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# OSCOLLATION 0F THE PLATE WITH CUSPED EDGES

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## *Abstract*

Admissible static and dynamical problems are investigated for a cusped plate. The setting of boundary conditions at the plates ends depends on the geometry of sharpenings of plates ends, while the setting of initial conditions is independent of them.

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In 1955 I.Vekua [1-3] raised the problem of investigation of cusped plates, i.e. such ones whose thickness on the part of plate boundary or on the whole one vanishes. The problem mathematically leads to the question of setting and solving of boundary value problems (BVP) for even order equations and systems of elliptic type with the order degeneration in the statical case and of initial boundary value problems (IBVP) for even order equations and systems of hyperbolic type with the order degeneration in the dynamical case. The first work concerning classical bending of cusped elastic plates was done by S. Mikhlin [4] and Makhover [5, 6]. Since, a wide literature devoted to such plates. A brief survey of results and references can be found in [7].

If we consider cylindrical bending of a plate with the rectangular projection $a \leq x_1 \leq b$, $0 \leq x_2 \leq \ell$, we actually get results also for cusped beams.

In this chapter we will consider a plate, whose projection on $x_3 = 0$ occupies the domain $\Omega$

$$\Omega = \{(x_1, x_2, x_3) :\ -\infty < x_1 < \infty,\ \ 0 < x_2 < l,\ \ x_3 = 0\}.$$

The equation of bending vibration has the following form (see, e.g., [8])

$$(D(x_2)w,_{22}(x_2,t)),_{22} = q(x_2,t) - 2\rho h(x_2)\frac{\partial^2 w(x_2,t)}{\partial t^2}, \qquad 0 < x_2 < l, \quad (1.1)$$

where $w(x_2)$ is a deflection of the plate, $q(x_2)$ is an intensity of a lateral load, $\rho$ is a density of the shell, $D(x_2)$ is a flexural rigidity,

$$D(x_2) := \frac{2Eh^3(x_2)}{3(1-\nu^2)}, \tag{1.2}$$

where $E$ is the Young's modulus, $\nu$ is the Poison's ratio, and $2h(x_2)$ is the thickness of the shell. Let $E$ =const, $\nu$ =const, and

$$D(x_2) = D_0 x_2^{\alpha}(l-x_2)^{\beta}, \quad D_0,\ \alpha,\ \beta = \text{const}, \quad D_0 > 0, \quad \alpha,\ \beta \geq 0. \tag{1.3}$$

Then

$$2h(x_2) = h_0 x_2^{\alpha/3}(l-x_2)^{\beta/3}, \quad h_0 = \text{const} > 0.$$

In the case $\alpha^2 + \beta^2 > 0$ equation (1.1) becomes degenerate one. Such plates are called cusped plates.

In the case under consideration (see [8])

$$M_2(x_2,t) := \quad -D(x_2)w_{,22}(x_2,t), \tag{1.4}$$

$$Q_2(x_2,t) := \quad M_{2,2}(x_2,t), \tag{1.5}$$

where $M_2(x_2,t)$ is a bending moment, $Q_2(x_2,t)$ is an intersecting force.

We suppose that $q(x_2) \in C([0,l])$.

**Remark 1.1.** Since $q(x_2) \in C([0,l])$, it is easy to prove that (see [9]), $w(\cdot,t) \in C^4(]0,l[)$, and

$$\begin{aligned} Q_2(\cdot,t),\ \ M_2(\cdot,t) &\in C([0,l]), \\ w(\cdot,t),\ \ w_{,2}(\cdot,t) &\in C(]0,l[), \end{aligned}$$

the behaviour of the $w_{,2}(x_2)$ and $w(x_2)$ when $x_2 \to 0_+$ and $x_2 \to l_-$ depends on $\alpha$ and $\beta$, as follows:

$$\begin{array}{lll} w \in C^1([0,l)) \ \ (w \in C^1((0,l])) & \text{if} & \alpha < 1,\ \beta > 1\ (\alpha > 1,\ \beta < 1); \\ w \in C([0,l)) \ \ (w \in C((0,l])) & \text{if} & \alpha < 2,\ \beta > 2\ (\alpha > 2,\ \beta < 2); \\ w \in C^1([0,l]) & \text{if} & \alpha,\ \beta < 1; \\ w \in C([0,l]) & \text{if} & \alpha,\ \beta < 2; \end{array}$$

$$w \in C^1([0,l)) \cap C([0,l]), \ \ (w \in C^1((0,l]) \cap C([0,l])) \quad \text{if} \ \ \alpha < 1,\ \beta < 2\ (\alpha < 2,\ \beta < 1).$$

We consider equation (1.1) under the initial conditions (ICs)

$$w(x_2,0) = \varphi_1(x_2), \ \ w_{,t}(x_2,0) = \varphi_2(x_2), \quad x_2 \in ]0,l[, \tag{1.6}$$

where $\varphi_i(x_2) \in C^4(]0,l[), \quad i = 1,2$ are given functions.

Let us consider the following boundary value problems (BVP):

**Problem 1.** Let $0 \leq \alpha,\ \beta < 1$. Find

$$\begin{aligned} &w(\cdot,t) \in C^4(]0,l[) \cap C^1([0,l]), \\ &w(x_2,\cdot) \in C^1(t \geq 0) \cap C^2(t > 0), \ \ w(x_2,t) \in C(0 \leq x_2 \leq l,\ t \geq 0), \end{aligned}$$

satisfying equation (1.1), the boundary conditions (BCs)

$$w(0,t) = w_{,2}(0,t) = w(l,t) = w_{,2}(l,t) = 0,$$

and ICs (1.6), where

$$\varphi_i(x_2) \in C^4(]0,l[) \cap C^1([0,l])$$

$$\varphi_i(0) = \varphi_i'(0) = \varphi_i(l) = \varphi_i'(l) = 0, \quad i = 1,2.$$

**Problem 2.** Let $0 \le \alpha,\ \beta < 1$. Find

$$\begin{aligned} &w(\cdot,t) \in C^4(]0,l[) \cap C^1([0,l]),\\ &w(x_2,\cdot) \in C^1(t \ge 0) \cap C^2(t > 0), \quad w(x_2,t) \in C(0 \le x_2 \le l,\ t \ge 0), \end{aligned}$$

satisfying equation (1.1), the BCs

$$w(0,t) = w_{,2}(0,t) = w_{,2}(l,t) = Q_2(l,t) = 0,$$

and ICs (1.6), where

$$\varphi_i(x_2) \in C^4(]0,l[) \cap C^1([0,l])$$

$$\varphi_i(0) = \varphi_i'(0) = \varphi_i'(l) = (-D(x_2)\varphi_i''(x_2))'\,|_{x_2=l_-} = 0,\ i = 1,2.$$

**Problem 3.** Let $0 \le \alpha, < 1,\ 0 \le \beta < 2$. Find

$$\begin{aligned} &w(\cdot,t) \in C^4(]0,l[) \cap C^1([0,l[) \cap C([0,l]),\\ &w(x_2,\cdot) \in C^1(t \ge 0) \cap C^2(t > 0), \quad w(x_2,t) \in C(0 \le x_2 \le l,\ t \ge 0), \end{aligned}$$

satisfying equation (1.1), the BCs

$$w(0,t) = w_{,2}(0,t) = w(l,t) = M_2(l,t) = 0,$$

and ICs (1.6), where

$$\varphi_i(x_2) \in C^4(]0,l[) \cap C^1([0,l[) \cap C([0,l]),$$

$$\varphi_i(0) = \varphi_i'(0) = \varphi_i(l) = (-D(x_2)\varphi_i''(x_2))\,|_{x_2=l_-} = 0,\ i = 1,2.$$

**Problem 4.** Let $0 \le \alpha < 1,\ \beta \ge 0$. Find

$$\begin{aligned} &w(\cdot,t) \in C^4(]0,l[) \cap C^1([0,l[),\\ &w(x_2,\cdot) \in C^1(t \ge 0) \cap C^2(t > 0), \quad w(x_2,t) \in C(0 \le x_2 < l,\ t \ge 0), \end{aligned}$$

satisfying equation (1.1), the BCs

$$w(0,t) = w_{,2}(0,t) = M_2(l,t) = Q_2(l,t) = 0,$$

and ICs (1.6), where

$$\varphi_i(x_2) \in C^4(]0,l[) \cap C^1([0,l[),$$

$$\begin{aligned}\varphi_i(0) &= \varphi_i'(0) = (-D(x_2)\varphi_i''(x_2))\,|_{x_2=l_-}\\ &= (-D(x_2)\varphi_i''(x_2))'\,|_{x_2=l_-} = 0,\ i=1,2.\end{aligned}$$

**Problem 5.** Let $0 \le \alpha,\ \beta < 1$. Find

$$\begin{array}{l} w(\cdot,t) \in C^4(]0,l[) \cap C^1([0,l]),\\ w(x_2,\cdot) \in C^1(t \ge 0) \cap C^2(t>0),\ \ w(x_2,t) \in C(0 \le x_2 \le l,\ t \ge 0),\end{array}$$

satisfying equation (1.1), the BCs

$$w_{,2}(0,t) = Q_2(0,t) = w(l,t) = w_{,2}(l,t) = 0,$$

and ICs (1.6), where

$$\varphi_i(x_2) \in C^4(]0,l[) \cap C^1([0,l]),$$

$$\varphi_i'(0) = (-D(x_2)\varphi_i''(x_2))'\,|_{x_2=0_+} = \varphi_i(l) = \varphi_i'(l) = 0,\ i=1,2.$$

**Problem 6.** Let $0 \le \alpha < 1,\ 0 \le \beta < 2$. Find

$$\begin{array}{l} w(\cdot,t) \in C^4(]0,l[) \cap C^1([0,l[) \cap C([0,l]),\\ w(x_2,\cdot) \in C^1(t \ge 0) \cap C^2(t>0),\ \ w(x_2,t) \in C(0 \le x_2 \le l,\ t \ge 0),\end{array}$$

satisfying equation (1.1), the BCs

$$w_{,2}(0,t) = Q_2(0,t) = w(l,t) = M_2(l,t) = 0,$$

and ICs (1.6), where

$$\varphi_i(x_2) \in C^4(]0,l[) \cap C^1([0,l[) \cap C([0,l]),$$

$$\begin{aligned}\varphi_i'(0) &= (-D(x_2)\varphi_i''(x_2))'\,|_{x_2=0_+} = \varphi_i(l)\\ &= (-D(x_2)\varphi_i''(x_2))\,|_{x_2=l_-} = 0,\ i=1,2.\end{aligned}$$

**Problem 7.** Let $0 \le \alpha < 2,\ 0 \le \beta < 1$. Find

$$\begin{array}{l} w(\cdot,t) \in C^4(]0,l[) \cap C^1(]0,l]) \cap C([0,l]),\\ w(x_2,\cdot) \in C^1(t \ge 0) \cap C^2(t>0),\ \ w(x_2,t) \in C(0 \le x_2 \le l,\ t \ge 0),\end{array}$$

satisfying equation (1.1), the BCs

$$w(0,t) = M_2(0,t) = w(l,t) = w_{,2}(l,t) = 0,$$

and ICs (1.6), where

$$\varphi_i(x_2) \in C^4(]0,l[) \cap C^1(]0,l]) \cap C([0,l]),$$

$$\varphi_i(0) = (-D(x_2)\varphi_i''(x_2))\,|_{x_2=0_+} = \varphi_i(l) = \varphi_i'(l) = 0,\ i=1,2.$$

**Problem 8.** Let $0 \le \alpha < 2$, $0 \le \beta < 1$. Find

$$\begin{gathered} w(\cdot,t) \in C^4(]0,l[) \cap C([0,l]) \cap C^1(]0,l]) \\ w(x_2,\cdot) \in C^1(t \ge 0) \cap C^2(t>0), \\ w(x_2,t) \in C(0 \le x_2 \le l,\ t \ge 0), \end{gathered} \tag{1.7}$$

satisfying equation (1.1), the BCs

$$w(0,t) = M_2(0,t) = w,_2(l,t) = Q_2(l,t) = 0, \tag{1.8}$$

and ICs (1.6), where

$$\varphi_i(x_2) \in C^4(]0,l[) \cap C([0,l]) \cap C^1(]0,l]), \quad i=1,2. \tag{1.9}$$

$$\begin{aligned} \varphi_i(0) &= -D(x_2)\varphi_i''(x_2)|_{x_2=0_+} = \varphi_i'(l) \\ &= (-D(x_2)\varphi_i''(x_2))'|_{x_2=l_-} = 0, i=1,2. \end{aligned} \tag{1.10}$$

**Problem 9.** Let $0 \le \alpha,\ \beta < 2$. Find

$$\begin{aligned} &w(\cdot,t) \in C^4(]0,l[) \cap C([0,l]), \\ &w(x_2,\cdot) \in C^1(t \ge 0) \cap C^2(t>0), \quad w(x_2,t) \in C(0 \le x_2 \le l,\ t \ge 0), \end{aligned}$$

satisfying equation (1.1), the BCs

$$w(0,t) = M_2(0,t) = w(l,t) = M_2(l,t) = 0,$$

and ICs (1.6), where

$$\varphi_i(x_2) \in C^4(]0,l[) \cap C([0,l]),$$

$$\begin{aligned} \varphi_i(0) &= (-D(x_2)\varphi_i''(x_2))\,|_{x_2=0_+} = \varphi_i(l) \\ &= (-D(x_2)\varphi_i''(x_2))\,|_{x_2=l_-} = 0,\ i=1,2. \end{aligned}$$

**Problem 10.** Let $\alpha \ge 0,\ 0 < \beta < 1$. Find

$$\begin{aligned} &w(\cdot,t) \in C^4(]0,l[) \cap C^1(]0,l]), \\ &w(x_2,\cdot) \in C^1(t \ge 0) \cap C^2(t>0), \quad w(x_2,t) \in C(0 < x_2 \le l,\ t \ge 0), \end{aligned}$$

satisfying equation (1.1), the BCs

$$M_2(0,t) = Q_2(0,t) = w(l,t) = w,_2(l,t) = 0,$$

and ICs (1.6), where

$$\varphi_i(x_2) \in C^4(]0,l[) \cap C^1(]0,l]),$$

$$\begin{aligned} (-D(x_2)\varphi_i''(x_2)) &= (-D(x_2)\varphi_i''(x_2))'\,|_{x_2=0_+} \\ &= \varphi_i(l) = \varphi_i'(l) = 0,\ i=1,2. \end{aligned}$$

Let us solve typical one.

**Solution of the Problem 8.**

Using the Fourier method, we look for $w(x_2,t)$ in the following form

$$w(x_2,t) = X(x_2)T(t). \tag{1.11}$$

Let firstly $q(x_2,t) \equiv 0$. Then from (1.1) we get

$$\frac{(D(x_2)X''(x_2))''}{g(x_2)X(x_2)} = -\frac{T''(t)}{T(t)} = \lambda = \text{const.}$$

Hence,

$$T''(t) + \lambda T(t) = 0, \tag{1.12}$$

and

$$(D(x_2)X''(x_2))'' = \lambda g(x_2)X(x_2), \tag{1.13}$$

where $g(x_2) := 2\rho h(x_2)$.

From (1.8) for $X(x_2)$ we obtain the following BCs

$$X(0) = -D(x_2)X''(x_2)|_{x_2=0} = X'(l) = (-D(x_2)X''(x_2))'|_{x_2=l} = 0. \tag{1.14}$$

Now, in view of (1.7), we have to solve the following BVP:

Find

$$X(x_2) \in C^4(]0,l[) \cap C([0,l]) \cap C^1(]0,l]), \tag{1.15}$$

which satisfies equation (1.13) and BCs (1.14). Above BVP can be reduced to the following integral equation (see [9])

$$X(x_2) = \lambda \int_0^l g(\xi)K(x_2,\xi)X(\xi)d\xi, \tag{1.16}$$

where

$$K(x_2,\xi) = \begin{cases} K_3(\xi,x_2), & 0 \le \xi \le x_2, \\ K_3(x_2,\xi), & x_2 \le \xi \le l, \end{cases} \tag{1.17}$$

$$K_3(x_2,\xi) := -x_2\int_\xi^{x_2} \eta D^{-1}(\eta)d\eta + \int_0^{x_2} \eta^2 D^{-1}(\eta)d\eta + x_2\xi\int_\xi^l D^{-1}(\eta)d\eta. \tag{1.18}$$

Obviously, $K_3(x_2,\xi) \in C([0,l]\times[0,l])$ and, therefore $K(x_2,\xi) \in C([0,l]\times \times[0,l])$.

**Proposition 1.2.** *$K(x_2,\xi)$ is a symmetric with respect to $x_2$ and $\xi$.*

**Proof.** For $z_1$ and $z_2$, such that $0 \le z_1,\ z_2 \le l$ we have

$$K(z_1,z_2) = \begin{cases} K_3(z_2,z_1), & 0 \le z_2 \le z_1 \le l, \\ K_3(z_1,z_2), & 0 \le z_1 \le z_2 \le l, \end{cases}$$

$$K(z_2,z_1) = \begin{cases} K_3(z_1,z_2), & 0 \le z_1 \le z_2 \le l, \\ K_3(z_2,z_1), & 0 \le z_2 \le z_1 \le l, \end{cases}$$

i.e.,

$$K(z_1,z_2)=K(z_2,z_1),\quad \text{for any } z_1,z_2\in[0,l].$$

□

(1.16) can be rewritten as follows

$$Y(x_2)=\lambda\int_0^l R(x_2,\xi)Y(\xi)d\xi, \tag{1.19}$$

where

$$Y(x_2)=\sqrt{g(x_2)}X(x_2),\quad R(x_2,\xi)=\sqrt{g(x_2)}K(x_2,\xi)\sqrt{g(\xi)}. \tag{1.20}$$

(1.19) is an integral equation with a symmetric and continuous kernel.

**Remark 1.3.** For all other BVPs (see Problems 1-7, 9, 10) we get (1.16) type integral equations. In all these cases kernel of the integral equation is symmetric. Let write down typical ones:

**Problem 1.**

$$\begin{aligned} K_3(x_2,\xi) &= \int_0^{x_2}(\eta-x_2)(\eta-\xi)D^{-1}(\eta)d\eta \\ &+ \left\{\int_0^{\xi}(\xi-\eta)D^{-1}d\eta\int_0^{x_2}(x_2-\eta)\eta D^{-1}(\eta)d\eta\right. \\ &+ \left.\int_0^{\xi}\eta(\xi-\eta)D^{-1}(\eta)d\eta\int_0^{x_2}(x_2-\eta)D^{-1}(\eta)d\eta\right\}\frac{\int_0^l\eta D^{-1}(\eta)d\eta}{\Delta} \\ &- \int_0^{\xi}(\xi-\eta)\eta D^{-1}(\eta)d\eta\int_0^{x_2}(x_2-\eta)\eta D^{-1}(\eta)d\eta\frac{\int_0^l D^{-1}(\eta)d\eta}{\Delta} \\ &+ \int_0^{\xi}(\xi-\eta)D^{-1}(\eta)d\eta\int_0^{x_2}(x_2-\eta)D^{-1}(\eta)d\eta\frac{\int_0^l\eta^2D^{-1}(\eta)d\eta}{\Delta}, \end{aligned} \tag{1.21}$$

where

$$\Delta:=\left[\int_0^l\xi D^{-1}(\xi)d\xi\right]^2-\int_0^l D^{-1}(\xi)d\xi\int_0^l\xi^2D^{-1}(\xi)d\xi<0,$$

The last assertion follows from the Hölder inequality which is strong since $\xi D^{-\frac{1}{2}}(\xi)$ and $D^{-\frac{1}{2}}(\xi)$ are positive on $]0,l[$, and $\xi^2D^{-1}(\xi)$ and $D^{-1}(\xi)$ differ from each other by a nonconstant factor $\xi^2$.

**Problem 2.**

$$K_3(x_2,\xi) = \int_0^{x_2} (x_2-\eta)(\xi-\eta)D^{-1}d\eta \tag{1.22}$$
$$- \frac{1}{\int_0^l D^{-1}(\eta)d\eta} \int_0^{\xi} (\xi-\eta)D^{-1}(\eta)d\eta \int_0^{x_2} (x_2-\eta)D^{-1}(\eta)d\eta.$$

**Problem 9.**

$$K_3(x_2,\xi) = \frac{x_2\xi}{l^2}\int_{\xi}^{l} (l-\eta)D^{-1}(\eta)d\eta + \frac{x_2(l-\xi)}{l^2}\int_{\xi}^{x_2} (l-\eta)\eta D^{-1}(\eta)d\eta$$
$$+ \frac{(l-x_2)(l-\xi)}{l^2}\int_0^{x_2} \eta^2 D^{-1}(\eta)d\eta. \tag{1.23}$$

**Problem 10.**

$$K_3(x_2,\xi) = -\int_{\eta}^{l} (x_2-\eta)(\eta-\xi)D^{-1}(\eta)d\eta. \tag{1.24}$$

Recall the following three Hilbert-Schmidt theorems (see, e.g., [10])

**Theorem 1.4.** *If $u(x_2)$ has the form*

$$u(x_2) = \lambda \int_0^l R(x_2,\xi)f(\xi)d\xi,$$

*with $f \in C([0,l])$ and symmetric Kernel $R(x_2,\xi) \in C([0,l]\times[0,l])$, then*

$$u(x_2) = \sum_{n=1}^{\infty}(u,Y_n)Y_n(x_2), \tag{1.25}$$

*where*

$$(u,Y_n) := \int_0^l u(x_2)Y_n(x_2)dx_2,$$

*$Y_n$ is an eigenfunction of $R(x_2,\xi)$, and the series on the right hand side of (1.25) is convergent absolutely and uniformly on $[0,l]$.*

**Theorem 1.5.** *If the number of eigenvalues $\lambda_n$ of the symmetric and continuous kernel is finite then*

$$R(x_2,\xi) = \sum_{n=1}^{N}\frac{Y_n(x_2)Y_n(\xi)}{\lambda_n}.$$

**Theorem 1.6.** *If $f(x_2) \in C([0,l])$, then*

$$\int\limits_0^l R(x_2,\xi)f(\xi)d\xi = \sum_{n=1}^{\infty} \frac{(f,Y_n)}{\lambda_n} Y_n,$$

*and the series is convergent absolutely and uniformly, here $R(x_2,\xi)$ is a symmetric and continuous kernel with respect to $x_2$; $\xi$, and $Y_n$ are eigenfunctions of $R$ corresponding to the eigenvalues $\lambda_n$.*

**Proposition 1.7.** Let $Y_n(x_2) \in C^4(]0,l[)$. Number of eigenvalues $\lambda_n$ of (1.19) is not finite.

**Proof.** Let it be finite, and $n = \overline{1,m}$. Then we can express $R(x_2,\xi)$ as follows (see Theorem )

$$R(x_2,\xi) = \sum_{n=1}^{m} \frac{Y_n(x_2)Y_n(\xi)}{\lambda_n},$$

where $Y_n(x_2) \in C^4(]0,l[)$, i.e.,

$$R(x_2,\xi) \in C^4\left(]0,l[\times]0,l[\right). \tag{1.26}$$

On the other hand, by virtue of (1.18),

$$K'''_{x_2}(x_2,\xi)|_{\xi\to x_2-} - K'''_{x_2}(x_2,\xi)|_{\xi\to x_2+} = \frac{1}{D(x_2)},$$

then kernel

$$R(x_2,\xi) \notin C^4\left(]0,l[\times]0,l[\right). \tag{1.27}$$

But, (1.26) and (1.27) contradict each other, thus the number of $\lambda_n$ is not finite. □

**Proposition 1.8.** All $\lambda_n$ are positive.

**Proof.** Obviously, if we denote by $Y_n$ orthonormalized eigenfunctions (it can be assumed without loss of generality) of (1.19), then

$$X_n(x_2) = \frac{Y_n(x_2)}{\sqrt{g(x_2)}}$$

are eigenfunctions of (1.16) (i.e., of (1.13)). Hence,

$$(D(x_2)X''_n(x_2))'' = \lambda_n g(x_2)X_n(x_2). \tag{1.28}$$

Let us multiply both sides of (1.28) by $X_n(x_2)$ and integrate it from 0 to $l$. Taking into account of the first expression of (1.20), we obtain

$$\begin{aligned}\int\limits_0^l X_n(x_2)(D(x_2)X''_n(x_2))''dx_2 &= \lambda_n \int\limits_0^l g(x_2)X_n(x_2)X_n(x_2)dx_2 \\ &= \lambda_n \int\limits_0^l Y_n(x_2)Y_n(x_2)dx_2 = \lambda_n.\end{aligned}$$

Further,

$$\begin{aligned}\lambda_n &= \int\limits_0^l X_n(x_2)(D(x_2)X_n''(x_2))''dx_2 = X_n(x_2)(D(x_2)X''(x_2))'\Big|_0^l \\ &- \int\limits_0^l X_n'(x_2)(D(x_2)X_n''(x_2))'dx_2 \\ &\text{(by virtue of the BCs (1.14))} \\ &= -\int\limits_0^l X_n'(x_2)(D(x_2)X_n''(x_2))'dx_2 = X_n'(x_2)(D(x_2)X''(x_2))\Big|_0^l \\ &+ \int\limits_0^l D(x_2)(X_n'')^2(x_2)dx_2 = \int\limits_0^l D(x_2)(X_n'')^2(x_2)dx_2 \geq 0.\end{aligned}$$

Hence, $\lambda_n > 0$ for any $n$, since in non trivial case $X_n \not\equiv 0$. □

We can write the solution of (1.12) as follows

$$T_n(t) = b_1^n \sin\left(\sqrt{\lambda_n}t\right) + b_2^n \cos\left(\sqrt{\lambda_n}t\right), \quad b_i^n = \text{const}, \quad i = 1, 2.$$

Now, we can find a solution of the Problem 8 in the form as follows

$$w(x_2,t) = \sum_{n=1}^{\infty} \frac{Y_n(x_2)}{\sqrt{g(x_2)}} \left(b_1^n \sin\left(\sqrt{\lambda_n}t\right) + b_2^n \cos\left(\sqrt{\lambda_n}t\right)\right) \tag{1.29}$$

or, taking into account (1.20), in the following form

$$w(x_2,t) = \sum_{n=1}^{\infty} X_n(x_2) \left(b_1^n \sin\left(\sqrt{\lambda_n}t\right) + b_2^n \cos\left(\sqrt{\lambda_n}t\right)\right). \tag{1.30}$$

In view of initial conditions (1.6), we formally have

$$\sum_{n=1}^{\infty} Y_n(x_2) b_2^n = \varphi_1(x_2)\sqrt{g(x_2)}, \quad \sum_{n=1}^{\infty} \sqrt{\lambda_n} Y_n(x_2) b_1^n = \varphi_2(x_2)\sqrt{g(x_2)}. \tag{1.31}$$

If $\psi_i(x_2) := \frac{(D\varphi_i'')''}{\sqrt{g(x_2)}} \in C[0,l]$, $(i = 1, 2)$, then after integration of the last expression, $\sqrt{g(x_2)}\varphi_i(x_2)$ can be expressed as follows

$$\sqrt{g(x_2)}\varphi_i(x_2) = \int\limits_0^l \sqrt{g(x_2)g(\xi)} K(x_2,\xi)\psi_i(\xi)d\xi,$$

i.e.,

$$\sqrt{g(x_2)}\varphi_i(x_2) = \int\limits_0^l R(x_2,\xi)\psi_i(\xi)d\xi.$$

Hence, by virtue of Theorem , since $\psi_i(\xi) \in C([0,l])$ and symmetric $R(x_2,\xi) \in C([0,l] \times [0,l])$, we get absolutly and uniformly convergence of the series

$$\sqrt{g(x_2)}\varphi_i(x_2) = \sum_{n=1}^{\infty} \int_0^l \sqrt{g(\xi)}\varphi_i(\xi)Y_n(\xi)d\xi \cdot Y_n(x_2),$$

i.e., of (1.31) on $[0,l]$, and

$$b_1^n = \frac{1}{\sqrt{\lambda_n}} \int_0^l g(x_2)X_n(x_2)\varphi_2(x_2)dx_2, \quad b_2^n = \int_0^l g(x_2)X_n(x_2)\varphi_1(x_2)dx_2. \tag{1.32}$$

Further, taking into account (1.15), $X(x_2) \in C([0,l])$. Then, by virtue of (1.20), we can rewrite (1.31) as follows

$$\varphi_1(x_2) = \sum_{n=1}^{\infty} X_n(x_2)b_2^n, \quad \varphi_2(x_2) = \sum_{n=1}^{\infty} \sqrt{\lambda_n}X_n(x_2)b_1^n. \tag{1.33}$$

Evidently, last series will be absolutely and uniformly convergent on $]0,l[$. Since there exists positive minimum of eigenvalues, from the convergence of the second series follows absolute and uniform convergence on $]0,l[$ of the series $\sum\limits_{n=1}^{N} X_n(x_2)b_1^n$. Therefore, the series (1.30) is absolutely and uniformly convergent on $]0,l[$.

After formal differentiation of (1.30) with respect to $t$ we get

$$w_{,t}(x_2,t) = \sum_{n=1}^{\infty} X_n(x_2)\sqrt{\lambda_n}\left(b_1^n \cos(\sqrt{\lambda_n}t) - b_2^n \sin(\sqrt{\lambda_n}t)\right), \tag{1.34}$$

$$w_{,tt}(x_2,t) = -\sum_{n=1}^{\infty} X_n(x_2)\lambda_n\left(b_1^n \sin(\sqrt{\lambda_n}t) + b_2^n \cos(\sqrt{\lambda_n}t)\right). \tag{1.35}$$

**Theorem 1.9.** *(1.33) and (1.30) converge absolutely and uniformly on* $[0,l]$*, and (1.34) - (1.35) converge absolutely and uniformly on* $]0,l[$ *if*

$$\Psi_i(x_2) := \frac{\psi_i(x_2)}{\sqrt{g(x_2)}}, \quad i = 1,2, \tag{1.36}$$

*are satisfying conditions (1.10) and*

$$\chi_i(x_2)\sqrt{g(x_2)} := \left(D(x_2)\Psi_i''(x_2)\right)'', \quad i = 1,2, \tag{1.37}$$

*are integrable functions on* $]0,l[$ *(for this, e.g., it is sufficient that* $\varphi_i^{(j)}(x_2) = O(x_2^{\gamma_{ij}}), x_2 \to 0_+$, $\gamma_{ij} = \text{const} > 7 - j - \frac{5\alpha}{3}$, $\varphi_i^{(j)}(x_2) = O((l-x_2)^{\delta_{ij}})$, $x_2 \to l_-$, $\delta_{ij} = \text{const} > 7 - j - \frac{5\beta}{3}$, $i = 1,2$; $j = \overline{2,8}$*).*

**Proof.** Substituting in (1.32) the function $g(x_2)X_n(x_2)$ found from (1.28), we get

$$b_1^n = \frac{1}{\lambda_n\sqrt{\lambda_n}}\int_0^l (D(x_2)X_n''(x_2))''\varphi_2(x_2)dx_2$$

(after integrating by parts 4-times, taking into account BCs, (1.10), (1.14), and (1.20))

$$= \frac{1}{\lambda_n\sqrt{\lambda_n}}\left\{(D(x_2)X_n''(x_2))'\varphi_2(x_2)|_0^l - \int_0^l (D(x_2)X_n''(x_2))'\varphi_2'(x_2)dx_2\right\}$$
$$= \frac{1}{\lambda_n\sqrt{\lambda_n}}\left\{-D(x_2)X_n''(x_2)\varphi_2'(x_2)|_0^l + \int_0^l D(x_2)X_n''(x_2)\varphi_2''(x_2)dx_2\right\}$$
$$= \frac{1}{\lambda_n\sqrt{\lambda_n}}\int_0^l X_n''(x_2)D(x_2)\varphi_2''(x_2)dx_2 = \frac{1}{\lambda_n\sqrt{\lambda_n}}\left\{X_n'(x_2)D(x_2)\varphi''(x_2)|_0^l\right.$$
$$\left.-\int_0^l X_n'(x_2)(D(x_2)\varphi_2''(x_2))'dx_2\right\} = \frac{1}{\lambda_n\sqrt{\lambda_n}}\left\{-X_n(x_2)(D(x_2)\varphi_2''(x_2))'|_0^l\right.$$
$$\left.+\int_0^l X_n(x_2)(D(x_2)\varphi_2''(x_2))''dx_2\right\} = \frac{1}{\lambda_n\sqrt{\lambda_n}}\int_0^l X_n(x_2)(D(x_2)\varphi_2''(x_2))''dx_2$$
$$= \frac{1}{\lambda_n\sqrt{\lambda_n}}\int_0^l Y_n(x_2)\psi_2(x_2)dx_2. \tag{1.38}$$

Analogously,

$$b_2^n = \frac{1}{\lambda_n}\int_0^l Y_n(x_2)\psi_1(x_2)dx_2. \tag{1.39}$$

In view of (1.37), $\Psi_i(x_2)$ can be expressed as follows

$$\Psi_i(x_2) = \int_0^l K(x_2,\xi)\sqrt{g(\xi)}\chi_i(\xi)d\xi, \quad i=1,2,$$

and by virtue of (1.36), (1.20) we obtain

$$\psi_i(x_2) = \int_0^l R(x_2,\xi)\chi_i(\xi)d\xi, \quad i=1,2.$$

According to the Theorem , the following series

$$\sum_{n=1}^{\infty}\beta_i^n Y_n(x_2),$$

where

$$\beta_i^n = \int_0^l Y_n(x_2)\psi_i(x_2)dx_2, \quad i = 1, 2, \tag{1.40}$$

is convergent absolutely and uniformly on $]0, l[$, i.e.,

$$\sum_{n=1}^{\infty}|\beta_i^n||Y_n(x_2)| < +\infty. \tag{1.41}$$

By view of $K(x_2, \xi)\sqrt{g(\xi)} \in C([0, l] \times [0, l])$, there exists such $M$ that

$$M := \max_{0\le x_2,\,\xi\le l}\left|K(x_2, \xi)\sqrt{g(\xi)}\right| < +\infty.$$

Using (1.19), (1.39), (1.40) we have

$$\begin{aligned}
|X_n(x_2)b_2^n| &= \left|\lambda_n \int_0^l K(x_2, \xi)\sqrt{g(\xi)}Y_n(\xi)b_2^n d\xi\right| \\
&= \left|\int_0^l K(x_2, \xi)\sqrt{g(\xi)}Y_n(\xi)\beta_2^n d\xi\right| \\
&\le \int_0^l |K(x_2, \xi)\sqrt{g(\xi)}||Y_n(\xi)||\beta_2^n|d\xi := c_n^1
\end{aligned}$$

On the other hand in virtue of (1.41) we obtain

$$\begin{aligned}
\sum_{n=1}^{\infty} c_n^1 &= \sum_{n=1}^{\infty} c_n^1 \int_0^l |K(x_2, \xi)\sqrt{g(\xi)}||Y_n(\xi)||\beta_2^n|d\xi \\
&\le M\int_0^l \sum_{n=1}^{\infty}|Y_n(\xi)||\beta_2^n|d\xi \le MM_1 l < \infty.
\end{aligned}$$

From the last two uniquality we get

$$|\varphi_1| \le \sum_{n=1}^{\infty}|X_n(x_2)b_2^n| \le \sum_{n=1}^{\infty} c_n^1 < +\infty.$$

Which means that $\varphi_1$ can be expressed as absolutely and uniformly convergent series. Analoguously, we can prove that $\varphi_2$ converges absolutely and uniformly on $[0, l]$.

Let, now consider (1.34) series. It is obviously that

$$|w(x_2,t)| \le \sum_{n=1}^{\infty} |X_n(x_2) b_1^n| + \sum_{n=1}^{\infty} |X_n(x_2) b_2^n|,$$

and from the convergent of $\varphi_1$ and $\varphi_2$ we obtain that (1.30) converges absolutely and uniformly on $[0,l]$.

Further, from (1.34)

$$\begin{aligned}
|w,_t(x_2,t)| &= \left|\sum_{n=1}^{\infty} X_n(x_2)\sqrt{\lambda_n}\left(b_1^n \cos(\sqrt{\lambda_n}t) - b_2^n \sin(\sqrt{\lambda_n}t)\right)\right| \\
&\le \left|\sum_{n=1}^{\infty} X_n(x_2)\sqrt{\lambda_n} b_1^n \cos(\sqrt{\lambda_n}t)\right| \\
&+ \left|\sum_{n=1}^{\infty} X_n(x_2)\sqrt{\lambda_n} b_2^n \sin(\sqrt{\lambda_n}t)\right| \\
&\le \sum_{n=1}^{\infty} \left|X_n(x_2)\sqrt{\lambda_n} b_1^n\right| + \sum_{n=1}^{\infty} \left|X_n(x_2)\sqrt{\lambda_n} b_2^n\right|. \qquad (1.42)
\end{aligned}$$

According to Proposition , all of $\lambda_n$ are positive. Therefore, we can find $\lambda_0$ such that $\lambda_0 \le \min\limits_{1\le i\le\infty}\{\lambda_i\}$, and by virtue of (1.20), (1.38)-(1.41), we obtain

$$\begin{aligned}
\sum_{n=1}^{\infty} \left|X_n(x_2)\sqrt{\lambda_n} b_2^n\right| &= \frac{1}{\sqrt{g(x_2)}} \sum_{n=1}^{\infty} \left|Y_n \sqrt{\lambda_n} \frac{1}{\lambda_n} \beta_1^n\right| \\
&\le \frac{1}{\sqrt{\lambda_0}} \frac{1}{\sqrt{g(x_2)}} \sum_{n=1}^{\infty} |Y_n||\beta_1^n| < \infty, \\
\sum_{n=1}^{\infty} \left|X_n(x_2)\sqrt{\lambda_n} b_1^n\right| &= \frac{1}{\sqrt{g(x_2)}} \sum_{n=1}^{\infty} \left|Y_n \sqrt{\lambda_n} \frac{1}{\lambda_n\sqrt{\lambda_n}} \beta_2^n\right| \\
&\le \frac{1}{\lambda_0} \frac{1}{\sqrt{g(x_2)}} \sum_{n=1}^{\infty} |Y_n||\beta_2^n| < \infty, \quad x_2 \in ]0,l[.
\end{aligned}$$

Hence, the series in (1.42) are convergent. Thus, (1.34) is convergent absolutely and uniformly on $]0,l[$. Similarly, we get the absolute and uniform convergence of (1.35) on $]0,l[$. □

Let us now differentiate (1.30) formally $i$-times with respect to $x_2$ and consider the following expressions

$$w_{x_2}^{(i)}(x_2,t) = \sum_{n=1}^{\infty} X_n^{(i)}(x_2)\left(b_1^n \sin(\sqrt{\lambda_n}t) + b_2^n \cos(\sqrt{\lambda_n}t)\right), \quad i = 1,2,3,4, \qquad (1.43_{\mathrm{i}})$$

$$(D(x_2)w,_{x_2x_2}(x_2,t))_{x_2}^{(i-1)} = \sum_{n=1}^{\infty} (D(x_2)X_n''(x_2))^{(i-1)}\left(b_1^n \sin(\sqrt{\lambda_n}t) + b_2^n \cos(\sqrt{\lambda_n}t)\right), \ i = 1,2 \qquad (1.44_{\mathrm{i}})$$

**Theorem 1.10.** *The series* $(1.43_i)$ $(i = 1, ..., 4)$ *are convergent absolutely and uniformly on* $]0, l[$. *The series* $(1.44_i)$ $(i = 1, 2)$ *are convergent absolutely and uniformly on* $[0, l]$.

**Proof.** Obviously, in view of (1.14), after integration of (1.28), we get

$$X'_n(x_2) = \lambda_n \int_0^l R_1(x_2, \xi) X_n(\xi) d\xi, \tag{1.45}$$

where

$$R_1(x_2, \xi) = \begin{cases} \xi \int\limits_{x_2}^{l} D^{-1}(\eta) d\eta, \quad 0 \le \xi \le x_2, \\ -\int\limits_{\xi}^{x_2} \eta D^{-1}(\eta) d\eta + \xi \int\limits_{\xi}^{l} D^{-1}(\eta) d\eta, \quad x_2 \le \xi \le l, \end{cases}$$

and

$$R_1(x_2, \xi) \in C([0, l] \times [0, l]), \tag{1.46}$$

because of $0 \le \alpha < 2$, $0 \le \beta < 1$.

Substituting (1.45) into $(1.43_1)$ for $i = 1$, we obtain

$$w'_{x_2}(x_2, t) = \sum_{n=1}^{\infty} \lambda_n \int_0^l R_1(x_2, \xi) X_n(\xi) d\xi \left( b_1^n \sin(\sqrt{\lambda_n} t) + b_2^n \cos(\sqrt{\lambda_n} t) \right) =$$

$$= \int_0^l R_1(x_2, \xi) \left[ \sum_{n=1}^{\infty} X_n(\xi) \lambda_n \left( b_1^n \sin(\sqrt{\lambda_n} t) + b_2^n \cos(\sqrt{\lambda_n} t) \right) \right] d\xi, \tag{1.47}$$

since (1.35) is absolutely and uniformly convergent on $]0, l[$ and in view of (1.46) and $X_n(x_2) \in C([0, l])$ we conclude that the corresponding integral in (1.47) is absolutely convergent on $]0, l[$. Similarly, we can prove the convergence of the series $(1.43_2)$, $(1.43_3)$, $(1.43_4)$, on $]0, l[$ and $(1.44_i)$ $(i = = 1, 2)$ on $[0, l]$. □

Thus, (1.30) is the solution of the Problem 8 in the case $q(x_2, t) \equiv 0$.

Now, let us consider Problem 8 when $q(x_2, t) \not\equiv 0$, $\varphi_i = 0$, $i = 1, 2$, and let $\frac{q}{\sqrt{g}}(\cdot, t) \in L_2(0, l)$. Then $q(x_2, t)$ can be represented as convergent series in $L_2(0, l)$:

$$\frac{q(x_2, t)}{\sqrt{g(x_2)}} = \sum_{n=1}^{\infty} \left( \frac{q(x_2, t)}{\sqrt{g(x_2)}}, Y_n \right) Y_n = \sum_{n=1}^{\infty} (q, X_n) X_n \sqrt{g},$$

hence,

$$q(x_2,t)=\sum_{n=1}^{\infty} g(x_2)X_n(x_2)q_n(t),\ \ q_n(t):=\int\limits_0^l q(x_2,t)X_n(x_2)dx_2.$$

Further, we look for the solution in the form

$$w(x_2,t)=\sum_{n=1}^{\infty} w_n(x_2,t),$$

where $w_n(x_2,t)$ is a solution of the Problem 8 with $q(x_2,t)$ replaced by $g(x_2)X_n(x_2)q_n(t)$. Using the method of separation of variables, we get

$$w_n(x_2,t)=X_n(x_2)T_{1n}(t),$$

where

$$T''_{1n}(t)+\lambda_n T_{1n}(t)=q_n(t)$$

and $X_n(x_2)$ satisfies (1.15).

Therefore, $w(x_2,t)$ can be expressed as follows

$$w(x_2,t)=\sum_{n=1}^{\infty}\frac{1}{\sqrt{\lambda_n}}X_n\int\limits_0^t \sin(\sqrt{\lambda_n}(t-\tau))q_n(\tau)d\tau. \tag{1.48}$$

Now, similarly to the proofs of Theorems 1.8 and 1.9, if the following conditions are fulfilled

$$\tau(x_2,t):=\frac{1}{\sqrt{g(x_2)}}\left(D(x_2)\left(\frac{q(x_2,t)}{g(x_2)}\right)_{,x_2x_2}\right)_{,x_2x_2}\in C[0,l],$$

$$\frac{\tau}{\sqrt{g}}(0,t)=-D(x_2)\left(\frac{\tau(x_2,t)}{\sqrt{g(x_2)}}\right)_{,x_2x_2}\Bigg|_{x_2=0_+}=\left(\frac{\tau(x_2,t)}{\sqrt{g(x_2)}}\right)_{,x_2}\Bigg|_{x_2=l} \tag{1.49}$$

$$=\left(-D(x_2)\left(\frac{\tau(x_2,t)}{\sqrt{g(x_2)}}\right)_{,x_2x_2}\right)_{,x_2}\Bigg|_{x_2=l_-}=0,$$

(for this, e.g., it is sufficient that $q^{(j)}(\cdot,t)=O(x_2^{\gamma_j})$ $x_2\to 0_+$, $\gamma_j>7-j-\frac{2\alpha}{3}$, $q^{(j)}(\cdot,t)=O((l-x_2)^{\delta_j})$ $x_2\to l_-$, $\gamma_j>7-j-\frac{2\beta}{3}$, $j=\overline{0,8}$) we have the absolute and uniform convergence of the series (1.48) and

$$(D(x_2)w_{,x_2x_2}(x_2,t))^{(i)}_{x_2}=\sum_{n=1}^{\infty}(D(x_2)X''_n)^{(i)}(x_2)T_{1n}(t),\ \ i=0,1,$$

on $[0,l]$, and the absolute and uniform convergence of the series

$$w_{x_2}^{(i)}(x_2,t)=\sum_{n=1}^{\infty}X_n^{(i)}(x_2)T_{1n}(t),\ \ i=1,...,4,$$

$$w_t^{(i)}(x_2,t)=\sum_{n=1}^{\infty}X_n(x_2)T_{1n}^{(i)}(t),\ \ i=1,2,$$

on $]0,l[$.

**Remark 1.11.** Solution of the Problem 8 in case $q(x_2,t)$, $\varphi_i\not\equiv 0$ can be expressed as follows

$$w(x_2,t)=\sum_{n=1}^{\infty}w_n(x_2,t),$$

where

$$w_n(x_2,t)=X_n(x_2)(T_{1n}(t)+T_n(t)).$$

**Remark 1.12.** Similarly, we can solve the initial boundary value problems which correspond to the Problems 1-7, 9, 10.

We can avoid the restrictions (1.49) on $q(x_2,t)$ if we consider harmonic vibration. In this case

$$w(x_2,t)=e^{i\omega\, t}w_0(x_2),\ \ q(x_2,t)=e^{i\omega\, t}q_0(x_2),$$

where $\omega=const$ is an oscillation frequency, $q_0(x_2)\in C([0,l])$ is a given function. Now, for $w_0(x_2)$ from (1.1) we get the following equation

$$(D(x_2)w_0''(x_2))''\ \ =\ \ q_0(x_2)+2\omega^2\rho h(x_2)w_0(x_2),$$

which we solve under one of the following boundary conditions (BCs)

1. $w_0(0)=w_0'(0)=w_0(l)=w_0'(l)=0,\ \ 0\le\alpha,\ \beta<1,$
   $w_0(x_2)\in C^4(]0,l[)\cap C^1([0,l]).$

2. $w_0(0)=w_0'(0)=w_0'(l)=Q_2(l)=0,\ \ 0\le\alpha,\ \beta<1,$
   $w_0(x_2)\in C^4(]0,l[)\cap C^1([0,l]).$

3. $w_0(0)=w_0'(0)=w_0(l)=M_2(l)=0,\ \ 0\le\alpha<1,\ 0\le\beta<2,$
   $w_0(x_2)\in C^4(]0,l[)\cap C^1([0,l[)\cap C([0,l]).$

4. $w_0(0)=w_0'(0)=M_2(l)=Q_2(l)=0,\ \ 0\le\alpha<1,\ \beta\ge 0,$
   $w_0(x_2)\in C^4(]0,l[)\cap C^1([0,l[).$

5. $w_0'(0)=Q_2(0)=w_0(l)=w_0'(l)=0,\ \ 0\le\alpha\,\beta<1,$

$w_0(x_2) \in C^4(]0,l[) \cap C^1([0,l]).$

6. $w_0'(0) = Q_2(0) = w_0(l) = M_2(l) = 0, \;\; 0 \le \alpha < 1, \; 0 \le \beta < 2,$
$w_0(x_2) \in C^4(]0,l[) \cap C^1([0,l[) \cap C([0,l]).$

7. $w_0(0) = M_2(0) = w_0(l) = w_0'(l) = 0, \;\; 0 \le \alpha < 2, \; 0 \le \beta < 1,$
$w_0(x_2) \in C^4(]0,l[) \cap C^1(]0,l]) \cap C([0,l]).$

8. $w_0(0) = M_2(0) = w_0'(l) = Q_2(l) = 0, \;\; 0 \le \alpha < 2, \; 0 \le \beta < 1,$
$w_0(x_2) \in C^4(]0,l[) \cap C([0,l]) \cap C^1(]0,l]).$

9. $w_0(0) = M_2(0) = w_0(l) = M_2(l) = 0, \;\; 0 \le \alpha, \; \beta < 2,$
$w_0(x_2) \in C^4(]0,l[) \cap C([0,l]).$

10. $M_2(0) = Q_2(0) = w_0(l) = w_2'(l) = 0, \;\; \alpha \ge 0, \; 0 \le \beta < 1,$
$w_0(x_2) \in C^4(]0,l[) \cap C^1(]0,l]).$

Here, by $M_2(x_2)$, $Q_2(x_2)$ are denoted the following expressions

$$M_2(x_2) := -D(x_2) w_{0,22}(x_2), \;\; Q_2(x_2) := M_{2,2}(x_2),$$

The above BVPs are equivalent to the integral equation

$$w_0(x_2) - \omega^2 \int\limits_0^l K(x_2,\xi)\, g(\xi)\, w_0(\xi) d\xi = F(x_2), \tag{1.50}$$

where

$$F(x_2) := \int\limits_0^l K(x_2,\xi)\, q_0(\xi) d\xi,$$

and $K(x_2,\xi)$ is given by the expression (1.17), $K_3(x_2,\xi)$ is different for the different BCs, e.g., for the BCs 1., 2., 8., 9., and 10. it has the form (1.22), (1.23), (1.18), (1.23), (1.24), respectively.

Introducing a new unknown function

$$w_1(x_2) = w_0(x_2)\sqrt{g(x_2)}$$

we can reduce (1.50) to the following integral equation

$$w_1(x_2) - \omega^2 \int\limits_0^l R(x_2,\xi)\, w_1(\xi) d\xi = F(x_2)\sqrt{g(x_2)} \tag{1.51}$$

with $R(x_2, \xi)$ given by (1.20). If $\omega^2 \neq \lambda_n$, the unique solution of (1.51) can be written as follows (see, e.g., [10], Theorem XVIII, p.140)

$$\begin{aligned} w_1(x_2) &= F(x_2)\sqrt{g(x_2)} \\ &+\omega^2 \sum_{n=1}^{\infty} \left[ \frac{1}{\lambda_n - \omega^2} \int_0^l F(\xi)\sqrt{g(x_2)}\, Y_n(\xi) d\xi \right] Y_n(x_2), \end{aligned} \tag{1.52}$$

where the series in the right hand side of (1.52) is absolutely and uniformly convergent on $[0, l]$.

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