{4}{æ}K^{(1)}-N_\theta^{(1)}, \tag{8}$$

Using the method of separation of variables and taking into consideration the first two equalities from (5) we have

$$\varphi_1^{(1)}=\sum_{n=1}^{\infty}A_{1n}^{(1)}e^{-n\theta}\cos(n\alpha),\quad \varphi_2^{(1)}=\sum_{n=1}^{\infty}A_{2n}^{(1)}e^{-n\theta}\sin(n\alpha). \tag{9}$$

From the third and fourth equality of (5) with (7) and (8) in mind we have

$$\left.\varphi_2^{(1)}\right|_{\theta=0}=\frac{f_{01}(\alpha)}{\sin\alpha},\qquad \left.\frac{\partial\varphi_1^{(1)}}{\partial\theta}\right|_{\theta=0}=\frac{F_{02}(\alpha)}{\sin\alpha}+(æ-2)\left.\frac{\partial\varphi_2^{(1)}}{\partial\alpha}\right|_{\theta=0}. \tag{10}$$

After substituting (9) into (10) we shall obtain two equalities: the right-hand side of the first equality can be expanded into Fourier series, which will provide coefficients $A_{2n}^{(1)}$ or functions $\varphi_2^{(1)}$; the right-hand side of the second equality can also be expanded into Fourier series, which will provide coefficients $A_{1n}^{(1)}$ or functions $\varphi_1^{(1)}$.

Hence at any point of the domain $\Omega$, which is a part of the considered domain $\Omega^*$, components of the displacement vector $u^{(1)}$, $v^{(1)}$ and stress tensor $N_\theta^{(1)}$, $N_\alpha^{(1)}$, $S^{(1)}$ can be found.

If we substitute the functions $\left.\varphi_1^{(1)}\right|_{\theta=\theta_1}$ and $\left.\varphi_2^{(1)}\right|_{\theta=\theta_1}$ obtained above into (8), we shall have the functions $\widetilde{F}_{01}(\alpha)=\left.\frac{h_0^2}{\mu}N_\theta^{(1)}\right|_{\theta=\theta_1}$ and $\widetilde{F}_{02}(\alpha)=$ $=\left.\frac{h_0^2}{\mu}S^{(1)}\right|_{\theta=\theta_1}$, which will provide the solution of the second problem.

**Solution of Problem 2).**

The solution of system (1) of equilibrium equations ( on the domain $\Omega$) using two harmonics $\varphi_1^{(1)}$ and $\varphi_2^{(1)}$ will be expressed as follows:

$$K^{(2)}=\frac{æ\mu}{h_0^2}\left[\cosh\theta\sin\alpha\, L_{11}^{(2)}(\varphi_1,\varphi_2)+\sinh\theta\cos\alpha\, L_{21}^{(2)}(\varphi_1,\varphi_2)\right],$$

$$B^{(2)}=\frac{æ\mu}{h_0^2}\left[\sinh\theta\cos\alpha\, L_{11}^{(2)}(\varphi_1,\varphi_2)-\cosh\theta\sin\alpha\, L_{21}^{(2)}(\varphi_1,\varphi_2)\right];$$

$$
\begin{aligned}
\overline{u}^{(2)} &= \left[\sinh^2\theta_1 \coth\theta\, L_{21}^{(2)}(\varphi_1,\varphi_2) + (æ-1)\varphi_2\right]\sinh\theta\cos\alpha- \\
&\quad -\left[\cosh^2\theta_1 \tanh\theta\, L_{11}^{(2)}(\varphi_1,\varphi_2) - (æ-1)\varphi_1\right]\cosh\theta\sin\alpha, \\
\overline{v}^{(2)} &= -\left[\cosh^2\theta_1 \tanh\theta\, L_{21}^{(2)}(\varphi_1,\varphi_2) + (æ-1)\varphi_2\right]\cosh\theta\sin\alpha- \\
&\quad -\left[\sinh^2\theta_1 \coth\theta\, L_{11}^{(2)}(\varphi_1,\varphi_2) - (æ-1)\varphi_1\right]\sinh\theta\cos\alpha,
\end{aligned}
\tag{11}
$$

while the components of the stress tensor have the following form:

$$
\begin{aligned}
\frac{h_0^2}{\mu}N_\theta^{(2)} &= \left[2\sinh^2\theta_1\coth\theta\frac{\partial L_{11}^{(2)}(\varphi_1,\varphi_2)}{\partial\alpha} + L_{2k}^{(2)}(\varphi_1,\varphi_2)\right]\sinh\theta\cos\alpha+ \\
&\quad +\left[2\cosh^2\theta_1\tanh\theta\frac{\partial L_{21}^{(2)}(\varphi_1,\varphi_2)}{\partial\alpha} + L_{1k}^{(2)}(\varphi_1,\varphi_2)\right]\cosh\theta\sin\alpha- \\
&\quad -\frac{4\sinh(\theta_1+\theta)\sinh(\theta_1-\theta)}{h_0^2}\Big[\cosh\theta\sin\alpha\, L_{11}^{(2)}(\varphi_1,\varphi_2)+ \\
&\quad + \sinh\theta\cos\alpha\, L_{21}^{(2)}(\varphi_1,\varphi_2)\Big], \\
\frac{h_0^2}{\mu}S^{(2)} &= -\left[2\cosh^2\theta_1\tanh\theta\frac{\partial L_{11}^{(2)}(\varphi_1,\varphi_2)}{\partial\alpha} + L_{2k}^{(2)}(\varphi_1,\varphi_2)\right]\cosh\theta\sin\alpha+ \\
&\quad +\left[2\sin h^2\theta_1\coth\theta\frac{\partial L_{21}^{(2)}(\varphi_1,\varphi_2)}{\partial\alpha} + L_{1k}^{(2)}(\varphi_1,\varphi_2)\right]\sinh\theta\cos\alpha+ \\
&\quad +\frac{4\sinh(\theta_1+\theta)\sinh(\theta_1-\theta)}{h_0^2}\Big[\sinh\theta\cos\alpha\, L_{11}^{(2)}(\varphi_1,\varphi_2)- \\
&\quad - \cosh\theta\sin\alpha\, L_{21}^{(2)}(\varphi_1,\varphi_2)\Big], \\
N_\alpha^{(2)} &= \frac{4}{æ}K^{(2)} - N_\theta^{(2)},
\end{aligned}
\tag{12}
$$

Using the method of separation of variables and taking into consideration the first four equalities of (6) we obtain

$$
\varphi_1^{(2)} = \sum_{n=1}^{\infty} A_{1n}^{(2)}\frac{\sinh(n\theta)}{\cosh(n\theta_1)}\sin(n\alpha), \quad \varphi_2^{(2)} = \sum_{n=1}^{\infty} A_{2n}^{(2)}\frac{\cosh(n\theta)}{\cosh(n\theta_1)}\cos(n\alpha). \tag{13}
$$

The conditions $\theta=\theta_1$ on the boundary will be substituted by the following equivalent conditions

$$
\left[\sinh(2\theta_1)\frac{\partial L_{21}^{(2)}(\varphi_1,\varphi_2)}{\partial\alpha} + L_{1k}^{(2)}(\varphi_1,\varphi_2)\right]_{\theta=\theta_1} =
$$

$$= \left(\frac{2}{h_0^2}\right)_{\theta=\theta_1} \left\{\cosh\theta_1 \sin\alpha \left[F_{11}(\alpha) - \widetilde{F}_{01}(\alpha)\right] + \right.$$

$$\left. + \sinh\theta_1 \cos\alpha \left[F_{12}(\alpha) - \widetilde{F}_{02}(\alpha)\right]\right\},$$

$$\left[\sinh(2\theta_1)\frac{\partial L_{11}^{(2)}(\varphi_1,\varphi_2)}{\partial\alpha} + L_{2k}^{(2)}(\varphi_1,\varphi_2)\right]_{\theta=\theta_1} = \tag{14}$$

$$= \left(\frac{2}{h_0^2}\right)_{\theta=\theta_1} \left\{\sinh\theta_1 \cos\alpha \left[F_{11}(\alpha) - \widetilde{F}_{01}(\alpha)\right] - \right.$$

$$\left. - \cosh\theta_1 \sin\alpha \left[F_{12}(\alpha) - \widetilde{F}_{02}(\alpha)\right]\right\}.$$

Substituting (13) into (14) and expanding the right-hand side into Fourier series by equalization of the expressions for similar trigonometric functions we shall obtain an infinite system of linear algebraic equations with respect to the unknown $A_{1n}^{(2)}$ and $A_{2n}^{(2)}$ which for each $n$ can be solved by means of the two following equations in two unknowns

$$\left[-n^2\sinh(2\theta_1)\tanh(n\theta) - æn\right]A_{1n}^{(2)} - \left[n^2\sinh(2\theta_1) + (æ-2)n\right]A_{2n}^{(2)} = b_{1n},$$

$$\left[n^2\sinh(2\theta_1) - (æ-2)n\tanh(n\theta)\right]A_{1n}^{(2)} + \left[n^2\sinh(2\theta_1) + æn\right]A_{2n}^{(2)} = b_{2n},$$

where $b_{1n}$ and $b_{2n}$ are Fourier series expansion coefficients for the right-hand side of (14).

Hence at any point of the domain $\Omega$ the components of the displacement vector $u^{(2)}$, $v^{(2)}$ and stress tensor $N_\theta^{(2)}$, $N_\alpha^{(2)}$, $S^{(2)}$ can be found.

Finally, at any point of the domain $\Omega$ the corresponding components of the displacement vector and stress tensor of problem (1), (2) (4$a$) will be expressed as follows

$$\overline{u} = \overline{u}^{(1)} + \overline{u}^{(2)},\ \overline{v} = \overline{v}^{(1)} + \overline{v}^{(2)},\ N_\theta = N_\theta^{(1)} + N_\theta^{(2)},\ N_\alpha = N_\alpha^{(1)} + N_\alpha^{(2)},$$

$$S = S^{(1)} + S^{(2)}.$$

In a similar way one can solve any of boundary value problems (1),(2) (4) or (1), (3), (4).

The boundary value problems of elasticity considered in the article can also be solved using the theory of complex variable functions [3], but the approach described in the article provides an easier solution technique.

**R E F E R E N C E S**

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