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ON UNBOUNDED SOLUTIONS OF A DEGENERATE ELLIPTIC EQUARION IN NON-SMOOTH DOMAINS

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Abstract

Second order differential equation which order degenerate on the boundary of the angular domain is considered. In the cace of strong degenerate classes of unbounded solutiones is described.

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1. Introduction

Consider the equation

$$Lu = \Delta u - \frac{2k^2mx}{y^2 - k^2x^2}\frac{\partial u}{\partial x} + \frac{2my}{y^2 - k^2x^2}\frac{\partial u}{\partial y} + c(x,y)u = 0, \tag{1.1}$$

which is a constituent equation of the system of equilibrium equations for the prismatic shell, considered by I.Vekua in the case of zero approximation (sec [1]-[3]).

We are going to consider the equation in the following domain:

$$\Omega = \{(x,y) : y > -kx, \ \ y > kx, \ \ y < \sigma(x)\}$$

where $m > 0, \ k > 0, \ \sigma(x)$ is a defined on the interval (a, b), sufficiently smooth function $(a < 0, \ b > 0)$, $\sigma(a) = -ka$, $\sigma(b) = kb$, $\sigma(x) \neq$ $\neq -kx$ and $\sigma(x) \neq kx$ for any x from interval (a, b).

Those segments of boundary $\partial\Omega$, which lie on lines $y = -kx$ and $y = kx$ will be denoted by $\Gamma_1$and Γ_2 respectively and the rest of the boundary will be denoted by Γ_3.

Equation (1.1) in domain Ω is a second order elliptic differential equation. On the part $(\Gamma_1 \cup \Gamma_2)\backslash(0, 0)$ of the boundary it is under order degeneration, and at point $(0, 0)$ it is not a differential equation.

Below it will be assumed everywhere that $c(x, y) \in H^\delta(\bar{\Omega})(0 < \delta \leq$ $\leq 1)$ and $c(x, y) \leq 0$ in Ω.

For equation (1.1) problems of Dirichlet and Keldysh type were considered by us in paper [4]. It is interesting to describe some classes of non-bounded solutions of (1,1) in the case $m \leq 1$. One of the means to describe a non- bounded solution is to solve a problem with weight for degenerate elliptic equation, which was first set by A. Bitsadze (see [5],[6]).

Below we will use the term of a regular solution of equation (1.1) under which we meon such a solution, that has the second order continuous derivation in domain Ω. Let us seek the solution of this equation in the following form:

$$u(x,y) = \omega(x,y)v(x,y) \tag{1.2}$$

If we insert (1.2) expression of $u(x,y)$ in equation (1.1), we will receive the following equation towards v

$$Mv = \Delta v - \frac{2k^2x}{y^2-k^2x^2}\left(m - \frac{y^2-k^2x^2}{k^2x}\frac{\omega_x}{\omega}\right)\frac{\partial v}{\partial x} +$$

$$+\frac{2y}{y^2-k^2x^2}\left(m + \frac{y^2-k^2x^2}{y}\frac{\omega_y}{\omega}\right)\frac{\partial v}{\partial y} + \frac{L\omega}{\omega}v = 0 \tag{1.3}$$

To achieve our main gool it is necessary, afterselecting the proper function $\omega(x,y)$, to set required boundary problem for equation (1.3) and to show the axistence and uniqueness of its bounded solution. Below we are going to consider cases of $m > 1$ and $m = 1$ separately.

2. The case of $m > 1$

If $m > 1$ then let us consider the following weight

$$\omega(x,y) = (y^2-k^2x^2)^\alpha - dy^{2\alpha} ln(l(y^2-k^2x^2)),$$

were $\alpha < 0$ and $d, l > 0$ are yet unknown numbers.

Let us calculate $L\omega$ and reguire that $L\omega < 0$ in Ω.

$$\frac{\partial \omega}{\partial x} = -2\alpha k^2 x(y^2-k^2x^2)^{\alpha-1} + 2k^2dxy^{2\alpha}(y^2-k^2x^2)^{-1},$$

$$\frac{\partial \omega}{\partial y} = 2\alpha y(y^2-k^2x^2)^{\alpha-1} - 2dy^{2\alpha+1}(y^2-k^2x^2)^{-1} - 2d\alpha y^{2\alpha-1} ln(l(y^2-k^2x^2)),$$

$$\frac{\partial^2 \omega}{\partial x^2} = 4k^4\alpha(\alpha-1)x^2(y^2-k^2x^2)^{\alpha-2} - 2k^2\alpha(y^2-k^2x^2)^{\alpha-1} +$$

$$+4dk^4x^2y^{2\alpha}(y^2-k^2x^2)^{-2} + 2dk^2y^{2\alpha}(y^2-k^2x^2)^{-1},$$

$$\frac{\partial^2 \omega}{\partial y^2} = 4\alpha(\alpha-1)y^2(y^2-k^2x^2)^{\alpha-2} + 2\alpha(y^2-k^2x^2)^{\alpha-1} +$$

$$+4dy^{2\alpha+2}(y^2-k^2x^2)^{-2} - 8d\alpha y^{2\alpha}(y^2-k^2x^2)^{-1} - 2dy^{2\alpha}(y^2-k^2x^2)^{-1} -$$

$$-2d\alpha(2\alpha-1)y^{2\alpha-2} ln(l(y^2-k^2x^2)),$$

$$\Delta\omega = 4\alpha(\alpha-1)(y^2-k^2x^2)^{\alpha-2}(y^2+k^4x^2) + 2(1-k^2)\alpha(y^2-k^2x^2)^{\alpha-1} +$$

$$+4dy^{2\alpha}(y^2-k^2x^2)^{-2}(y^2+k^4x^2)+$$
$$+2dy^{2\alpha}(y^2-k^2x^2)^{-1}(k^2-4\alpha-1)-2d\alpha(2\alpha-1)y^{2\alpha-2}ln(l(y^2-k^2x^2)).$$
$$L\omega=\Delta\omega+4k^4m\alpha x^2(y^2-k^2x^2)^{\alpha-2}+4m\alpha y^2(y^2-k^2x^2)^{\alpha-2}-$$
$$-4dk^4mx^2y^{2\alpha}(y^2-k^2x^2)^{-2}-$$
$$-4dmy^{2\alpha+2}(y^2-k^2x^2)^{-2}-4d\alpha my^{2\alpha}(y^2-k^2x^2)^{-1}ln(l(y^2-k^2x^2))+c\omega$$
$$=\Delta\omega+4m\alpha(y^2-k^2x^2)^{\alpha-2}(y^2+k^4x^2)-4dmy^{2\alpha}(y^2-k^2x^2)^{-2}(y^2+k^4x^2)-$$
$$-4d\alpha my^{2\alpha}(y^2-k^2x^2)^{-1}ln(l(y^2-k^2x^2))+c\omega,$$
$$L\omega=4\alpha(y^2-k^2x^2)^{\alpha-2}(y^2+k^4x^2)(m+\alpha-1)+2(1-k^2)\alpha(y^2-k^2x^2)^{\alpha-1}+$$
$$+4d(1-m)y^{2\alpha}(y^2-k^2x^2)^{-2}(y^2+k^4x^2)+2dy^{2\alpha}(y^2-k^2x^2)^{-1}(k^2-4\alpha-1)-$$
$$-2d\alpha y^{2\alpha}(y^2-k^2x^2)^{-1}ln(l(y^2-k^2x^2))[(2\alpha-1)y^{-2}(y^2-k^2x^2)+2m]+c\omega. \tag{2.1}$$

Consider restriction of $L\omega$ on all $l_p=\{(x,y):x=py,\ |p|<\frac{1}{k}\}$ Lines.

$$L\omega|_{l_p}=2\alpha y^{2(\alpha-1)}(1-k^2p^2)^{\alpha-1}[2(1-k^2p^2)^{-1}(1+k^4p^2)(m+\alpha-1)+$$
$$+2d\alpha^{-1}(1-m)(1-k^2p^2)^{-\alpha-1}(1+k^4p^2)+1-k^2]-$$
$$-2dy^{2(\alpha-1)}(1-k^2p^2)^{-1}[\alpha ln(ly^2(1-k^2p^2))((2\alpha-1)(1-k^2p^2)+2m)-$$
$$-k^2+4\alpha+1]+(c\omega)|_{l_p}. \tag{2.2}$$

If we duly select constants d and l, then $L\omega|_{l_p}<0$ for every $|p|<\dfrac{1}{k}$ under the following conditions:

$$I.\begin{cases}\alpha=1-m\\ 1<m<2\end{cases}\qquad II.\begin{cases}\alpha>1-m\\ m>1\end{cases}$$

Thus, under conditions I and II d and l constants can be selected so that $\omega(x,y)>0$ and $L\omega<0$ in Ω, i.e. $\frac{L\omega}{\omega}<0$ in Ω.

Insert function $\omega(x,y)$ into equation (1.3). We receive the following equation with respect to v

$$Mv=\Delta v-\frac{2k^2xn_1(x,y)}{y^2-k^2x^2}\frac{\partial v}{\partial x}+\frac{2yn_2(x,y)}{y^2-k^2x^2}\frac{\partial v}{\partial y}+\frac{L\omega}{\omega}v=0, \tag{2.3}$$

where

$$n_1(x,y)=m+2\alpha(y^2-k^2x^2)^{\alpha}\omega^{-1}-2dy^{2\alpha}\omega^{-1}, \tag{2.4}$$
$$n_2(x,y)=n_1(x,y)-2d\alpha y^{2(\alpha-1)}ln(l(y^2-k^2x^2))(y^2-k^2x^2)\omega^{-1}.$$

There could be found such l that for arbitrary $\alpha\geq 1-m$ and $m>1$ the following estimation holds:

$$m+2\alpha-\varepsilon<n_1(x,y)<m,(x,y)\in\Omega \tag{2.5}$$

where $\varepsilon > 0$ can be made indenfinitaly small by selecting l. It is easy to see that $n_2(x,y)$ is bounded in Ω for every $\alpha \geq 1-m$ and $m > 1$.

Lemma 2.1. *If $1 < m < 2$ and $\alpha = 1-m$ then there exists the unique solution of (2.3), bounded and regular in domain Ω, which satisfies the following boundary condition $v|_{\partial\Omega\setminus\{(0,0)\}} = \varphi$, where $\varphi \in C(\partial\Omega)$.*

Proof. It is known that if at a certain point of the boundary a barrier can be found, then Viner solution of equation (2.3) will satisfy the boundary condition at this point (sec [7-11]). Function $f(x,y)$ is called a barrier at a point $Q(x_0,y_0) \in \partial\Omega$, if it satisfies the following conditions:
1. $f(x,y)$ continuous in $\Omega_Q = B_Q^\rho \cap \bar{\Omega}$ where $B_Q^\rho = \{(x,y) : (x-x_0)^2 + (y-$ $-y_0)^2 = \rho^2\}$, $\rho > 0$,
2. $f(x,y) \geq 0$ in Ω_Q and $f(x,y) = 0$ only at Q.
3. $Mf < 0$ in Ω_Q.

Let us show that at every point (x_0,y_0) of $(\Gamma_1 \cup \Gamma_2)\setminus\{(0,0)\}$ there exists a barrier. Consider the following equation

$$f(x,y) = (y^2 - k^2x^2)^\beta + (x-x_0)^2, \quad \beta > 0.$$

$f(x,y)$ is continuous, $f(x,y) \geq 0$ in the neighbourhood of (x_0,y_0) and $f(x,y) = 0$ only at (x_0,y_0). Now we have to show that $Mf < 0$ in some domain of (x_0,y_0). Indeed,

$$\frac{\partial f}{\partial x} = -2k^2\beta x(y^2 - k^2x^2)^{\beta-1} + 2(x-x_0),$$

$$\frac{\partial f}{\partial y} = 2\beta y(y^2 - k^2x^2)^{\beta-1},$$

$$\frac{\partial^2 f}{\partial x^2} = 4k^4\beta(\beta-1)x^2(y^2-k^2x^2)^{\beta-2} - 2k^2\beta(y^2-k^2x^2)^{\beta-1} + 2,$$

$$\frac{\partial^2 f}{\partial y^2} = 4\beta(\beta-1)y^2(y^2-k^2x^2)^{\beta-2} + 2\beta(y^2-k^2x^2)^{\beta-1}.$$

$$Mf = 4\beta(y^2-k^2x^2)^{\beta-2}(y^2+k^4x^2)(\beta-1+n_1(x,y)) +$$
$$+2\beta(1-k^2)(y^2-k^2x^2)^{\beta-1} - 4k^2n_1(x,y)x(x-x_0)(y^2-k^2x^2)^{-1} -$$
$$-8d\beta(1-m)y^{2(1-m)}ln(l(y^2-k^2x^2))(y^2-k^2x^2)^{\beta-1}\omega^{-1} + 2 + \frac{L\omega}{\omega}f.$$

In the small vicinity of (x_0,y_0) the plus or minus sign of Mf is determined by its first component, the sign of which depends on the sign of $\beta - 1 + n_1(x,y)$. From (2.4) it is clear that when $(x,y) \to (x_0,y_0)$, then $n_1(x,y) \to 2-m$, therefore $\lim\limits_{(x,y)\to(x_0,y_0)} = \beta + 1 - m$.

With duly selected β, $\beta + 1 - m < 0$. e.g. $Mf < 0$ for small enough vicinity of (x_0,y_0). Thus, function $f(x,y)$ is a barier at every point $(x_0,y_0) \in$ $\in (\Gamma_1 \cup \Gamma_2)\setminus\{(0,0)\}$.

Assume $(x_0, y_0) \in \Gamma_3$. Let us show that at (x_0, y_0) the barier has the following form:

$$f_1(x,y) = (x - x_0)^{2l_1} + (y - y_0)^{2l_2},$$

where $l_1, l_2 > 0$ are yet undertermined constants. It is easy to see that $f_1(x,y)$ satisfies conditions 1. and 2. for a barier. We have to show that $Mf_1 < 0$ for a sertain vicinity of (x_0, y_0).

$$\frac{\partial f_1}{\partial x} = 2l_1(x - x_0)^{2l_1 - 1},$$

$$\frac{\partial f_1}{\partial y} = 2l_2(y - y_0)^{2l_2 - 1},$$

$$\frac{\partial^2 f_1}{\partial x^2} = 2l_1(2l_1 - 1)(x - x_0)^{2(l_1 - 1)},$$

$$\frac{\partial^2 f_1}{\partial y^2} = 2l_2(2l_2 - 1)(y - y_0)^{2(l_2 - 1)}.$$

$$Mf_1 = \Delta f_1 - \frac{4k^2 l_1 n_1(x,y) x (x - x_0)^{2l_1 - 1}}{y^2 - k^2x^2} +$$

$$+ \frac{4l_2 n_1(x,y) y (y - y_0)^{2l_2 - 1}}{y^2 - k^2x^2} -$$

$$-8dl_2(1 - m)y^{1-2m}(y - y_0)^{2l_2 - 1} ln(l(y^2 - k^2x^2))\omega^{-1} + \frac{L\omega}{\omega} f_1 =$$

$$= 2l_1(x - x_0)^{2(l_1 - 1)} \left[2l_1 - 1 - \frac{2k^2 n_1(x,y) x (x - x_0)}{y^2 - k^2x^2}\right] +$$

$$+ 2l_2(y - y_0)^{2(l_2 - 1)} \left[2l_2 - 1 + \frac{2n_1(x,y) y (y - y_0)}{y^2 - k^2x^2}\right] -$$

$$-8dl_2(1 - m)y^{1-2m}(y - y_0)^{2l_2 - 1} ln(l(y^2 - k^2x^2))\omega^{-1} + \frac{L\omega}{\omega} f_1.$$

From received expression one can sec that when $l_1 < \frac{1}{2}$ and $l_2 < \frac{1}{2}$ then $Mf_1 < 0$ in a small enough vieinity of (x_0, y_0). So, $f_1(x,y)$ is a barrier for every $(x_0, y_0) \in \Gamma_3$.

Thus, according to the principle of barrier, Viner solution of equation (2.3) will satisfy the boundary condition set in Lemma.

Let us show the uniqueness of the solution. Consider $\psi(x,y) = -ln(x^2 + y^2) + q$ function, where $q > 0$ is a constant number. $\psi(x,y)$ satisfies the following conditions:

$$1.\ \ \psi(x,y) > 0, \quad (x,y) \in \Omega, \tag{2.6}$$

$$2.\ \ \lim_{(x,y)\to(0,0)} \psi(x,y) = +\infty \tag{2.7}$$

Let us show that $M\psi < 0$ in domain Ω.

$$\frac{\partial\psi}{\partial x} = -\frac{2x}{x^2+y^2}, \quad \frac{\partial\psi}{\partial y} = -\frac{2y}{x^2+y^2}, \quad \Delta\psi = 0.$$

$$M\psi = \frac{4k^2 n_1(x,y)x^2}{(y^2-k^2x^2)(x^2+y^2)} - \frac{4n_1(x,y)y^2}{(y^2-k^2x^2)(x^2+y^2)} +$$

$$+8d(1-m)y^{2(1-m)}(x^2+y^2)^{-1}ln(l(y^2-k^2x^2))\omega^{-1} + \frac{L\omega}{\omega}\psi =$$

$$= -4n_1(x,y)(x^2+y^2)^{-1} + 8d(1-m)y^{2(1-m)}(x^2+y^2)^{-1}ln(l(y^2-k^2x^2))\omega^{-1} +$$

$$+\frac{L\omega}{\omega}\psi = -4n_1(x,y)(x^2+y^2)^{-1} +$$

$$+8d(1-m)y^{2(1-m)}(x^2+y^2)^{-1}ln(l(y^2-k^2x^2))\omega^{-1} +$$

$$+\Theta(x,y)\psi\omega^{-1} + (L\omega - \Theta(x,y))\psi\omega^{-1},$$

Where $\Theta(x,y) = -d(1-m)y^{2(1-m)}(y^2-k^2x^2)^{-1}ln(l(y^2-k^2x^2))[(1-$ $-2m)y^{-2}(y^2-k^2x^2)+2m]$.

Taking into account (2.5) we will receive that the first component of $M\psi$ is negative in Ω. From (2.1) and (2.2) it is clear that $L\omega - \Theta(x,y) < 0$ in Ω. Consider the sum the second and the third components of $M\psi$.

$$8d(1-m)y^{2(1-m)}(x^2+y^2)^{-1}ln(l(y^2-k^2x^2))\omega^{-1} + \Theta(x,y)\psi\omega^{-1} =$$

$$= d(1-m)y^{2(1-m)}ln(l(y^2-k^2x^2))\omega^{-1}[8(x^2+y^2)^{-1} - (y^2-k^2x^2)^{-1}(-ln(x^2+y^2)+$$

$$+q)((1-2m)y^{-2}(y^2-k^2x^2)+2m)].$$

It is clear that for big enough q the received expression is less than 0 in Ω. So we get that $M\psi < 0$ in Ω.

Suppose the boundary problem bas two distinct solutions v_1 and v_2 then $v_0 \equiv v_1 - v_2$ will be the solution of uniform boundary problem. Consider function $\varepsilon\psi \pm v_0$. For arbitrary $\varepsilon > 0$ in domoin Ω the folowing inequality holds true:

$$L(\varepsilon\psi \pm v_0) < 0 \tag{2.8}$$

Let us define the domains $\Omega'_n \subset \Omega$ in this way ;

$$\Omega'_n = \{(x,y) : y > -kx, y > kx, y > \frac{1}{n}, y < \sigma(x)\} \;\; n \in N.$$

Take any point Q from domain Ω. For arbitrary $\varepsilon > 0$ there exists a domain $Q \in \Omega'_n$ such that on its boundary the following inequality holds true:

$$\varepsilon\psi \pm v_0 > 0$$

From (2.8)we receive that $L(\varepsilon\psi \pm v_0) < 0$ in domain Ω'_n. According to the principle of maximum $\varepsilon\psi \pm v_0 > 0$ in Ω'_n. So, for arbitrary $\varepsilon > 0$ the inequality $\varepsilon\psi(Q) \pm v_0(Q) > 0$ holds true. As a result

$$|v_0(Q)| < \varepsilon\psi(Q) \quad \forall\varepsilon > 0.$$

i.e. $v_0(Q) = 0$. As Q was an arbitrary point from Ω, so $v_0 = 0$ in Ω, with this Lemma 2.1. is proved.

From Lemma 2.1. it follows that the following theorem holds true:

Theorem 2.1. *If $1 < m < 2$ and $\alpha = 1-m$ then there exists the unique regular solution of equation (1.1) in Ω, which satisfies the following boundary condition:* $\lim\limits_{(x,y)\to(x_0,y_0)} u(x,y)\omega^{-1} = \varphi(x_0,y_0),\ \forall(x_0,y_0) \in \partial\Omega\backslash\{(0,0)\},\ \varphi \in$ $\in C(\partial\Omega)$.

Lemma 2.2. *If $m > 1$ and $\alpha > 1-m$ then there exists the uniqul bounded regular solution of equation (2.3) in domain Ω, which satisfies the following boundary condition $v|_{\partial\Omega\backslash\{0,0\}} = \varphi$, which $\varphi \in C(\partial\Omega)$ and $\varphi\backslash_{\Gamma_1\cup\Gamma_2}$=0.*

Proof. Let us show that at every point (x_0,y_0) of $\Gamma_1\cup\Gamma_2\backslash\{0,0\}$ there exists a barrier. Consider $f(x,y) = (y^2-k^2x^2)^\beta + (x-x_0)^2,\ \beta > 0$ function. $f(x,y)$ satisfies 1. and 2. conditions of barrier. Let us show validity of 3. Indeed,

$$Mf = 4\beta(y^2-k^2x^2)^{\beta-2}(y^2+k^4x^2)(\beta-1+n_1(x,y))+2\beta(1-k^2)(y^2-k^2x^2)^{\beta-1}-$$
$$-4k^2n_1(x,y)x(x-x_0)(y^2-k^2x^2)^{-1}-$$
$$-8d\beta\alpha y^{2\alpha}(y^2-k^2x^2)^{\beta-1}ln(l(y^2-k^2x^2))\omega^{-1}+2+\frac{L\omega}{\omega}f.$$

In the vicinity of (x_0,y_0) $\frac{L\omega}{\omega}f = O((y^2-k^2x^2)^{\beta-2})$. Therefore for a ny $\alpha >$ $> 1-m$ with duly selected β $Mf < 0$ in the small enough vicinity of (x_0,y_0). Thus, at every point (x_0,y_0) of $(\Gamma_1\cup\Gamma_2)\backslash\{0,0\}$ there exists a barrier. According to barrier principle, Viner solution of equation (2.3) will satisty the uniform boundary condition.

At every point (x_0,y_0) of Γ_3 there exists a barrier and it has the form: $f_1(x,y) = (x-x_0)^{2l_1} + (y-y_0)^{2l_2},\ l_1 > 0,\ l_2 > 0$. With this the existence of the solution of the boundary problem set in Lemma is proved.

Let us show the uniqueness of the solution. Consider $\psi(x,y) = -ln(x^2+$ $+y^2) + q,\ q > 0$ function. $\psi(x,y)$ satisfies conditions (2.6) and (2.7). Let us show that $M\psi < 0$ in Ω.

$$M\psi = -4n_1(x,y)(x^2+y^2)^{-1}+8d\alpha y^{2\alpha}(x^2+y^2)^{-1}ln(l(y^2-k^2x^2))\omega^{-1}+\frac{L\omega}{\omega}\psi =$$
$$= -4(x^2+y^2)^{-1}\Big[m-2\alpha(dy^{2\alpha}(y^2-k^2x^2)^{-\alpha}ln(l(y^2-k^2x^2))-1)^{-1}-2dy^{2\alpha}\omega^{-1}-$$
$$-2\alpha(d^{-1}y^{-2\alpha}(y^2-k^2x^2)^{\alpha}ln^{-1}(l(y^2-k^2x^2))-1)^{-1}\Big]+\frac{L\omega}{\omega}\psi =$$
$$= -4(x^2+y^2)^{-1}[m+2\alpha-2dy^{2\alpha}\omega^{-1}]+\frac{L\omega}{\omega}\psi =$$
$$= -4(x^2+y^2)^{-1}[m+2\alpha-2dy^{2\alpha}\omega^{-1}]+\Theta_1(x,y)\psi\omega^{-1}+(L\omega-\Theta_1(x,y))\psi\omega^{-1},$$

where $\Theta_1(x,y) = 2\alpha(y^2 - k^2x^2)^{\alpha-2}(y^2 + k^4x^2)(m + \alpha - 1)$. From (2.1) and (2.2) it is clear that $L\omega - \Theta_1(x,y) < 0$ in Ω. Let us introduce the following rotation:

$$g(x,y) \equiv -4(x^2+y^2)^{-1}[m + 2\alpha - 2dy^{2\alpha}\omega^{-1}] + \Theta_1(x,y))\psi\omega^{-1}.$$

$$g(x,y) = -4(x^2+y^2)^{-1}[m+2\alpha-2(d^{-1}y^{-2\alpha}(y^2-k^2x^2)^{\alpha}-ln(l(y^2-k^2x^2)))^{-1}]+$$
$$+\frac{2\alpha(m+\alpha-1)(y^2+k^4x^2)(-ln(x^2+y^2)+q)}{(y^2-k^2x^2)^2-dy^{2\alpha}(y^2-k^2x^2)^{2-\alpha}ln(l(y^2-k^2x^2))} =$$
$$= -2(x^2+y^2)^{-1}[2(m+2\alpha) - 4(d^{-1}y^{-2\alpha}(y^2-k^2x^2)^{\alpha} - ln(l(y^2-k^2x^2)))^{-1} -$$
$$-\frac{\alpha(m+\alpha-1)(y^2+k^4x^2)(x^2+y^2)(-ln(x^2+y^2)+q)}{(y^2-k^2x^2)^2-dy^{2\alpha}(y^2-k^2x^2)^{2-\alpha}ln(l(y^2-k^2x^2))}].$$

Consider limitation of $g(x,y)$ on l_p lines.

$$g(x,y)|_{l_p} = -2y^{-2}(p^2+1)^{-1}\Big[2(m+2\alpha)-4(d^{-1}(1-k^2p^2)^{\alpha}-ln(ly^2(1-k^2p^2)))^{-1}-$$
$$-\frac{\alpha(m+\alpha-1)(1+k^4p^2)(p^2+1)(-ln(y^2(p^2+1))+q)}{(1-k^2p^2)^2-d(1-k^2p^2)^{2-\alpha}ln(y^2(1-k^2p^2))}\Big].$$

From the received expression it is clear that for big enough q, $g|_{l_p} < 0$ for arbitrary $|p| < \frac{1}{k}$, i.e. $g(x,y) < 0$ in Ω. Thus we receive that $M\psi < 0$ in Ω.

By using reasoning, analogous to the reasoning given in the proof of Lemma 2.1, the uniquenesss of the solution of the set boundary problem is proved. So, Lemma 2.2 is proved.

From Lemma 2.2 validity of the following theorem follows.

Theorem 2.2. *If $m > 0$ and $\alpha > 1 - m$ then there exists the unique regular solution of equation (1.2) in Ω, which satisfies the following boundary condition:*

$$\lim_{(x,y)\to(x_0,y_0)} u(x,y)\omega^{-1} = \varphi(x_0,y_0),\ \forall(x_0,y_0) \in \partial\Omega\backslash\{(0,0)\},\ \varphi|_{\Gamma_1\cup\Gamma_2} = 0.$$

3. $m = 1$ case

When $m = 1$ then let us take the following $\omega(x,y)$ weight.

$$\omega(x,y) = (-ln((y^2-k^2x^2)l))^{\gamma} + (-ln(ly))^{\gamma+1}$$

where x and l are positive constants. Let us require that $\omega(x,y) > 0$ and $L\omega < 0$ in Ω.

$$\frac{\partial\omega}{\partial x} = 2k^2\gamma x(-ln((y^2-k^2x^2)l))^{\gamma-1}(y^2-k^2x^2)^{-1},$$

$$\frac{\partial\omega}{\partial y} = -2\gamma y(-ln((y^2-k^2x^2)l))^{\gamma-1}(y^2-k^2x^2)^{-1} - (\gamma+1)y^{-1}(-ln(ly))^{\gamma},$$

$$\frac{\partial^2\omega}{\partial x^2} = 4k^4\gamma x^2(y^2-k^2x^2)^{-2}(-ln((y^2-k^2x^2)l))^{\gamma-1}\times$$
$$\times[1-(\gamma-1)ln^{-1}((y^2-k^2x^2)l)]+$$
$$+2k^2\gamma(y^2-k^2x^2)^{-1}(-ln((y^2-k^2x^2)l))^{\gamma-1}$$
$$\frac{\partial^2\omega}{\partial y^2} = 4\gamma y^2(y^2-k^2x^2)^{-2}(-n((y^2-k^2x^2)l))^{\gamma-1}\times$$
$$\times[1-(\gamma-1)ln^{-1}((y^2-k^2x^2)l)]-$$
$$-2\gamma(y^2-k^2x^2)^{-1}(-ln((y^2-k^2x^2)l))^{\gamma-1}-(\gamma+1)y^{-2}(-ln(ly))^{\gamma-1}(ln(ly)-\gamma).$$
$$\Delta\omega = 4\gamma(y^2-k^2x^2)^{-2}(y^2+k^4x^2)(-ln((y^2-k^2x^2)l))^{\gamma-1}\times$$
$$\times\Big[1-(\gamma-1)ln^{-1}((y^2-k^2x^2)l)\Big]+$$
$$+2\gamma(k^2-1)(y^2-k^2x^2)^{-1}(-ln((y^2-k^2x^2)l))^{\gamma-1}-$$
$$-(\gamma+1)y^{-2}(-ln(ly))^{\gamma-1}(ln(ly)-\gamma)$$
$$L\omega = \Delta\omega - 4k^4\gamma x^2(y^2-k^2x^2)^{-2}(-ln((y^2-k^2x^2)l))^{\gamma-1}-$$
$$-4\gamma y^2(y^2-k^2x^2)^{-2}(-ln((y^2-k^2x^2)l))^{\gamma-1}-2(\gamma+1)(y^2-k^2x^2)^{-1}(-ln(ly))^{\gamma}+$$
$$+c\omega = \Delta\omega - 4\gamma(y^2-k^2x^2)^{-2}(y^2+k^4x^2)(-ln((y^2-k^2x^2)l))^{\gamma-1}-$$
$$-2(\gamma+1)(y^2-k^2x^2)^{-1}(-ln(ly))^{\gamma}+c\omega$$
$$= 4\gamma(\gamma-1)(y^2-k^2x^2)^{-2}(y^2+k^4x^2)(-ln((y^2-k^2x^2)l))^{\gamma-2}+$$
$$+2(y^2-k^2x^2)^{-1}\Big[\gamma(k^2-1)(-ln((y^2-k^2x^2)l))^{\gamma-2}-(\gamma+1)(-ln(ly))^{\gamma}\Big]-$$
$$-(\gamma+1)y^{-2}(-ln(ly))^{\gamma-1}(ln(ly)-\gamma)+c\omega.$$

Consider limitation of $L\omega$ on every $L_p = \{(x,y) : x = py, |p| < \dfrac{1}{k}\}$ line.

$$L\omega|_{l_p} = 4\gamma(\gamma-1)y^{-2}(1-k^2p^2)^{-2}(1+k^4p^2)(-ln(ly^2(1-k^2p^2)))^{\gamma-2}+$$
$$+2y^{-2}(1-k^2p^2)^{-1}[\gamma(k^2-1)(-ln(ly^2(1-k^2p^2)))^{\gamma-1}-(\gamma+1)(-ln(ly))^{\gamma}]-$$
$$-(\gamma+1)y^{-2}(-ln(ly))^{\gamma-1}(ln(ly)-\gamma)+(c\omega)|_{l_p} =$$
$$= y^{-2}(1-k^2p^2)^{-1}[4\gamma(\gamma-1)(1-k^2p^2)^{-1}(1+k^4p^2)(-ln(ly^2(1-k^2p^2)))^{\gamma-2}-$$
$$-(\gamma+1)(-ln(ly))^{\gamma-1}(ln(ly)-\gamma)(1-k^2p^2)-$$
$$-2(\gamma+1)(-ln(ly))^{\gamma}+2\gamma(k^2-1)(-ln(ly^2(1-k^2p^2)))^{\gamma-1}]+(c\omega)|_{l_p} =$$
$$= y^{-2}(1-k^2p^2)^{-1}[4\gamma(\gamma-1)(1-k^2p^2)^{-1}(1+k^4p^2)(-ln(ly^2(1-k^2p^2)))^{\gamma-2}+$$
$$+(\gamma+1)(-ln(ly))^{\gamma}((1-k^2p^2)-2)+$$
$$+\gamma(\gamma+1)(-ln(ly))^{\gamma-1}(1-k^2p^2)+2\gamma(k^2-1)(-ln(ly^2(1-k^2p^2)))^{\gamma-1}]+(c\omega)|_{l_p}.$$

From this it is clear that for small enough l, when $0 < \gamma \leq 1$ for every $|p| < \frac{1}{k}$, the inequality $L\omega|_{l_p} < 0$ holds true. Therefore $L\omega < 0$ in Ω, i.e.

$\dfrac{L\omega}{\omega} < 0$ in Ω. Insert $\omega(x,y)$ in (1.3). We will receive the following equality with respect to v:

$$Mv = \Delta v - \frac{2k^2 n_1(x,y)x}{y^2-k^2x^2}\frac{\partial v}{\partial x} + \frac{2n_2(x,y)y}{y^2-k^2x^2}\frac{\partial v}{\partial y} + \frac{L\omega}{\omega}v = 0, \tag{3.1}$$

where:

$$n_1(x,y) = 1 - 2\gamma(-ln(l(y^2-k^2x^2)))^{\gamma-1}\omega^{-1},$$
$$n_2(x,y) = n_1(x,y) - (\gamma+1)y^{-2}(-ln(ly))^{\gamma}(y^2-k^2x^2)\omega^{-1}$$

If we take small enough l then for arbitrary $0 < \gamma \leq 1$ the following estimation holds true

$$0 < n_1(x,y) < 1, \quad (x,y) \in \Omega.$$

It is clear that $n_2(x,y)$ is bounded in Ω.

Lemma 3.1. *If $m = 1$ and $\gamma = 1$ then there exists the only bounded regular solution in domain Ω which satisfies the following boundary condition $v|_{\partial\Omega\setminus\{(0,0)\}} = \varphi$, $\varphi \in C(\partial\Omega)$.*

Proof. Let us show that at any point (x_0,y_0) of $(\Gamma_1 \cup \Gamma_2)\setminus\{(0,0)\}$ there exists a barrier. Consider a function of the following tupe: $f(x,y) = ln^{-1}(\beta(y^2-k^2x^2)) + (x-x_0)^2$, where $\beta > 0$ is yet undetermined constant number. After duly selecting β, $f(x,y)$ will satisfy 1.2. conditions of barrier. It remains to prove that the inequality $Mf < 0$ holds true for $f(x,y)$.

$$\frac{\partial f}{\partial x} = -2k^2x(y^2-k^2x^2)^{-1}ln^{-2}(\beta(y^2-k^2x^2)) + 2(x-x_0),$$

$$\frac{\partial f}{\partial y} = 2y(y^2-k^2x^2)^{-1}ln^{-2}(\beta(y^2-k^2x^2)),$$

$$\frac{\partial^2 f}{\partial x^2} = -4k^4x^2(y^2-k^2x^2)^{-2}ln^{-2}(\beta(y^2-k^2x^2))(1+2ln^{-1}(\beta(y^2-k^2x^2))-$$
$$-2k^2(y^2-k^2x^2)^{-1}ln^{-2}(\beta(y^2-k^2x^2)) + 2.$$

$$\frac{\partial^2 f}{\partial y^2} = -4y^2(y^2-k^2x^2)^{-2}ln^{-2}(\beta(y^2-k^2x^2))(1+2ln^{-1}(\beta(y^2-k^2x^2)))+$$
$$+2(y^2-k^2x^2)^{-1}ln^{-2}(\beta(y^2-k^2x^2)),$$

$$\Delta f = -4(y^2-k^2x^2)^{-2}ln^{-2}(\beta(y^2-k^2x^2))(1+2ln^{-1}(\beta(y^2-k^2x^2)))(y^2+k^4x^2)-$$
$$-2(y^2-k^2x^2)^{-1}ln^{-2}(\beta(y^2-k^2x^2))(k^2-1)+2$$

$$Mf = \Delta f + 4k^4n_1(x,y)x^2(y^2-k^2x^2)^{-2}ln^{-2}(\beta(y^2-k^2x^2))-$$
$$-4k^2n_1(x,y)(y^2-k^2x^2)^{-1}x(x-x_0)+$$
$$+4n_2(x,y)y^2(y^2-k^2x^2)^{-2}ln^{-2}(\beta(y^2-k^2x^2)) + \frac{L\omega}{\omega}f =$$

$$= \Delta f + 4n_1(x,y)(y^2 - k^2x^2)^{-2}(y^2 + k^4x^2)ln^{-2}(\beta(y^2 - k^2x^2)) +$$
$$+8(y^2 - k^2x^2)^{-1}ln(ly)ln^{-2}(\beta(y^2 - k^2x^2))\omega^{-1} - 4k^2n_1(x,y)(y^2 - k^2x^2)^{-1}x(x - x_0) +$$
$$+\frac{L\omega}{\omega}f = 4(y^2 - k^2x^2)^{-2}(y^2 + k^4x^2)ln^{-2}(\beta(y^2 - k^2x^2)) \times$$
$$\times[n_1(x,y) - 2ln^{-1}(\beta(y^2 - k^2x^2)) - 1] -$$
$$-2(y^2 - k^2x^2)^{-1}ln^{-2}(\beta(y^2 - k^2x^2))(k^2 - 1) +$$
$$+8(y^2 - k^2x^2)^{-1}ln(ly)ln^{-2}(\beta(y^2 - k^2x^2))\omega^{-1} -$$
$$-4k^2n_1(x,y)(y^2 - k^2x^2)^{-1}x(x - x_0) + 2 + \frac{L\omega}{\omega}f -$$
$$= -8(y^2 - k^2x^2)^{-2}(y^2 + k^4x^2)ln^{-3}(\beta(y^2 - k^2x^2))\omega^{-1}[ln\frac{\beta}{l} + ln^2(ly)] -$$
$$-2(y^2 - k^2x^2)^{-1}ln^{-2}(\beta(y^2 - k^2x^2))(k^2 - 1) +$$
$$+8(y^2 - k^2x^2)^{-1}ln(ly)ln^{-2}(\beta(y^2 - k^2x^2))\omega^{-1} -$$
$$-4k^2n_1(x,y)(y^2 - k^2x^2)^{-1}x(x - x_0) + 2 + \frac{L\omega}{\omega}f.$$

The sign of Mf in a small vicinity of any poing of $(\Gamma_1 \cup \Gamma_2)\backslash\{(0,0)\}$ is determined by the first component of the last expression, which is negative for small enough β. Thus $f(x,y)$ satisfies three conditions of a barrier.

At every (x_0, y_0) point of Γ_3 there exists a barrier and it has $f_1(x,y) = (x - x_0)^{2l_1} + (y - y_0)^{2l_2}$ form. A ccording to the principle of barrier, the boundary problem set in Lemma has a solution.

Let us show the uniqueness of the solution. Consider $\psi(x,y) = -ln(x^2 + y^2) + q, q > 0$ function. $\psi(x,y) > 0$ in Ω and $\lim\limits_{(x,y)\to(0,0)} \psi(x,y) = +\infty$. Let us show that $M\psi < 0$ in Ω.

$$M\psi = -4n_1(x,y)(x^2 + y^2)^{-1} - 8(y^2 + x^2)^{-1}ln(ly)\omega^{-1} + \frac{L\omega}{\omega}\psi =$$
$$= -4(x^2 + y^2)^{-1}[1 - 2\omega^{-1} + 2ln(ly)\omega^{-1}] + \frac{L\omega}{\omega}\psi.$$

For small enough l the expression in square brackets is positive in Ω, so $M\psi < 0$ in Ω.

By using the reasoning analogous to that in Lemma 2.1 the uniqueness of the solution can be proved.

From Lemma 3.1. the validity of the following theorem follows:

Theorem 3.1. If $m = 1$ and $\gamma = 1$ then there exists the unique regular solution of (1.1) in Ω which satisfies the following condition
$\lim\limits_{(x,y)\to(x_0,y_0)} u(x,y)\omega^{-1} = \varphi(x_0,y_0),\ \forall(x_0,y_0) \in \partial\Omega\backslash\{(0,0)\},\ \varphi \in C(\partial\Omega).$

Lemma 3.2. *If $m = 1$ and $0 < \gamma < 1$ then there exists the uniqul bounded solution of equation (3.1) in domain Ω which satisfies the following boundary condition $v|_{\partial\Omega\backslash\{(0,0)\}} = \varphi,\ \varphi \in C(\partial\Omega),\ \varphi|_{\Gamma_1\cup\Gamma_2} = 0.$*

Proof. Let us show that at every (x_0, y_0) point of $(\Gamma_1 \cup \Gamma_2)\backslash\{(0,0)\}$ the barrier has the following form $f(x,y) = -ln^{-1}(\beta(y^2-k^2x^2)) + (x-x_0)^2$ where $\beta > 0$ is a constant number. It is to be shon that $\alpha,\ Mf < 0$ inequality holds true in a neighbourhood of (x_0, y_0). Indeed

$$Mf = \Delta f + 4k^4 n_1(x,y)x^2(y^2-k^2x^2)^{-2}ln^{-2}(\beta(y^2-k^2x^2))-$$

$$-4k^2n_1(x,y)(y^2-k^2x^2)^{-1}x(x-x_0)+$$

$$+4n_2(x,y)y^2(y^2-k^2x^2)^{-2}ln^{-2}(\beta(y^2-k^2x^2)) + \frac{L\omega}{\omega}f =$$

$$= \Delta f + 4n_1(x,y)(y^2-k^2x^2)^{-2}(y^2+k^4x^2)ln^{-2}(\beta(y^2-k^2x^2))-$$

$$-4(\gamma+1)(y^2-k^2x^2)^{-1}(-ln(ly))^{\gamma}ln^{-2}(\beta(y^2-k^2x^2))\omega^{-1}-$$

$$-4k^2n_1(x,y)(y^2-k^2x^2)^{-1}x(x-x_0) + \frac{L\omega}{\omega}f =$$

$$= 4(y^2-k^2x^2)^{-2}(y^2+k^4x^2)ln^{-2}(\beta(y^2-k^2x^2))[n_1(x,y)-2ln^{-1}(\beta(y^2-k^2x^2))-1]-$$

$$-2(y^2-k^2x^2)^{-1}ln^{-2}(\beta(y^2-k^2x^2))(k^2-1)-$$

$$-4(\gamma+1)(y^2-k^2x^2)^{-1}(-ln(ly))^{\gamma}ln^{-2}(\beta(y^2-k^2x^2))\omega^{-1}-$$

$$-4k^2n_1(x,y)(y^2-k^2x^2)^{-1}x(x-x_0) + 2 + \frac{L\omega}{\omega}f =$$

$$= -8(y^2-k^2x^2)^{-2}(y^2+k^4x^2)ln^{-3}(\beta(y^2-k^2x^2))\omega^{-1}\times$$

$$\times[(-ln(y^2-k^2x^2)^{\gamma-1}ln(\frac{\beta^{\gamma}}{l}(y^2-k^2x^2)^{\gamma-1}) + (-ln(ly))^{\gamma+1}]-$$

$$-2(y^2-k^2x^2)^{-1}ln^{-2}(\beta(y^2-k^2x^2))(k^2-1)-$$

$$-4(\gamma+1)(y^2-k^2x^2)^{-1}(-ln(ly))^{\gamma}ln^{-2}(\beta(y^2-k^2x^2))\omega^{-1}-$$

$$-4k^2n_1(x,y)(y^2-k^2x^2)^{-1}x(x-x_0) + 2 + \frac{L\omega}{\omega}f.$$

In a small vicinity of (x_0, y_0) the sign of Mf is determined by the component $\frac{L\omega}{\omega}f$, i.e. $Mf < 0$ in a sertain neighbourhood of (x_0, y_0) point the Viner solution of equation (3.1) will satisfy an uniform boundary condition. also, at every (x_0, y_0) point of Γ_3 there exists a barrier and it has the form: $f_1(x,y) = (x-x_0)^{2l_1} + (y-y_0)^{2l_2},\ l_1 > 0,\ l_2 > 0.$

Thus, existenu of the solution of the boundary problem set in Lemma is proved.

Let us show the uniqueness of the solution. For this it is enough to show the existence of such $\psi(x,y)$ function that $M\psi < 0$ in Ω and which satisfies (2.6), (2.7) conditions. Consider $\psi(x,y) = -ln(x^2+y^2) + q$ function where $q > 0$.

It is clear that,after duly selecting q, $\psi(x,y)$ will satisfy conditions (2.6) and (2,7). $M\psi$ has the following form:

$$M\psi = -4n_1(x,y)(x^2+y^2)^{-1} + 4(\gamma+1)(x^2+y^2)^{-1}(-ln(ly))^{\gamma}\omega^{-1} + \frac{L\omega}{\omega}\psi =$$

$$= -4(x^2+y^2)^{-1}[1-2\gamma(-ln((y^2-k^2x^2)l))^{\gamma-1}\omega^{-1}-(\gamma+1)(-ln(ly))^{\gamma}\omega^{-1}]+\frac{L\omega}{\omega}\psi.$$

From this we can sec that for small enough l $M\psi < 0$ in Ω. Thus Lemma is proved.

From Lemma 3.1. it follows the validity of the following theorem:

Theorem 3.2. *If $m = 1$ and $0 < \gamma < 1$ then there exists the unique regular solution of equatiom (1.1) in domain Ω which satisfies the following boundary condition* $\lim\limits_{(x,y)\to(x_0,y_0)} u(x,y)\omega^{-1} = \varphi(x_0,y_0)$, $(x_0,y_0) \in \partial\Omega\backslash\{(0,0)\}$, $\varphi \in C(\partial\Omega)$ $\varphi|_{\Gamma_1\cup\Gamma_2} = 0$.

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