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# STATIC AND DYNAMICAL PROBLEMS FOR A CUSPED BEAM

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## *Abstract*

The paper deals with admissible static and dynamical problems for an Euler-Bernoulli cusped beam with a continuously varying cross-section of an arbitrary form. The beam may have both the ends cusped. The setting of boundary conditions at the beam ends depends on the geometry of sharpening of beam ends (they can become weighted or disappear completely), while the setting of initial conditions is independent of the beam ends geometry. An up-to-date survey of results concerning cusped beams is also given.

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## 1. Introduction

In the 1950's, I.Vekua [1, 2] constructed his hierarchic models of prismatic shells (i.e., plates of variable thickness). At the same time, he recommended to investigate well-posedness of boundary value problems (BVPs) for cusped plates (cusped plates have a thickness which is zero at least on some subset of the plate's boundary). I.Vekua foresaw that the setting of BVPs for cusped plates, in general, fell outside the limits of the classical formulation. A brief survey of results and corresponding references concerned cusped plates can be found in [3]. If we consider cylindrical bending of a cusped plate, with a rectangular projection $a \leq x_1 \leq b$, $0 \leq x_2 \leq \ell$, we actually get results also for cusped beams. In 1976 (see [4, 5]), studying static problems for plates, using the Kirchhoff-Love model, whose thickness has the form

$$2h(x_1, x_2) = h_0 x_2^{\varkappa}, \quad a \leq x_1 \leq b,\ 0 \leq x_2 \leq l, \qquad a, b, l, h_0, \varkappa = \text{const} > 0, \tag{1.1}$$

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(in particular, $x_2$ coincides with the beam axis), the well-posedness of BVPs by cylindrical bending was completely resolved. In particular, it was established that the cusped end of the beam can be:

– clamped, i.e.,

$$w(0) = 0, \quad w'(0) = 0 \text{ if and only if (iff) } \varkappa < \frac{1}{3}; \tag{1.2}$$

– freely supported, i.e.,

$$w(0) = 0, \quad M(0) = -(EIw,_{22})|_{x_2=0} = 0 \text{ iff } \varkappa < \frac{2}{3}; \tag{1.3}$$

– free, i.e.,

$$M(0) = 0, \quad Q(0) = -(EIw,_{22}),_2|_{x_2=0} = 0 \text{ for all values of } \varkappa, \tag{1.4}$$

where $E$ is Young's modulus, $I$ is a moment of inertia, $w$ is a deflection, $M$ is a bending moment, and $Q$ is a shearing force. All admissible BVPs were solved in the explicit form. Geometry (profiles and projections) of cusped edges are discussed in [5, 6].

In 1980-1986 S. Uzunov [7, 8] numerically solved the problem of bending of the cusped circular beam on an elastic foundation with constant compliance. The moment of inertia of the beam had the form

$$I(x_2) = \pi\frac{r^4}{4}, \quad r = cx_2^\gamma, \quad c, \gamma = \text{const} > 0, \quad \gamma < 1$$

($r$ is the cross-section radius). The cusped end was free and the non-cusped end was clamped.

In 1990-1995 the bending vibration of homogeneous Euler-Bernoulli cone beams and beams of continuously varying rectangular cross-sections, when one side of the cross-section is constant, while the other side is proportional to $x_2^\varkappa$, $\varkappa = \text{const} > 0$ (compare with (1.1)), where $x_2$ is the axial coordinate measured from the cusped end, were considered by S. Naguleswaran [9 - 13]. Firstly, the concrete cases of $\varkappa = 1, 1/2$, and finally, the general case of $\varkappa$ were investigated. In these investigations the cusped end is always free (compare with (1.2)-(1.4)); direct analytical solutions were constructed for the mode shape equation and the frequences were also tabulated.

In 1995-1999 the classical bending and vibration of the cusped beam with two cusped ends when the variable thickness has the following form

$$2h(x_2) := h_0 x_2^\varkappa(\ell - x_2)^\delta, \quad h_0, l, \varkappa, \delta = \text{const} > 0,$$

was studied by N.Chinchaladze [14, 15]. The restrictions on $\varkappa$ coincide with (1.2)-(1.4) and on $\delta$ are the same as on $\varkappa$ in (1.2)-(1.4).

In 1998-2001 (see [16, 17]), generalizing I.Vekua's dimension reduction method, the hierarchic models of beams were constructed. Peculiarities of setting of BVPs were investigated in the case of a continuously varying rectangular cross-section when all sides of the cross-section were arbitrary differentiable on $]0, l[$ functions of the axial coordinate and, in particular, when the cross-sectional area was zero at least on one of the ends $x_2 = 0$ and $x_2 = l$ of the beam.

In 1999-2001 two contact problems were considered by N. Shavlakadze, [18, 19], namely, the contact problem for an unbounded elastic medium composed of two half-planes $x_1 > 0$ and $x_1 < 0$ having different elastic constants and strengthened on the semi-axis $x_2 > 0$ by an inclusion of variable thickness (cusped beam) with constant Young's modulus and Poisson's ratio. It was assumed that the plate is subjected to plane deformation, the flexural rigidity $D$ had the form

$$D = D_0 x_2^{\varkappa}, \quad D_0, \varkappa = \text{const} > 0,$$

and the cusped end $x_2 = 0$ of the beam was free (compare with (1.2)-(1.4)).

The second contact problem considered in [18,19] was the problem of bending of an isotropic plate of constant thickness reinforced by a finite elastic rib (beam) with the flexural rigidity $D$ of the form

$$D = (a^2 - x_2^2)^{n+\frac{1}{2}} P(x_2),$$

where $a$ =const$> 0$, $n \geq 1$ was an integer and $P(x_2)$ was a polynomial which satisfies certain restrictions. It was assumed that the rib was not loaded.

The aim of the present paper is to consider an elastic cusped beam with a continuously varying cross-section of an arbitrary form. Let the barycenters of an cross-sections lie on the axis $x_2$ of the Cartesian system of coordinates $0x_1x_2x_3$. The dynamical bending equation of such a beam (i.e., Euler-Bernoulli beam) has the following form (see, e.g., [20])

$$(D(x_2)w,_{22}),_{22} = f(x_2, t) - \rho\sigma(x_2)\frac{\partial^2 w}{\partial t^2}, \quad 0 \leq x_2 \leq l, \tag{1.5}$$

where, as before, $w(x_2, t)$ is a deflection of the beam, $f(x_2, t)$ is an intensity of the load,

$$D(x_2) := E(x_2)I(x_2), \tag{1.6}$$

$E(x_2)$ is Young's modulus, $I(x_2)$ is the moment of inertia with respect to the barycentric axis normal to the plane $x_2x_3$, $\rho(x_2)$ is a density, $\sigma(x_2)$ is the area of a transverse section lying in the plane $x_1x_3$, and index 2 after comma means differentiation with respect to $x_2$. Such a beam will be called cusped if $I(x_2)$ vanishes at least on one of the ends $x_2 = 0$, $l$ of the beam.

Let us remark that if we consider a cylindrical bending of the cusped plate with the flexural rigidity

$$D(x_2) := \frac{2E(x_2)h^3(x_2)}{3(1 - \nu^2)}, \tag{1.7}$$

where $\nu$ is Poisson's ratio and $2h(x_2)$ is a thickness of the plate then the bending equation for the plate coincides with (1.5), where $\sigma(x_2)$ should be replaced by $2h$.

The paper is organized as follows.

Section 2 is devoted to the investigation of properties of solutions of Euler-Bernoulli equation (see Theorem 2.1).

Section 3 deals with the well-posedness and correct formulation of all admissible BVPs for cusped elastic beams. In contrast to the case of non-cusped beams, when the beam end can always be either clamped or freely supported, for cusped beams this is not the case. The admissibility of these boundary conditions (BCs) depends on the geometry of the beam end sharpening, which is expressed by the convergence-divergence of the integrals $I_k^0$, $I_k^l$, $k = 0, 1, 2, ...$ (see Theorem 3.1). For the indicated cases of the beam end sharpening some BCs completely disappear and are replaced by the boundedness of the deflection and its derivative. In particular, mechanically free ends are also free of mathematical BCs (see Remark 3.3). The BVPs formulated in Theorem 3.1 are solved in the explicit form.

Let us note that a bending vibration of the cusped beam is considered in [21]. The investigation is based on the Lax-Milgram theorem. It is established that BCs preserve their peculiarities from the static case, while the presence of cusped ends does not affect the setting of initial conditions.

## 2. Properties of the General Solution of the Euler-Bernoulli Equation

In the static case, the equation (1.5) becomes

$$(D(x_2)w,_{22}),_{22} = f(x_2). \tag{2.1}$$

But (2.1) coincides with the equation of cylindrical bending of the cusped plate with the flexural rigidity (1.7) and projection $\omega := \{x_1, x_2 : \ -\infty < < x_1 < +\infty,\ 0 < x_2 < l\}$ on the plane $x_3 = 0$.

The well-posedness of BVPs for such plates when the thickness can vanish only at points $(-\infty < x_1 < +\infty, x_2 = 0)$ was investigated in [22]. After reformulation of the corresponding results for (2.1), where $D$ is given by (1.7), the case $I(0) = 0$, $I(l) \neq 0$ will be completely studied. Below in an analogous way we consider the general case when both $I(0) = 0$ and $I(l) = 0$ are admissible. Obviously, the results will be applicable also for cylindrical bending of a plate (2.1), where $D$ is given by (1.7), with the cusped edges, i.e., both $h(x_1, 0) = 0$ and $h(x_1, l) = 0$ for arbitrary $x_1$ will be admissible.

Now, let us consider (2.1), where $D$ is given by (1.6), with $D(x_2) \in \in C([0, l]) \cap C^2(]0, l[)$ and recall that the bending moment and shearing force are:

$$M_2 = -Dw_{,22}, \tag{2.2}$$

$$Q_2 = M_{2,2}. \tag{2.3}$$

At the ends of a beam, where $I(x_2)$ vanishes all quantities will be defined as limits from inside of $]0,l[$.

From (2.1)-(2.3) follows

$$Q_{2,2}=-f(x_2),\quad M_{2,22}=-f(x_2),$$

$$Dw_{,22}=\int\limits_{x_0}^{x_2}(x_2-t)f(t)dt-C_1(x_2-x_0)-C_2,$$

where fixed $x_0\in]0,l[$ and $C_1,C_2=\mathrm{const}$.

Hence

$$Q_2 = -\int\limits_{x_0}^{x_2} f(t)dt+C_1, \tag{2.4}$$

$$M_2 = -\int\limits_{x_0}^{x_2}(x_2-t)f(t)dt+C_1(x_2-x_0)+C_2, \tag{2.5}$$

$$\begin{aligned} w_{,2} &= \int\limits_{x_0}^{x_2} K_1(\tau)D^{-1}(\tau)d\tau+\int\limits_{x_0}^{x_2} K_2(\tau)\tau D^{-1}(\tau)d\tau+C_3 \\ &= \int\limits_{x_0}^{x_2} K(\tau)D^{-1}(\tau)d\tau+C_3, \end{aligned} \tag{2.6}$$

$$\begin{aligned} w &= \int\limits_{x_0}^{x_2}(x_2-\tau)K_1(\tau)D^{-1}(\tau)d\tau \\ &+ \int\limits_{x_0}^{x_2}(x_2-\tau)K_2(\tau)\tau D^{-1}(\tau)d\tau+C_3(x_2-x_0)+C_4 \\ &= \int\limits_{x_0}^{x_2}(x_2-\tau)K(\tau)D^{-1}(\tau)d\tau+C_3(x_2-x_0)+C_4, \end{aligned} \tag{2.7}$$

where

$$K(\tau):=K_1(\tau)+\tau K_2(\tau) \tag{2.8}$$

with

$$K_1(\tau) := C_1x_0-C_2-\int\limits_{x_0}^{\tau} f(t)tdt, \tag{2.9}$$

$$K_2(\tau) := -C_1+\int\limits_{x_0}^{\tau} f(t)dt. \tag{2.10}$$

Clearly,

$$K'(\tau) = K_2(\tau).$$

From (2.4), (2.5), (2.8), (2.9), (2.10) we conclude that

$$\begin{array}{l} K_2(\tau) = -Q_2(\tau), \quad K_1(\tau) = \tau Q_2(\tau) - M_2(\tau), \\ K(\tau) = -M_2(\tau). \end{array} \tag{2.11}$$

For $f$ summable on $]0,l[$, i.e., $f \in L(]0,l[)$, obviously,

$$Q_2, \quad M_2 \in C([0,l]); \quad w, \ w_{,2} \in C(]0,l[);$$

the behavior of

$$w_{,2} \quad \text{and} \quad w \quad \text{when} \quad x_2 \to 0+, \, l-$$

depends, in view of (2.6), (2.7), on the convergence of the integrals

$$I_i^0 := \int\limits_0^{\varepsilon} \tau^i D^{-1}(\tau) d\tau, \quad I_i^l \quad := \quad \int\limits_{l-\varepsilon}^{l} (l-\tau)^i D^{-1}(\tau) d\tau,$$
$$i = 0, 1, 2, \cdots, \quad l > \varepsilon \quad = \quad \text{const} > 0.$$

Evidently, for any nonnegative integer $i$ :

$$\text{if } \ I_i^{0(l)} \ < \ +\infty, \ \text{ then } \ I_{i+1}^{0(l)} < +\infty, \quad i \geq 0,$$

and

$$\text{if } \ I_i^{0(l)} \ = \ +\infty \ \text{ then } \ I_{i-1}^{0(l)} = +\infty, \ \ i \geq 1.$$

**Theorem 2.1.** *Let $f \in L(]0,l[)$, $D \in C^2(]0,l[) \cap C([0,l])$, and $w \in$ $\in C^4(]0,l[)$ be a solution of equation (2.1).*

**Case I.** *If $I_0^0$ $(I_0^l) < +\infty$, then*

$$w, \ \ w_{,2} \in C([0,l[) \ (C(]0,l])). \tag{2.12}$$

**Case II.** $I_0^0$ $(I_0^l) = +\infty$, $I_1^0$ $(I_1^l) < +\infty$.

*If either $D \in C^2([0,l[)(C^2(]0,l]))$, or the value of the first or second order derivative of $D$ at the point $0(l)$ tends to infinity and $f$ is bounded in some neighborhood $]0,\varepsilon]([l-\varepsilon,l[)$ of $0(l)$,* (2.13)

*then*

$$w \in C([0,l[) \ (C(]0,l])). \tag{2.14}$$

*If*

$$K(0) = 0 \ (K(l) = 0), \tag{2.15}$$

*then*

$$w_{,2} = O(1) \ \text{ as } \ x_2 \to 0+ \ \ (l-) \tag{2.16}$$

*(condition (2.15) is necessary and sufficient).*

**Case III.** *If* $I_1^0(I_1^l) = +\infty$, $I_2^0(I_2^l) < +\infty$, *and either* $D \in C^3([0,l[)$ $(C^3(]0,l]))$ *or the value of the first, second or third order derivative of* $D$ *at the point* $0(l)$ *tends to infinity, and* $f$ *is bounded with its first derivative in some right (left) neighborhood of the point* $0(l)$ *then*

$$w = O(1) \text{ as } x_2 \to 0+ \ (l-), \tag{2.17}$$

*if and only if (iff) (2.15) is fulfilled.*

**Case IV.** *If* $I_2^0(I_2^l) = +\infty$, *and moreover, for the fixed* $k \geq 2$

$$I_k^0(I_k^l) = +\infty, \quad I_{k+1}^0(I_{k+1}^l) < +\infty; \tag{2.18}$$

$$\begin{gathered} f^{(j)}(0) = 0 \ \ (f^{(j)}(l) = 0), \quad j = 0, 1, ..., k-2, \\ f^{k-1}(x_2) \quad \textit{is continuous at} \ \ 0(l), \end{gathered} \tag{2.19}$$

*then* (2.17) *is valid iff*

$$K(0) = 0, \ \ K_2(0) = 0 \ \ (K(l) = 0, \ \ K_2(l) = 0) \tag{2.20}$$

*hold.*

**Case V.** *If* $I_1^0\,(I_1^l) = +\infty$ *and either (2.18), (2.19) are fulfilled for* $k \geq 2$, *or (2.18) is fulfilled for* $k = 1$, *and* $f(x_2)$ *is continuous at* $0(l)$) *then (2.16) is valid iff (2.20) holds.*

In order to prove Theorem 2.1. beforehand we prove some lemmas

**Lemma 2.1.** *If*

$$I_0^0(I_0^l) = +\infty \tag{2.21}$$

*and moreover, for the fixed integer* $k \geq 0$

$$I_k^0(I_k^l) = +\infty, \ \ I_{k+1}^0(I_{k+1}^l) < +\infty; \tag{2.22}$$

$$f^{(j)}(0) = 0 \ (f^{(j)}(l) = 0), \ \ j = 0, 1, ..., k-2 \ (\textit{for the case } k \geq 2); \tag{2.23}$$

$$f^{(k-1)}(x_2) \ \ \textit{is continuous at}\, 0(l) \ (\textit{for the case } k \geq 1), \tag{2.24}$$

*then*

$$\left| \int\limits_{x_2}^{x_0} K(\tau) D^{-1}(\tau) d\tau \right| \leq \ \textit{const} < +\infty \ \ \forall x_2 \in ]0, x_0] \ \ (\forall \ \ x_2 \in [x_0, l[) \tag{2.25}$$

*iff (2.15) and (2.20) are fulfilled for* $k = 0$, *and* $k \geq 1$, *respectively.*

**Proof.** Obviously, in the case $k = 0$

$$\begin{aligned} \left| \int\limits_{x_2}^{x_0} K(\tau) D^{-1}(\tau) d\tau \right| &= \left| \int\limits_{x_2}^{x_0} \frac{K(\tau)}{\tau} \tau D^{-1}(\tau) d\tau \right| \\ &\leq C \int\limits_{0}^{x_0} \tau D^{-1}(\tau) d\tau = \text{const} < +\infty \quad \forall \, x_2 \in ]0, x_0], \end{aligned} \tag{2.26}$$

since, by virtue of $K(0)=0$ and $K'(\tau)=K_2(\tau)$,

$$\lim_{\tau\to 0+}\frac{K(\tau)}{\tau}=K'(0)=K_2(0)<+\infty,$$

i.e.,

$$\left|\frac{K(\tau)}{\tau}\right|\le C\ \ \forall\tau\in]0,x_0].$$

Analogously,

$$\begin{aligned}\left|\int\limits_{x_0}^{x_2}K(\tau)D^{-1}(\tau)d\tau\right| &= \left|\int\limits_{x_0}^{x_2}\frac{K(\tau)}{l-\tau}(l-\tau)D^{-1}(\tau)d\tau\right|\\ &\le C\int\limits_{x_0}^{l}(l-\tau)D^{-1}(\tau)d\tau=\text{const}<+\infty\quad\forall x_2\in[x_0,l[, \end{aligned}\tag{2.27}$$

since, using the substitution $l-\tau=\xi$,

$$\lim_{\tau\to l-}\frac{K(\tau)}{l-\tau}=\lim_{\xi\to 0+}\frac{K(l-\xi)}{\xi}=-K'(l)=-K_2(l)<+\infty,$$

i.e.,

$$\left|\frac{K(\tau)}{l-\tau}\right|\le C\ \ \forall\tau\in[x_0,l[.$$

In case $k\ge 1$, evidently,

$$\begin{aligned}\left|\int\limits_{x_2}^{x_0}K(\tau)D^{-1}(\tau)d\tau\right| &= \left|\int\limits_{x_2}^{x_0}\frac{K(\tau)}{\tau^{k+1}}\tau^{k+1}D^{-1}(\tau)d\tau\right|\\ &\le C\int\limits_{0}^{x_0}\tau^{k+1}D^{-1}(\tau)d\tau=\text{const}<+\infty\ \ \forall x_2\in]0,x_0], \end{aligned}\tag{2.28}$$

since, in view of, (2.20), (2.23), (2.24),

$$\begin{aligned}\lim_{\tau\to 0+}\frac{K(\tau)}{\tau^{k+1}} &= \lim_{\tau\to 0+}\frac{K'(\tau)}{(k+1)\tau^k}=\lim_{\tau\to 0+}\frac{K_2(\tau)}{(k+1)\tau^k}\\ &= \lim_{\tau\to 0+}\frac{f^{(k-1)}(\tau)}{(k+1)!}=\frac{1}{(k+1)!}f^{(k-1)}(0),\end{aligned}$$

i.e., $\left|\dfrac{K(\tau)}{\tau^{k+1}}\right|\le C\ \ \forall\tau\in]0,x_0]$.

Analogously,

$$\left|\int\limits_{x_0}^{x_2}K(\tau)D^{-1}(\tau)d\tau\right|\le\text{const}<+\infty\ \ \forall x_2\in[x_0,l[,\tag{2.29}$$

since, using the substitution $l-\tau=\xi$,

$$
\begin{aligned}
& \lim_{\tau\to l-}\frac{K(\tau)}{(l-\tau)^{k+1}}=\lim_{\xi\to 0+}\frac{K(l-\xi)}{\xi^{k+1}} \\
= & \lim_{\xi\to 0+}\frac{-K'(l-\xi)}{(k+1)\xi^{k}}=-\lim_{\xi\to 0+}\frac{K_2(l-\xi)}{(k+1)\xi^{k}} \\
= & \frac{(-1)^{k+1}}{(k+1)!}f^{(k-1)}(l),
\end{aligned}
$$

i.e., $\left|\dfrac{K(\tau)}{(l-\tau)^{k+1}}\right|\leq C\ \ \forall\tau\in[x_0,l[$.

Let us consider the end $x_2=0$ and show that the condition (2.15) is also necessary for (2.25). In fact if we assume that (2.25) takes place and at the same time, without loss of generality, suppose that $K(0)>0$, then $K(\tau)>\tilde{C}=$ const $>0$ in some neighborhood $[0,\varepsilon]$ of 0, and

$$
+\infty>\ \text{const}\geq\int\limits_{x_2}^{\varepsilon}K(\tau)D^{-1}(\tau)d\tau>\tilde{C}\int\limits_{x_2}^{\varepsilon}D^{-1}(\tau)d\tau, \tag{2.30}
$$

whence,

$$
\int\limits_{x_2}^{\varepsilon}D^{-1}(\tau)d\tau\leq\ \text{const}<+\infty\quad\text{for } x_2\in]0,\varepsilon].
$$

But the last inequality would contradict (2.21). Thus, $K(0)=0$.

Analogously, we can show the necessity of the conditions (2.20) for the case $k\geq 1$. The necessity of $K(0)=0$ follows from the previous assertion. Now, let (2.25) be valid and let $K(0)=0$ but $K_2(0)>0$. Then, in view of (2.8), from $K(0)=0$ we have $K_1(0)=0$. By virtue of $K_1'(x_2)=-x_2f(x_2)$, similar to the proof of (2.28) we can show that

$$
\left|\int\limits_{x_2}^{x_0}K_1(\tau)D^{-1}(\tau)d\tau\right|\leq\ \text{const}<+\infty\ \ \forall x_2\in]0,x_0],\ \text{iff } K_1(0)=0. \tag{2.31}
$$

From (2.25) and (2.31), because of $\tau K_2(\tau)=K(\tau)-K_1(\tau)$, we immediately get

$$
\left|\int\limits_{x_2}^{x_0}\tau K_2(\tau)D^{-1}(\tau)d\tau\right|\leq\ \text{const}<+\infty\ \ \forall x_2\in]0,x_0]. \tag{2.32}
$$

But the necessary condition for (2.32) is the condition $K_2(0)=0$. Indeed, if $K_2(0)>0$, then similar to (2.30) we get

$$
\left|\int\limits_{x_2}^{\varepsilon}\tau D^{-1}(\tau)d\tau\right|\leq\ \text{const}<+\infty\ \ \forall x_2\in]0,\varepsilon],
$$

which contradicts $I_1^0\,(I_1^l)=+\infty$. Thus, $K_2(0)=0$.

Let us now consider the end $x_2=l$. The proof of necessity of the conditions (2.15) and (2.20) is similar to the case of the end $x_2=0$. In this case, when $k\geq 1$, we use the following identity

$$\begin{aligned}&\int\limits_{x_0}^{x_2}(l-\tau)K_2(\tau)D^{-1}(\tau)d\tau\\ &=\int\limits_{x_0}^{x_2}[K_1(\tau)+lK_2(\tau)]D^{-1}(\tau)d\tau\\ &-\int\limits_{x_0}^{x_2}K(\tau)D^{-1}(\tau)d\tau\ \ \forall x_2\in[x_0,l[.\end{aligned}\tag{2.33}$$

Which is obvious in view of (2.8). Bearing in mind that

$$K_1(l)+lK_2(l)=K(l)=0$$

and, hence,

$$\begin{aligned}&\lim_{\tau\to l-}\frac{K_1(\tau)+lK_2(\tau)}{(l-\tau)^{k+1}}=\lim_{\tau\to l-}\frac{-f(\tau)\tau+lf(\tau)}{-(k+1)(l-\tau)^k}\\ &=\lim_{\tau\to l-}\frac{-f(\tau)}{(k+1)(l-\tau)^{k-1}}=\lim_{\xi\to 0+}\frac{-f(l-\xi)}{(k+1)\xi^{k-1}},\end{aligned}$$

in the right hand side of (2.33) we prove the boundedness as $x_2\to l-$ of the first integral like the proof of (2.29). Therefore, taking into account that we assumed the validity of (2.25), the left hand side is bounded for $x_2\in[l-\varepsilon,l[$, since such is the right hand side of (2.33). But the necessary condition for it is $K_2(l)=0$.

**Corollary 2.1.** Under assumptions of Lemma 2.1 we have

$$\lim_{x_2\to 0+}x_2\int\limits_{x_0}^{x_2}K(\tau)D^{-1}(\tau)d\tau=0,\tag{2.34}$$

$$\int\limits_{x_0}^{x_2}K(\tau)\tau D^{-1}(\tau)d\tau\leq\ \text{const}<+\infty,\quad\text{as}\ \ x_2\to 0+,\tag{2.35}$$

$$\lim_{x_2\to l-}(x_2-l)\int\limits_{x_0}^{x_2}K(\tau)D^{-1}(\tau)d\tau=0,$$

$$\int\limits_{x_0}^{x_2}K(\tau)(l-\tau)D^{-1}(\tau)d\tau\leq\ \text{const}<+\infty,\quad\text{as}\ \ x_2\to l-.\tag{2.36}$$

**Lemma 2.2.** *If* $I_0^0 = +\infty$, $I_1^0 < +\infty$ $(I_0^l = +\infty,\ I_1^l < +\infty)$, *then*

$$\left| x_2 \int\limits_{x_2}^{x_0} K(\tau) D^{-1}(\tau) d\tau \right| \le \ const < +\infty \ \ \forall x_2 \in ]0, x_0] \tag{2.37}$$

$$\left( \left| (x_2 - l) \int\limits_{x_2}^{x_0} K(\tau) D^{-1}(\tau) d\tau \right| \le \ const < +\infty \ \ \forall x_2 \in [x_0, l[ \right). \tag{2.38}$$

**Proof.** Evidently,

$$\left| x_2 \int\limits_{x_2}^{x_0} K(\tau) D^{-1}(\tau) d\tau \right| = \left| \int\limits_{x_2}^{x_0} K(\tau) \frac{x_2}{\tau} \tau D^{-1}(\tau) d\tau \right|$$

$$\le C \int\limits_{0}^{x_0} \tau D^{-1}(\tau) d\tau = \ \text{const} < +\infty \quad \forall x_2 \in ]0, x_0],$$

because of

$$|K(\tau)| \le C, \quad \tau \in [0, x_0]; \quad 0 < \frac{x_2}{\tau} \le 1,$$

since $0 < x_2 \le \tau \le x_0$.

Taking into account that

$$0 < \frac{l - x_2}{l - \tau} \le 1,$$

because of

$$0 < x_0 \le \tau \le x_2,$$

i.e.,

$$l > l - x_0 \ge l - \tau \ge l - x_2 > 0,$$

we analogously prove (2.38).

**Lemma 2.3.** *If* $I_0^0 = +\infty$, $I_1^0 < +\infty$ $(I_0^l = +\infty,\ I_1^l < +\infty)$, *and either* $D \in C^2([0, l[)$ $(D \in C^2(]0, l]))$ *or the value of the first or second derivative of* $D$ *tends of infinity as* $x_2 \to 0+$ $(l-)$, *and* $f$ *is bounded in some neighborhood* $]0, \varepsilon]$ $([l - \varepsilon, l[)$ *of* $0(l)$, *then*

$$\lim_{x_2 \to 0+} x_2 \int\limits_{x_2}^{x_0} K(\tau) D^{-1}(\tau) d\tau$$

$$= \begin{cases} \dfrac{2K(0)}{D''(0)} & \textit{if } D'(0) = 0, \ \ D''(0) \ne 0; \\ 0 & \textit{in the other arising cases} \end{cases} \tag{2.39}$$

$$\left(\lim_{x_2\to l-}(x_2-l)\int\limits_{x_0}^{x_2} K(\tau)D^{-1}(\tau)d\tau\right.$$

$$=\left\{\begin{array}{ll} -\dfrac{2K(l)}{D''(l)} & \textit{if } D'(l)=0,\ \ D''(l)\neq 0, \\ 0 & \textit{in the other arising cases} \end{array}\right.\Bigg). \tag{2.40}$$

**Proof.** If $K(0)=0$, then according to Lemma 2.1 for $k=0$ we have (2.25), and, hence,

$$\lim_{x_2\to 0+} x_2\int\limits_{x_2}^{x_0} K(\tau)D^{-1}(\tau)d\tau=0.$$

Let now $K(0)\neq 0$. By virtue of

$$K'(x_2)=K_2(x_2), \tag{2.41}$$

we obtain

$$\lim_{x_2\to 0+} x_2\int\limits_{x_2}^{x_0} K(\tau)D^{-1}(\tau)d\tau=\lim_{x_2\to 0+}\frac{x_2^2K(x_2)}{D(x_2)}$$

$$=\lim_{x_2\to 0+}\frac{2x_2K(x_2)+x_2^2K_2(x_2)}{D'(x_2)}$$

$$=\left\{\begin{array}{ll} 0 \quad \text{if } D'(0)\neq 0 \quad \text{or } D'(0)=\infty; & \\ \lim\limits_{x_2\to 0+}\dfrac{2K(x_2)+4x_2K_2(x_2)+x_2^2f(x_2)}{D''(x_2)} & \text{if } D'(0)=0. \end{array}\right. \tag{2.42}$$

Therefore, when $D'(0)=0$, we obtain

$$\lim_{x_2\to 0+} x_2\int\limits_{x_2}^{x_0} K(\tau)D^{-1}(\tau)d\tau=\left\{\begin{array}{ll} 0 \quad \text{if} \quad \mathrm{D}''(0)=\infty; & \\ \dfrac{2K(0)}{D''(0)} \quad \text{if } \mathrm{D}''(0)\neq 0, & \end{array}\right.$$

and

$$\lim_{x_2\to 0+} x_2\int\limits_{x_2}^{x_0} K(\tau)D^{-1}(\tau)d\tau=\infty \quad \text{if} \quad \mathrm{D}''(0)=0,\ \mathrm{K}(0)\neq 0. \tag{2.43}$$

But $D''(0)$ cannot be equal to 0, when $K(0)\neq 0$, otherwise (2.37) and (2.43) will contradict each other. Hence, (2.43) is excluded.

Similarly, we can prove (2.40). If $K(l) = 0$, then according to to Lemma 2.1 for $k = 0$ we have (2.25), and, hence,

$$\lim_{x_2 \to l-} (x_2 - l) \int_{x_0}^{x_2} K(\tau) D^{-1}(\tau) d\tau = 0.$$

Let now $K(l) \neq 0$. Then by virtue of (2.41), we obtain

$$\lim_{x_2 \to l-} (x_2 - l) \int_{x_0}^{x_2} K(\tau) D^{-1}(\tau) d\tau = \lim_{x_2 \to l-} \frac{-(x_2 - l)^2 K(x_2)}{D(x_2)}$$

$$= - \lim_{x_2 \to l-} \frac{2(x_2 - l) K(x_2) + (x_2 - l)^2 K_2(x_2)}{D'(x_2)}$$

$$= \begin{cases} 0 \quad \text{if } \ D'(l) \neq 0 \quad \text{or } \ D'(l) = +\infty; \\ - \lim\limits_{x_2 \to l-} \dfrac{2K(x_2) + 4(x_2 - l) K_2(x_2) + (x_2 - l)^2 f(x_2)}{D''(x_2)} \quad \text{if } \ D'(l) = 0. \end{cases} \tag{2.44}$$

Hence, when $D'(l) = 0$, we have

$$\lim_{x_2 \to l-} (x_2 - l) \int_{x_0}^{x_2} K(\tau) D^{-1}(\tau) d\tau = \begin{cases} 0 \quad \text{if } \quad D''(l) = \infty; \\ -\dfrac{2K(l)}{D''(l)} \quad \text{if } \ D''(l) \neq 0; \end{cases}$$

and

$$\lim_{x_2 \to l-} (x_2 - l) \int_{x_0}^{x_2} K(\tau) D^{-1}(\tau) d\tau = \infty \text{ if } D''(l) = 0, \ K(l) \neq 0. \tag{2.45}$$

But $D''(l)$ can not be equal to 0, when $K(l) \neq 0$, otherwise (2.38) and (2.45) will contradict each other. Hence, (2.45) is excluded. □

**Lemma 2.4.** *If $K(0) = 0$ $(K(l) = 0)$, $I_1^0 = +\infty$ and $I_2^0 < +\infty$ $(I_1^l = +\infty,\ I_2^l < +\infty)$, then (2.37) ((2.38)) is valid.*

**Proof.** Evidently, by virtue of $I_2^0 < +\infty$, we have

$$\begin{aligned} \left| x_2 \int_{x_2}^{x_0} K(\tau) D^{-1}(\tau) d\tau \right| &= \left| \int_{x_2}^{x_0} \frac{K(\tau)}{\tau} \frac{x_2}{\tau} \tau^2 D^{-1}(\tau) d\tau \right| \\ &\leq C \int_0^{x_0} \tau^2 D^{-1}(\tau) d\tau \\ &= \text{const} < +\infty \ \ \forall x_2 \in ]0, x_0], \end{aligned}$$

because of

$$0 < \frac{x_2}{\tau} \leq 1$$

(since $0 < x_2 \le \tau \le x_0$) and

$$\left|\frac{K(\tau)}{\tau}\right| < C \quad \forall\tau \in ]0, x_0] \tag{2.46}$$

(since $\lim\limits_{\tau\to 0+} \frac{K(\tau)}{\tau} = K_2(0) < +\infty$).

Similarly, by virtue of $I_2^l < +\infty$, we have

$$\begin{aligned}\left|(x_2 - l)\int_{x_2}^{x_0} K(\tau)D^{-1}(\tau)d\tau\right| &= \left|\int_{x_2}^{x_0} \frac{K(\tau)}{l-\tau}\,\frac{x_2 - l}{l-\tau}(l-\tau)^2 D^{-1}(\tau)d\tau\right| \\ &\le C\int_{x_0}^{l}(l-\tau)^2 D^{-1}(\tau)d\tau \\ &= \text{const} < +\infty \ \ \forall x_2 \in [x_0, l[,\end{aligned}$$

because of

$$0 < \frac{l - x_2}{l-\tau} \le 1$$

(since $x_0 \le \tau \le x_2 < l$, i.e., $0 < l - x_2 \le l - \tau$) and

$$\left|\frac{K(\tau)}{l-\tau}\right| < C \quad \forall\tau \in [x_0, l[ \tag{2.47}$$

(since $\lim\limits_{\tau\to l-} \frac{K(\tau)}{l-\tau} = -\lim\limits_{\tau\to l-} K'(\tau) = -\lim\limits_{\tau\to l-} K_2(\tau) = -K_2(l) < +\infty$). $\square$

**Lemma 2.5.** *Let either $D \in C^3([0,\ell[)$ $(D \in C^3(]0,\ell]))$ or the value of the first, second or third order derivative of $D$ at the point $x_2 = 0$ $(l)$ tend to infinity. Let further $f$ be bounded with its first derivative in a neighborhood $]0,\varepsilon[$ $(\ ]l-\varepsilon, l[\ )$ of the point $x_2 = 0$ $(x_2 = l)$. If $I_1 = +\infty$ and $I_2 < +\infty$, then*

*1.*

$$\lim_{x_2\to 0+} x_2 \int\limits_{x_2}^{x_0} K(\tau)D^{-1}(\tau)d\tau = \begin{cases} 0 \textit{ when } D'(0) \ne 0 \textit{ or } D'(0) = \infty \\ \quad \textit{or } D'(0) = 0 \textit{ and } D''(0) = \infty; \\ \dfrac{2K(0)}{D''(0)} \textit{ when } D'(0) = 0 \\ \quad \textit{and } D''(0) \ne 0, \\ \infty \textit{ when } D'(0) = 0 \textit{ and } D''(0) = 0 \end{cases} \tag{2.48}$$

$$\left(\lim_{x_2\to l-} (x_2 - l) \int\limits_{x_0}^{x_2} K(\tau)D^{-1}(\tau)d\tau = \begin{cases} 0 \textit{ when } D'(l) \ne 0 \textit{ or } D'(l) = \infty) \\ \quad \textit{or } D'(l) = 0 \textit{ and } D''(l) = \infty; \\ \dfrac{2K(l)}{D''(l)} \textit{ when } D'(l) = 0 \\ \textit{and } D''(l) \ne 0, \\ \infty \textit{ when } D'(l) = 0 \textit{ and } D''(l) = 0 \end{cases}\right) \tag{2.49}$$

*if*

$$K(0) \ne 0; (K(l) \ne 0)$$

*2.*

$$\lim_{x_2\to 0+} x_2 \int\limits_{x_2}^{x_0} K(\tau)D^{-1}(\tau)d\tau$$
$$= \begin{cases} 0 \textit{ when either } D'(0)\neq 0 \textit{ or } D'(0)=\infty \\ \ \textit{or } D'(0)=0 \textit{ and } D''(0)=\infty \\ \ \textit{or } D'(0)=0 \textit{ and } D''(0)\neq 0 \\ \ \ \textit{or } D'(0)=0,\ D''(0)=0,\ \textit{and } D'''(0)=\infty \\ \ \text{or } \mathrm{D}'(0)=0,\ \mathrm{D}''(0)=0,\ \mathrm{D}'''(0)=0,\ \text{and } \mathrm{K}_2(0)=0; \\ \dfrac{6K_2(0)}{D'''(0)} \text{ when } \mathrm{D}'(0)=0,\ \mathrm{D}''(0)=0,\ \text{and } \mathrm{D}'''(0)\neq 0 \end{cases} \tag{2.50}$$

*[the case* $D'''(0)=0$ *and* $(Q_2w)(0)\neq 0$ *(at the same time) is exclude]*

$$\left(\lim_{x_2\to l-} (x_2-l) \int\limits_{x_0}^{x_2} K(\tau)D^{-1}(\tau)d\tau\right.$$
$$= \begin{cases} 0 \text{ when either } \mathrm{D}'(\mathrm{l})\neq 0 \text{ or } \mathrm{D}'(\mathrm{l})=\infty \\ \ \text{or } \mathrm{D}'(\mathrm{l})=0 \text{ and } \mathrm{D}''(\mathrm{l})=\infty \\ \ \text{or } \mathrm{D}'(\mathrm{l})=0 \text{ and } \mathrm{D}''(\mathrm{l})\neq 0 \\ \ \text{or } \mathrm{D}'(\mathrm{l})=0,\ \mathrm{D}''(\mathrm{l})=0,\ \text{and } \mathrm{D}'''(\mathrm{l})=\infty \\ \ \text{or } \mathrm{D}'(\mathrm{l})=0,\ \mathrm{D}''(\mathrm{l})=0,\ \mathrm{D}'''(\mathrm{l})=0,\ \text{and } \mathrm{K}_2(\mathrm{l})=0; \\ \dfrac{6K_2(l)}{D'''(l)} \text{ when } \mathrm{D}'(\mathrm{l})=0,\ \mathrm{D}''(\mathrm{l})=0,\ \text{and } \mathrm{D}'''(\mathrm{l})\neq 0 \end{cases} \tag{2.51}$$

*[the case* $D'''(l)=0$ *and* $K_2(l)\neq 0$ *(at the same time) is excluded]*
*if*

$$K(0)=0 \quad (K(l)=0).$$

**Proof.** In both the cases the reasonings (2.42) are valid. Therefore, (2.48) easily follows from (2.42) if $K(0)\neq 0$. If $K(0)=0$, when $D'(0)=0$, from (2.42) we get

$$\lim_{x_2\to 0+} x_2 \int\limits_{x_2}^{x_0} K(\tau)D^{-1}(\tau)d\tau = = \begin{cases} 0 \text{ if } \ \mathrm{D}''(0)\neq 0; \\ \lim\limits_{x_2\to 0+} \dfrac{6K_2(x_2)+6x_2f(x_2)+x_2^2f'(x_2)}{D'''(x_2)} \\ \text{if } \mathrm{D}''(0)=0. \end{cases}$$

Hence, when $D'(0)=0$, $D''(0)=0$, we have

$$\lim_{x_2\to 0+} x_2 \int\limits_{x_2}^{x_0} K(\tau)D^{-1}(\tau)d\tau \ = \begin{cases} 0 \quad \text{if} \quad \mathrm{D}'''(0)=\infty; \\ \dfrac{6K_2(0)}{D'''(0)} \quad \text{if} \quad \mathrm{D}'''(0)\neq 0, \end{cases}$$

$$\lim_{x_2\to 0+} x_2 \int\limits_{x_2}^{x_0} K(\tau)D^{-1}(\tau)d\tau = \infty \text{ if } \mathrm{D}'''(0)=0, \quad \mathrm{K}_2(0)\neq 0. \tag{2.52}$$

But $D'''(0)$ and $K_2(0) \neq 0$ cannot take place at the same time, otherwise (2.52) and (2.37) (see Lemma 2.4 which has been proved under the assumptions $I_1^0 = +\infty$, $I_2^0 < +\infty$, without any requirment of differentiability of $D(x_2)$) will contradict each other. Thus, (2.52) is excluded. When $D'(0) = 0$, $D''(0) = 0$, $D'''(0) = 0$, and $K_2(0) = 0$, then according to the Lemma 2.1 for $k = 1$, (2.25) holds iff (2.15) is valid. Therefore,

$$\lim_{x_2 \to 0+} x_2 \int\limits_{x_2}^{x_0} K(\tau) D^{-1}(\tau) d\tau = 0 \text{ if } \mathrm{D}'(0) = 0,$$
$$D''(0) = 0, \quad D'''(0) = 0, \quad \text{and} \quad \mathrm{K}_2(0) = 0.$$

So, (2.47) is proved.

Similarly, in both the cases the reasonings (2.44) are valid. Therefore, (2.49) easily follows if $K(l) \neq 0$. If $K(l) = 0$, when $D'(l) = 0$, from (2.44) we get

$$\lim_{x_2 \to l-} (x_2 - l) \int\limits_{x_0}^{x_2} K(\tau) D^{-1}(\tau) d\tau$$
$$= \begin{cases} 0 \quad \text{if} \quad D''(l) \neq 0; \\ \lim\limits_{x_2 \to 0+} \dfrac{6K_2(x_2) + 6(x_2 - l) f(x_2) + (x_2 - l)^2 f'(x_2)}{D'''(x_2)} \\ \text{if } D''(l) = 0. \end{cases}$$

Hence, when $D'(l) = 0$, $D''(l) = 0$, we have

$$\lim_{x_2 \to l-} (x_2 - l) \int\limits_{x_0}^{x_2} K(\tau) D^{-1}(\tau) d\tau = \begin{cases} 0 \quad \text{if} \quad D'''(l) = \infty; \\ \dfrac{6K_2(l)}{D'''(l)} \quad \text{if} \quad D'''(l) \neq 0, \end{cases}$$

$$\lim_{x_2 \to l-} (x_2 - l) \int_{x_2}^{x_0} K(\tau) D^{-1}(\tau) d\tau = \infty \text{ if } D'''(l) = 0, \quad K_2(l) \neq 0. \tag{2.53}$$

But $D'''(l)$ and $K_2(l) \neq 0$ cannot take place at the same time, otherwise (2.53) and (2.38) (see Lemma 2.4 which has been proved under the assumptions $I_1^l = +\infty$, $I_2^l < +\infty$, without any requirment of differentiability of $D(x_2)$) will contradict each other. Thus, (2.53) is excluded. When $D'(l) = 0$, $D''(l) = 0$, $D'''(l) = 0$, and $K_2(l) = 0$, then according to the Lemma 2.1 for $k = 1$, (2.25) holds iff (2.15) is valid. Therefore,

$$\lim_{x_2 \to 0+} x_2 \int\limits_{x_0}^{x_2} K(\tau) D^{-1}(\tau) d\tau = 0$$
$$\text{if } D'(l) = 0, \; D''(l) = 0, \; D'''(l) = 0, \; \text{and } K_2(l) = 0.$$

So, (2.51) is proved. □

**Lemma 2.6.** *If $I_1^0 = +\infty$ and $I_2^0 < +\infty$ ($I_1^l = +\infty$ and $I_2^l < +\infty$), then*

$$\lim_{x_2\to 0+}\int_{x_2}^{x_0} K(\tau)\tau D^{-1}(\tau)d\tau = \int_0^{x_0} K(\tau)\tau D^{-1}(\tau)d\tau < +\infty \tag{2.54}$$

$$\left(\lim_{x_2\to l-}\int_{x_0}^{x_2}(l-\tau)K(\tau)D^{-1}(\tau)d\tau = \int_{x_0}^{l}(l-\tau)K(\tau)D^{-1}(\tau)d\tau < +\infty\right) \tag{2.55}$$

*iff*

$$K(0) = 0 \quad (K(l) = 0). \tag{2.56}$$

**Proof.** For every $\tau \in ]0, x_0]$ we have

$$|K(\tau)\tau D^{-1}(\tau)d\tau| = \left|\frac{K(\tau)}{\tau}\right| |\tau^2 D^{-1}(\tau)| \le C|\tau^2 D^{-1}(\tau)|, \tag{2.57}$$

by virtue of (2.46). But the right hand side of (2.57) is integrable on $]0, x_0[$, because of $I_2^0 < +\infty$. Therefore, the left hand side of (2.57) will be also integrable on $]0, x_0[$, and so, we arrive at (2.54). The necessity of (2.15) we can show in usual way by a contradiction (see e.g., (2.30)).

Similarly, for every $\tau \in [x_0, l[$ we have

$$|K(\tau)(l-\tau)D^{-1}(\tau)d\tau| = \left|\frac{K(\tau)}{l-\tau}\right| |(l-\tau)^2 D^{-1}(\tau)| \le C|(l-\tau)^2 D^{-1}(\tau)|, \tag{2.58}$$

by virtue of (2.47). But the right hand side of (2.58) is integrable on $]0, x_0[$, because of $I_2^l < +\infty$. Therefore, the left hand side of (2.58) will be also integrable on $]0, x_0[$, and so, we arrive at (2.55). The necessity of (2.15) we can show in usual way by a contradiction (see e.g., (2.30)).

**Proof** of Theorem 2.1.

Case I is evident in view of (2.7), (2.6), and $I_0^0$ ($I_0^l$) $< +\infty$.

Case II. $I_0^0 = +\infty$, $I_1^0 < +\infty$ ($I_0^l = +\infty$, $I_1^l < +\infty$). Then, in view of Lemma 2.2, the estimate (2.37) ((2.38)) is valid. Taking into account the fact that the other term

$$-\int_{x_0}^{x_2} \tau K(\tau)D^{-1}(\tau)d\tau$$

in (2.7) is bounded on $]0, x_0]$ ($[x_0, l[$) because of $I_1^0 < +\infty$ ($I_1^l < +\infty$), we conclude that

$$w(x_2) = O(1) \quad \text{as } x_2 \to 0+ \, (l-).$$

Moreover, if (2.13) is fulfilled then (2.14) is valid. Indeed, from Lemma 2.3 it follows that

$$x_2 \int_{x_0}^{x_2} K(\tau)D^{-1}(\tau)d\tau \in C([0, l[) \;\; (C(]0, l])).$$

Obviously, the other term

$$-\int\limits_{x_0}^{x_2} \tau K(\tau) D^{-1}(\tau) d\tau$$

of (2.7) is also continuous on $[0,l[$ ($]0,l]$) and hence (2.14) is proved.

If (2.15) is fulfilled, then in view of (2.6), (2.25), obviously, $w_{,2}$ is bounded on $]0,l[$. So, (2.16) is proved. The necessity of condition (2.15) for (2.16) readily follows from (2.6) and Lemma 2.1 (see (2.25), (2.15)).

Case III. $I_1^0 = +\infty$, $I_2^0 < +\infty$ ($I_1^l = +\infty$, $I_2^l < +\infty$). Then, according to Corollary 2.1 for the case $k = 1$ (see also Lemma 2.1), from (2.7) we get (2.17). Let us note that in order to consider $x_2 \to l-$ we represent (2.7) as follows

$$w = \int\limits_{x_0}^{x_2} [(x_2 - l) + (l - \tau)] K(\tau) D^{-1}(\tau) d\tau + C_3(x_2 - x_0) + C_4.$$

Case IV. Proof immediately follows from Corollary 2.1 for the case $k \geq 2$.

Case V is evident in view of Lemma 2.1 (see (2.6), (2.25), (2.15), (2.20)).

**Remark 2.1.** In Theorem 2.1 the existence of $k$ was assumed such that $I_k^0 < +\infty$. If $I_k^0 = +\infty\ \forall k$, and $K(\tau) := K_1(\tau) + \tau K_2(\tau)$ is analytic in a right neighborhood of $\tau = 0$ ($l$), then, obviously, $w$ and $w_{,2}$ are unbounded when $x_2 \to 0+$ ($l-$). We prove this by contradiction. Indeed, e.g., consider (2.6):

$$w_{,2}(x_2) = \int\limits_{x_0}^{x_2} K(\tau) D^{-1}(\tau) d\tau + C_3.$$

Let this derivative be bounded when $x_2 \to 0+$, and $K(0) = 0$; the last condition is necessary for the boundednes of this derivative. Since the analytic function $K(\tau) \not\equiv 0$, there exists $k$ such that

$$K^{(j)}(0) = 0, \quad j = 0, 1, ..., k-1, \quad K^{(k)}(0) \neq 0.$$

Further

$$w_{,2}(x_2) = \int\limits_{x_0}^{x_2} \frac{K(\tau)}{\tau^k} \tau^k D^{-1}(\tau) d\tau + C_3,$$

where

$$\lim_{\tau \to 0+} \frac{K(\tau)}{\tau^k} = \frac{K^{(k)}(0)}{k!} \neq 0.$$

Then, taking into account the boundedness of $w_{,2}$, similar to the proof of Lemma 2.1 we can show

$$\left| \int\limits_{x_2}^{\varepsilon} \tau^k D^{-1}(\tau) d\tau \right| < +\infty \qquad \text{for} \quad x_2 \in ]0, \varepsilon],$$

which would be in contradiction with $I_k = +\infty\ \forall k$. Thus, $w_{,2}$ is bounded when $x_2 \to 0+$.

**Remark 2.2.** In the case of the cusped beam with only one cusped end $x_2 = 0$ Theorem 2.1 formulated in the slightly different form is proved in [22].

## 3. Solution of Boundary Value Problems

From Theorem 2.1 we conclude that:

On the cusped edge $x_2 = 0$ (correspondingly $x_2 = l$) we can admit the following classical BCs:

$$\begin{aligned} &w = w_0 \text{ (correspondingly } w_l),\\ &w_{,2} = w_0' \text{ (correspondingly } w_l') \end{aligned} \tag{3.1}$$

if and only if (iff) $I_0^0$ (correspondingly, $I_0^l$) $< +\infty$;

$$w_{,2} = w_0'\ (w_l'),\quad Q_2 = Q_0(Q_l) \text{ iff } I_0^0\ (I_0^l) < +\infty; \tag{3.2}$$

$$w = w_0\ (w_l),\quad M_2 = M_0(M_l) \text{ iff } I_1^0\ (I_1^l) < +\infty; \tag{3.3}$$

$$M_2 = M_0(M_l),\quad Q_2 = Q_0(Q_l) \text{ iff } I_0^0\ (I_0^l) \le +\infty, \tag{3.4}$$

and the following non-classical (in the sense of the bending theory) conditions (replacing BCs):

$$w = w_0\ (w_l),\quad w_{,2} = O(1) \text{ when } x_2 \to 0+\ \ (x_2 \to l-) \tag{3.5}$$

if

$$I_0^0\ (I_0^l) = +\infty,\quad I_1^0\ (I_1^l) < +\infty;$$

$$w = O\ (1),\quad w_{,2} = O(1) \text{ when } x_2 \to 0+\ \ (x_2 \to l-) \tag{3.6}$$

if

$$I_1^0\ (I_1^l) = +\infty,$$

where $w_0, w_l, w_0', w_l', M_0, M_l, Q_0, Q_l$ are given constants, $O$ is a Landau symbol ($O(1)$ means boundedness).

**Theorem 3.1.** *Let the conditions of Theorem 2.1 be fulfilled. Then the following BVPs are well-posed:*

1. $(2.1),\ (3.1)_0\ (3.1)_l,\quad w \in C^4(\,]0,l[\,) \cap C^1([0,l]);$
2. $(2.1),\ (3.2)_0\ (3.1)_l,\quad w \in C^4(\,]0,l[\,) \cap C^1([0,l]);$
3. $(2.1),\ (3.3)_0\ (3.1)_l,\quad w \in C^4(\,]0,l[\,) \cap C^1(\,]0,l]) \cap C([0,l]);$
4. $(2.1),\ (3.4)_0\ (3.1)_l,\quad w \in C^4(\,]0,l[\,) \cap C^1(]0,l]);$
5. $(2.1),\ (3.1)_0\ (3.2)_l,\quad w \in C^4(\,]0,l[\,) \cap C^1([0,l]);$
6. $(2.1),\ (3.3)_0\ (3.2)_l,\quad w \in C^4(\,]0,l[\,) \cap C^1(\,]0,l]) \cap C([0,l]);$
7. $(2.1),\ (3.1)_0\ (3.3)_l,\quad w \in C^4(\,]0,l[\,) \cap C^1([0,l[\,) \cap C([0,l]);$
8. $(2.1),\ (3.2)_0\ (3.3)_l,\quad w \in C^4(\,]0,l[\,) \cap C^1([0,l[) \cap C([0,l]);$
9. $(2.1),\ (3.3)_0\ (3.3)_l,\quad w \in C^4(\,]0,l[\,) \cap C([0,l]);$
10. $(2.1),\ (3.1)_0\ (3.4)_l,\quad w \in C^4(\,]0,l[\,) \cap C^1([0,l[\,).$

**Remark 3.1.** Indices 0 and $l$ at (3.1)-(3.5) mean the corresponding formulas for the points 0 and $l$, respectively.

**Remark 3.2.** Actually, conditions (3.6) and (3.5) are not BCs. They are the conditions on $w$ in a neighborhood of the boundary point. That is why we say that in these cases BCs disappear at the cusped end of the beam.

**Proof** of Theorem 3.1. Using Theorem 2.1, Corollary 2.1, and Lemmas 2.3, 2.5, 2.6, we solve all the BVPs 1-10 in the explicit form. The uniqueness of solutions is guaranted by their construction from the general representation (2.7) of the solution $w$ of the Euler-Bernoulli equation (2.1). The continuous dependence of the solution $w$ [in the case of BCs (2.1), $(3.4)_0$, $(3.1)_l$, ((2.1), $(3.1)_0$, $(3.4)_l$)] with the weight

$$[I_k^0(x_2)]^{-1} := \left[\int\limits_{x_2}^{x_0} t^k D^{-1}(t)dt\right]^{-1}, \quad x_2 \in ]0, x_0]$$

$$\left([I_k^l(x_2)]^{-1} := \left[\int\limits_{x_0}^{x_2} (l-t)^k D^{-1}(t)dt\right]^{-1}, \quad x_2 \in [x_0, l[\right)$$

on the boundary data easyly follows from the explicit representations of the solutions of BVPs.

Thus, all the BVPs 1-10 are well-posed in the Hadamard sense. Let us solve them.

SOLUTION of BVP 1. Since $I_0^0, I_0^l < +\infty$, obviously, we can take $x_0 = l$. Then, in view of (2.6), (2.7), from (3.1) we have

$$C_4 = w_l, \ \ C_3 = w_l'.$$

For determination of constants $C_1$, $C_2$, from (3.1) we have the following algebraic system

$$C_1 \int\limits_0^l \tau(\tau - l)D^{-1}(\tau)d\tau + C_2 \int\limits_0^l \tau D^{-1}(\tau)d\tau$$

$$= \int\limits_0^l \tau D^{-1}(\tau) \int\limits_l^\tau f(t)(\tau - t)dtd\tau - lw_l' + w_l - w_0,$$

$$-C_1 \int\limits_0^l (\tau - l)D^{-1}(\tau)d\tau - C_2 \int\limits_0^l D^{-1}(\tau)d\tau$$

$$= -\int\limits_0^l D^{-1}(\tau) \int\limits_l^\tau f(t)(\tau - t)dtd\tau + w_l' - w_0',$$

which is solvable as its determinant satisfies

$$\Delta := \left[\int_0^l \tau D^{-1}(\tau)d\tau\right]^2 - \int_0^l \tau^2 D^{-1}(\tau)d\tau \cdot \int_0^l D^{-1}(\tau)d\tau < 0.$$

The last assertion follows from the Hölder inequality which is strong since $\tau D^{-\frac{1}{2}}(\tau)$ and $D^{-\frac{1}{2}}(\tau)$ are positive on $]0,l[$, and $\tau^2 D^{-1}(\tau)$ and $D^{-1}(\tau)$ differ from each other by a nonconstant factor $\tau^2$.

SOLUTION of BVP 9. From (2.5), taking into account the second conditions from $(3.3)_0$, $(3.3)_l$, we obtain

$$\begin{aligned} \int_{x_0}^0 tf(t)dt - C_1x_0 + C_2 &= M_0, \\ -\int_{x_0}^l (l-t)f(t)dt + C_1(l-x_0) + C_2 &= M_l. \end{aligned}$$

Solving this system, we get

$$C_1 = \int_{x_0}^l f(t)dt + l^{-1}\left[\int_l^0 tf(t)dt + M_l - M_0\right], \tag{3.7}$$

$$\begin{aligned} C_2 &= M_0 - \int_{x_0}^0 tf(t)dt + x_0\int_{x_0}^l f(t)dt + \\ &+ \frac{x_0}{l}\left[\int_l^0 tf(t)dt + M_l - M_0\right]. \end{aligned} \tag{3.8}$$

Hence, in view of (2.5), (2.11), we have

$$\begin{aligned} K(x_2) = -M_2(x_2) = \int_{x_0}^{x_2}(x_2-t)f(t)dt + \int_{x_0}^0 tf(t)dt - M_0 \\ -x_2\int_{x_0}^l f(t)dt - \frac{x_2}{l}\left[\int_l^0 tf(t)dt + M_l - M_0\right]. \end{aligned} \tag{3.9}$$

Further, from (2.7), by virtue of the first conditions from $(3.3)_0$, $(3.3)_l$ and Lemma 2.3, we obtain

$$\begin{cases} \dfrac{2K(0)}{D''(0)} & \text{if } D'(0)=0, \quad D''(0)\neq 0; \\ 0 & \text{in the other arising cases} \end{cases}$$

$$-\int_{x_0}^0 \tau K(\tau)D^{-1}(\tau)d\tau - C_3x_0 + C_4 = w_0,$$

$$-\begin{cases} \dfrac{2K(l)}{D''(l)} & \text{if } D'(l)=0,\ \ D''(l)\neq 0; \\ 0 & \text{in the other arising cases} \end{cases}$$

$$+\int\limits_{x_0}^{l}(l-\tau)K(\tau)D^{-1}(\tau)d\tau + C_3(l-x_0)+C_4=w_l.$$

Solving this system, we get

$$C_3=-\int\limits_{x_0}^{l}K(\tau)D^{-1}(\tau)d\tau + l^{-1}\Bigg[\int\limits_{l}^{0}\tau K(\tau)D^{-1}(\tau)d\tau + w_l - w_0$$
$$+\begin{cases} \dfrac{2K(0)}{D''(0)} & \text{if } D'(0)=0,\ \ D''(0)\neq 0 \\ 0 & \text{in the other arising cases} \end{cases}$$
$$+\begin{cases} \dfrac{2K(l)}{D''(l)} & \text{if } D'(l)=0,\ \ D''(l)\neq 0 \\ 0 & \text{in the other arising cases} \end{cases}\Bigg], \tag{3.10}$$

$$C_4=w_0-\begin{cases} \dfrac{2K(0)}{D''(0)} & \text{if } D'(0)=0,\ \ D''(0)\neq 0 \\ 0 & \text{in the other arising cases} \end{cases} + \int\limits_{x_2}^{0}\tau K(\tau)D^{-1}(\tau)d\tau$$
$$-x_0\int\limits_{x_0}^{l}K(\tau)D^{-1}(\tau)d\tau + \frac{x_0}{l}\Bigg[-\int\limits_{l}^{0}\tau K(\tau)D^{-1}(\tau)d\tau + w_l - w_0$$
$$+\begin{cases} \dfrac{2K(0)}{D''(0)} & \text{if } D'(0)=0,\ \ D''(0)\neq 0 \\ 0 & \text{in the other arising cases} \end{cases}$$
$$+\begin{cases} \dfrac{2K(l)}{D''(l)} & \text{if } D'(l)=0,\ \ D''(l)\neq 0; \\ 0 & \text{in the other arising cases} \end{cases}\Bigg]. \tag{3.11}$$

Thus, the solution has the form (2.7) with $K(\tau)$, $C_3$, $C_4$ given by (3.9)-(3.11), respectively.

Let us note that if either $I_1^l<+\infty$, $I_1^0=+\infty$ but $I_2^0<+\infty$ and $M_2(0)=M_0=0$ or $I_1^0<+\infty$, $I_1^l=+\infty$ but $I_2^l<+\infty$ and $M_2(l)=M_l=0$ or $I_1^0=+\infty$, $I_1^l=+\infty$ but $I_2^0<+\infty$ and $I_2^l<+\infty$ and $M_2(0)=M_0=0$, $M_2(l)=M_l=0$, then BVP 9 (call your attention to the change of the restrictions on $I_k^0$, $I_k^l, k=0,1$) will be uniquely solvable. The proof is based on Lemmas 2.5 and 2.6. The first case is investigated in [22]. The other two cases we consider analogously. In these three cases BVP 9 is not well-posed since the arbitrarily small change of BCs $M_2(0)=0$, $M_2(l)=0$ implies the unsolvability of the BVP under consideration.

SOLUTION of BVP 10. Since $I_0^0 < +\infty$, without loss of generelity, we assume $x_0 = 0$. From $(3.1)_0, (3.4)_l$ we get

$$C_4 = w_0, \quad C_3 = w_0', \quad C_2 = M_l - lQ_l - \int\limits_0^l tf(t)dt, \;\; C_1 = Q_l + \int\limits_0^l f(t)dt.$$

Thus,

$$\begin{aligned} w(x_2) = \int\limits_0^{x_2} (x_2 - \tau) \Bigg[ (l-\tau)Q_l - M_l + \int\limits_\tau^l tf(t)dt \\ - \int\limits_\tau^l f(t)dt \Bigg] D^{-1}(\tau)d\tau + w_0' x_2 + w_0. \end{aligned} \tag{3.12}$$

Representing $(x_2 - \tau)$ in (3.12) as $(x_2 - l) + (l - \tau)$, it is not difficult to see that

$$|w(x_2)| \le (l - x_2) \left[ \tilde{C}_1 I_0^l(x_2) + |Q_l| I_1^l(x_2) \right]$$

$$+ \tilde{C}_1 I_1^l(x_2) + |Q_l| I_2^l(x_2) + |w_0'| x_2 + |w_0| \quad \text{for} \;\; x_2 \in [0, l[,$$

where

$$\tilde{C}_1 := |M_0| + \int\limits_0^l t|f(t)|dt + \int\limits_0^l |f(t)|dt.$$

Therefore,

$$\begin{aligned} |w(x_2)| \le 2\tilde{C}_1 I_1^l + |Q_l| \left[ (l - x_2) I_1^l(x_2) + I_2^l(x_2) \right] \\ + |w_0'| x_2 + |w_0| \quad \text{for} \quad x_2 \in [0, l[, \quad \text{if} \quad I_1^l < +\infty, \end{aligned} \tag{3.13}$$

and

$$\begin{aligned} | \left[ I_1^l(x_2) \right]^{-1} w(x_2)| \le 2\tilde{C}_1 + |Q_l| \left[ (l - x_2) + \tilde{C}_2 \right] \\ + \tilde{C}_3 (|w_0'| x_2 + |w_0|) \quad \text{for} \quad x_2 \in [0, l[ \;\; \text{if} \quad I_1^l = +\infty, \end{aligned} \tag{3.14}$$

since

$$(l - x_2) I_0^l(x_2) \le I_1^l(x_2)$$

(because of

$$I_1^l(x_2) - (l - x_2) I_0^l(x_2) = \int\limits_0^{x_2} (x_2 - t) D^{-1}(t)dt \ge 0);$$

$$I_2^l(x_2) \le \tilde{C}_2 I_1^l(x_2), \quad \tilde{C}_2 = \text{const} > 0, \quad \forall x_2 \in [0, l[$$

(because of

$$\lim_{x_2 \to l-} \frac{I_2^l(x_2)}{I_1^l(x_2)} = \lim_{x_2 \to l-} \frac{(l - x_2)^2 D^{-1}(x_2)}{(l - x_2) D^{-1}(x_2)} = \lim_{x_2 \to l-} (l - x_2) = 0$$

if $I_1^l := I_1^l(l) = +\infty$);

$$[I_1^l(x_2)]^{-1} \le \tilde{C}_3 = \mathrm{const} > 0 \quad \forall x_2 \in [0,l[.$$

The continuous dependence in the class of continuous on $[0,l]$ functions of the solution $w(x_2)$ and of $[I_1^l(x_2)]^{-1}w(x_2)$ for $I_1^l < +\infty$ and $I_1^l = +\infty$, respectively, on the boundary data and on the right hand side $f$ immediatly follows from the estimates (2.13) and (2.14), correspondingly. Let us note that for $I_1^l = +\infty$, the solution $w(x_2)$ for a fixed $x_2 \in [0,l[$ continuously dependens on the boundary data and the right hand side $f$.

The other BVPs 2-8 can be solved in an analogous way. □

**Remark 3.3.** According to (2.11)

$$K_2(0) = -Q_2(0), \;\; K(0) = -M_2(0), \;\; K_2(l) = -Q_2(l), \;\; K(l) = -M_2(l)$$

and conditions (2.15) and (2.20) can be rewritten in the form

$$M_2(0) = 0 \;\; (M_2(l) = 0)$$

and

$$M_2(0) = 0, \; Q_2(0) = 0 \;\; (M_2(l) = 0, \; Q_2(l) = 0),$$

respectively. Now, by virtue of Theorem 2.1 (see (2.15), (2.20), (2.16), (2.17)), the following assertions become evident:

1) if $I_0^0\,(I_0^l) = +\infty$, $I_1^0\,(I_1^l) < +\infty$, then conditions

$$w_{,2} = O(1), \quad x_2 \to 0+ \;\; (x_2 \to l-) \tag{3.15}$$

can be replaced by BCs

$$M_2(0) = 0 \;\; (M_2(l) = 0) \tag{3.16}$$

and vice versa, i.e., (3.15) and (3.16) are equivalent conditions.

2) if $I_0^0\,(I_0^l) = +\infty$, then conditions (3.6) can be replaced by BCs

$$M_2(0) = 0, \;\; Q_2(0) = 0 \;\; (M_2(l) = 0, \;\; Q_2(l) = 0)$$

and vice versa, i.e., the last conditions and (3.6) are equivalent conditions.

**Remark 3.4.** Let $D(0) = 0$, $D(l) > 0$. Homogeneous BVP 1 (see Theorem 3.1) corresponds to the three-dimensional problem when the lateral surfaces are loaded by surface forces, the edge $x_2 = l$ is fixed and the edge $x_2 = 0$ is glued to the absolutely rigid tangent plane. In the case of homogeneous BVP 3 the above mentioned plane is rigid parallel to the axis $x_3$. BVP 4 corresponds to the three-dimensional problem when along the edge $x_2 = 0$ the concentrated along the above edge force and moment are applied which are equal to $Q_0$ and $Q_l$ respectively.

For forces and moments concentrated along the line (in particular, at a point of a cusped edge) see [5].

**Remark 3.5.** By setting BVPs we have to take into account peculiarities of classical bending (see (2.4), (2.5)) that by the arbitrary load $f$, the shearing force $Q_2$ can be given only on one edge; from $Q_2(0)$ (or $Q_2(l)$), $M_2(0)$, $M_2(l)$ only two can participate in BCs on the both edges (these peculiarities are not caused by cusps they arise even in case of bending of a beam of a constant section). If we choose $f$ correspondingly (see (2.4), (2.5)), we can avoid these peculiarities but restriction on choice of $f$ would be artificial (in the mathematical sense but natural in the physical sense). Nevertheless, problems posed in this way can also make practical sense. Obviously, solutions to all these problems can be constructed in explicit forms. Some of them are unique, some defined either up to rigid translation along the axis $x_3$ or rigid rotation at the axis $x_1$ or general rigid motion (combination of above mentioned). We omit the exact formulation of these artificial BVPs. But for the sake of illustration, at the end of this section we set and solve the typical one.

**Remark 3.6.** From Theorem 3.1. and Remark 3.3 we arrive to the following conclusions. In the case of BVPs 3 and 6 the derivative of solution $w_{,2}$ is bounded if either $I_0^0 < +\infty$ or $I_0^0 = +\infty$ and $M_0 = 0$. In the case of BVP 4: the solution $w$ is bounded if either $I_1^0 < +\infty$ or $I_1^0 = +\infty$ and $\exists k \geq 2$ such that $I_k^0 < +\infty$ and $M_0 = 0$ (for $k \geq 2$), $Q_0 = 0$ (for $k \geq 3$); the derivative of solution $w_{,2}$ is bounded if either $I_0^0 < +\infty$ or $I_0^0 = +\infty$ and $\exists k \geq 1$ such that $I_k^0 < +\infty$ and $M_0 = 0$ (for $k \geq 1$), $Q_0 = 0$ (for $k \geq 2$). In the case of BVPs 7 and 8 the derivative of solution $w_{,2}$ is bounded if either $I_0^l < +\infty$ or $I_0^l = +\infty$ and $M_l = 0$. In the BVP 9 the derivative of solution $w_{,2}$ is bounded if either $I_0^0 < +\infty$, $I_0^l < +\infty$ or $I_0^0 = +\infty$ but $M_0 = 0$ and $I_0^l = +\infty$ but $M_l = 0$. In the case of BVP 10: the solution $w$ is bounded if either $I_1^l < +\infty$ or $I_1^l = +\infty$ and $\exists k \geq 2$ such that $I_k^l < +\infty$ and $M_l = 0$ (for $k \geq 2$), $Q_l = 0$ (for $k \geq 3$); the derivative of solution $w_{,2}$ is bounded if either $I_0^l < +\infty$ or $I_0^l = +\infty$ and $\exists k \geq 1$ such that $I_k^l < +\infty$ and $M_l = 0$ (for $k \geq 1$), $Q_l = 0$ (for $k \geq 2$).

**Remark 3.7.** If $I_1^0 = +\infty$, BVP 4 with $M_0 = 0$, $Q_0 = 0$ is equivalent to BVP

$$(2.1),\ (3.6)_0,\ (3.1)_l, \quad w \in C^4(]0,l[) \cap C^1(]0,l]).$$

If $I_1^l = +\infty$, BVP 10 with $M_l = 0$, $Q_l = 0$ is equivalent to BVP

$$(2.1),\ (3.1)_0,\ (3.6)_l, \quad w \in C^4(]0,l[) \cap C^1([0,l[).$$

If $I_0^0 = +\infty$, $I_1^0 < +\infty$, BVP 4 with $M_0 = 0$, is equivalent to BVP

$$\begin{aligned}(2.1),\ w_{,2} = O(1) \text{ as } x_2 \to 0+,\ Q_2(0) = Q_0,\ (3.1)_l,\\ w \in C^4(]0,l[) \cap C^1(]0,l]).\end{aligned}$$

If $I_0^l = +\infty$, $I_1^l < +\infty$, BVP 10 with $M_l = 0$ is equivalent to BVP

$$\begin{aligned}(2.1),\ (3.1)_0,\ w_{,2} = O(1) \text{ as } x_2 \to l-,\ Q_2(l) = Q_l,\\ w \in C^4(]0,l[) \cap C^1([0,l[).\end{aligned}$$

If $I_0^0=+\infty$, $I_1^0<+\infty$, BVP 3 with $M_0=0$, is equivalent to BVP

$$(2.1),\ (3.5)_0,\ (3.1)_l,\quad w\in C^4(]0,l[)\cap C^1(]0,l])\cap C([0,l]).$$

If $I_0^l=+\infty$, $I_1^l<+\infty$, BVP 7 with $M_l=0$ is equivalent to BVP

$$(2.1),\ (3.1)_0,\ (3.5)_l,\quad w\in C^4(]0,l[)\cap C^1([0,l[)\cap C([0,l]).$$

If $I_0^0=+\infty$, $I_0^l=+\infty$, $I_1^0<+\infty$, $I_1^l<+\infty$, BVP 9 with $M_0=0$ and $M_l=0$ is equivalent to BVP

$$(2.1),\ (3.5)_0,\ \ (3.5)_l,\quad w\in C^4(]0,l[)\cap C([0,l]).$$

Let us now consider an example mentioned in Remark 3.5. Let moments and shearing forces be applied at the both ends of the beam, i.e.,

$$M_2(0)=M_0, \tag{3.17}$$
$$Q_2(0)=Q_0, \tag{3.18}$$
$$M_2(l)=M_l, \tag{3.19}$$
$$Q_2(l)=Q_l. \tag{3.20}$$

In order to determine constants $C_1,\ C_2$ from (2.5), (3.17), (3.19) we get the following system

$$\begin{aligned}\int_{x_0}^{0} tf(t)dt-C_1x_0+C_2 &= M_0,\\ -\int_{x_0}^{l}(l-t)f(t)dt+C_1(l-x_0)+C_2 &= M_l,\end{aligned}$$

whence,

$$C_1 = \frac{1}{l}\left[M_l-M_0+l\int_{x_0}^{l} f(t)dt-\int_{0}^{l} tf(t)dt\right], \tag{3.21}$$

$$\begin{aligned}C_2 &= \frac{1}{l}\left[lM_0+x_0(M_l-M_0)-l\int_{x_0}^{0} tf(t)dt+\right.\\ &+ \left. lx_0\int_{x_0}^{l} f(t)dt-x_0\int_{0}^{l} tf(t)dt\right].\end{aligned} \tag{3.22}$$

In view of (2.5), (2.4), (3.21), (3.22) we have

$$M_2(x_2) = -\int_{x_0}^{x_2}(x_2-t)f(t)dt+(x_2-x_0)\frac{1}{l}\left[M_l-M_0+l\int_{x_0}^{l} f(t)dt\right.$$

$$
\begin{aligned}
&- \left. \int\limits_0^l tf(t)dt \right] + \frac{1}{l}\left[ lM_0 + x_0(M_l - M_0) - l\int\limits_{x_0}^0 tf(t)dt \right. \\
&+ \left. lx_0 \int\limits_{x_0}^l f(t)dt - x_0 \int\limits_0^l tf(t)dt \right] \\
&= x_2 \int\limits_{x_2}^l f(t)dt + \int\limits_0^{x_2} tf(t)dt \\
&+ \frac{x_2}{l}\left[ M_l - M_0 - \int\limits_0^l tf(t)dt \right] + M_0,
\end{aligned}
$$

$$
Q_2(x_2) = \int\limits_{x_2}^l f(t)dt + \frac{1}{l}\left[ M_l - M_0 - \int\limits_0^l tf(t)dt \right]. \tag{3.23}
$$

Now, we must find conditions on $f(t)$ which guarantee satisfaction of BCs (3.18), (3.20). To this end we substitute (3.23) in (3.18), (3.20):

$$
l\int\limits_0^l f(t)dt - \int\limits_0^l tf(t)dt + M_l - M_0 = lQ_0, \tag{3.24}
$$

$$
-\int\limits_0^l tf(t)dt + M_l - M_0 = lQ_l. \tag{3.25}
$$

The difference of (3.24) and (3.25)gives

$$
\int\limits_0^l f(t)dt = Q_0 - Q_l. \tag{3.26}
$$

(3.26) with either (3.24) or (3.25), yields the conditions we were looking for. These conditions are natural in the physical sense since they express the fact that the resultant vector and resultant moment of the applied forces should be equal to zero.

Let us observe that $C_3$, $C_4$ in (2.6), (2.7) remain arbitrary. This means that we found the solution up to the rigid translation along the axis $x_3$ and rigid rotation at the axis $x_1$, which are expressed by arbitrary $C_4$ and $C_3$ respectively.

In particular, let the both ends be free:

$$
M_2(0) = Q_2(0) = M_2(l) = Q_2(l) = 0.
$$

Then the conditions (3.26),(3.24)and their equivalent conditions (3.26),(3.25) become

$$\int_0^l f(t)dt = 0, \quad \int_0^l tf(t)dt = 0.$$

This means that the lateral load and its moment are self-balanced. It is easy to see that the above assertions are also true if at the ends of the beam either $\sigma(x_2) > 0$ and Young's modulus $E(x_2) = 0$ or both vanish. In particular, this means that the peculiarities of the cusped beams will be preserved if we consider a beam of uniform cross-section with an appropriately chosen variable Young's modulus which vanishes at the ends.

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