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ON SOME NONCLASSICAL PROBLEMS FOR DIFFERENTIAL EQUATIONS AND THEIR APPLICATIONS TO THE THEORY OF CUSPED PRISMATIC SHELLS

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Preface

The present Lecture Notes contains extended material mainly based on the lectures presented at the Workshop on Mathematical Methods for Elastic Cusped Plates and Bars (Tbilisi, September 27–28, 2001).

The work consists of the list of notation, introduction, three chapters and references.

The Introduction contains a survey of results related to the subject and a brief presentation of results of the present work.

In Chapter 1 some auxiliary materials are given which are used in Chapters 2 and 3.

Chapter 2 deals with the problems of cylindrical bending and bending vibration of a cusped plate. Bending problems of cusped plates fall outside of the limits of classical bending theory. The aim of this chapter is to study the problem of well-possedness of boundary value problems and initial boundary value problems in case of cylindrical bending of shells with two cusped edges and in some cases to solve these problems in explicit forms.

Chapter 3 is dedicated to the interface problem of the interaction of a plate with two cusped edges and a flow of an incompressible fluid.

Acknowledgments. The author is very grateful to Prof. G. Jaiani, Prof. S. Kharibegashvili, and Prof. D. Natroshvili for their useful discussions.

Author

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List of Notations

$\mathbb{N} := \{1, 2, \cdots\}$,
$\mathbb{N} := \{1, 2, \cdots\}$,
$\mathbb{R}^n$ $\quad n-$dimensional Euclidean space ($n \in \mathbb{N}$)
$\Omega := \{(x_1, x_2, x_3) : \ -\infty < x_1 < \infty, \ \ 0 < x_2 < l, \ \ x_3 = 0\}$ - the projection of a plate on the plane $x_3 = 0$
$I := \{[0, l] \times \{0\}\}$
$\Omega^f := \{x_1, x_2, x_3 : x_1 = 0, \ x_2 := (x_2, x_3) \in \mathbb{R}^2 \backslash I\}$ - space which occupies the fluid
$2h(x) := \overset{(+)}{h}(x) - \overset{(-)}{h}(x)$ - thickness of a plate at point x
ω - oscillation frequency
$D(x_2)$ - flexural rigidity
ρ - density of a plate
$w(x_2, t)$ - deflection of a plate
$q(x_2, t)$ - lateral load
$M_2(x_2, t)$ - bending moment
$Q_2(x_2, t)$ - intersecting force
E - Young's modulus
σ - Poisson's ratio
$F := (F_2, F_3)$ - plane volume forces
δ_{ij} - Kroneker Delta
ρ^f - density of a fluid
$u := (u_1, u_2, u_3)$ - displacement vector of a fluid
$v := (v_1, v_2, v_3)$ - velocity vector of a fluid
p - pressure of a fluid
$p(x_2, \overset{(+)}{h}(x_2), t)$ $(p(x_2, \overset{(-)}{h}(x_2), t))$ - the value of the pressure on the upper (lower) surface of the plate
$v_{3\infty}(t), \ p_\infty(t)$ - values of the velocity vector component and pressure at infinity
$\sigma^f_{jk} = -p\delta_{jk} + \mu \left(\dfrac{\partial v_j}{\partial x_k} + \dfrac{\partial v_k}{\partial x_j} \right)$ - stress tensor of a fluid
$\nu, \ \mu$ - coefficients of viscosity
$\Delta := \dfrac{\partial^2}{\partial x_2^2} + \dfrac{\partial^2}{\partial x_3^2}$
$w_{,t} := \dfrac{\partial w}{\partial t}$

$w_{,i} := \dfrac{\partial w}{\partial x_i}$, $i = 1, 2, 3$
$C^n(]0, l[)$ ($C^n([0, l])$) - n-times continuously differentiable functions in $]0, l[$ (on $[0, l]$)
$C^n(\Omega^f)$ - n-times continuously differentiable functions in Ω^f with respect to x_2 and x_3
$C(t > 0)$ - continuous functions with respect to t for $t > 0$
$H([0, l])$ - class of Hölder continuous functions
$L_2([0, l])$ - class of square integrable functions on $[0, l]$

Introduction

In 1955 I.Vekua [95]-[97] raised the problem of investigation of cusped plates, i.e. such ones whose thickness on the part of the plate boundary or on the whole one vanishes. The problem mathematically leads to the question of setting and solving of boundary value problems for even order equations and systems of elliptic type with the order degeneration in the statical case and of initial boundary value problems for even order equations and systems of hyperbolic type with the order degeneration in the dynamical case (for corresponding investigations see the survey [35] and also I. Vekua's comments in [97, p.86]). There exists a wide literature devoted to the theory of degenerate and mixed type equations (see, e.g., [5], [30]), which was developed intensively in the period from early 50-ies till early 70-ies but it could not cover the above equations and systems because of distinct peculiarities of the latter caused by the geometry of the mechanical problem.

The first work concerning classical bending of cusped elastic plates was done by E. Makhover [67], [68] and S. Mikhlin [71].

In 1957 E. Makhover [67], [68], by using the results of S. Mikhlin [71], had considered such a cusped plate with the stiffness $D(x_1, x_2)$ satisfying

$$D_1 x_2^{\kappa_1} \leq D(x_1, x_2) \leq D_2 x_2^{\kappa_1}, \quad D_1, D_2, \ \kappa_1 = const > 0, \tag{1}$$

within the framework of classical bending theory. She particularly studied in which cases the deflection ($\kappa_1 < 2$) or its normal derivative ($\kappa_1 < 1$) on the cusped edge of the plate can be given. In 1971, A. Khvoles [62] represented the forth order Airy stress function operator as the product of two second order operators in the case when the plate thickness $2h$ is given by

$$2h = h_0 x_2^{\kappa_2}, \ h_0, \kappa_2 = const > 0, \ x_2 \geq 0, \tag{2}$$

and investigated the general representation of corresponding solutions. Since 1972 the work of G. Jaiani in [36]–[51] is also devoted to these problems. By using more natural spaces than E. Makhover, G. Jaiani in [48] has analyzed in which cases the cusped edge can be freed ($\kappa_1 > 0$) or freely supported ($\kappa_1 < 2$). Moreover, he established well–posedness and the correct formulation of all admissible principal boundary value problems (BVPs). In [41], [42], [47] he also investigated the tension–compression problem of cusped plates, based on I. Vekua's model of shallow prismatic shells. G. Jaiani's results can be summarized as follows.

Let n be the inward normal of the plate boundary. In the case of the tension-compression problem on the cusped edge, where

$$0 \leq \frac{\partial h}{\partial n} < +\infty (\text{in the case (2) this means } \kappa_2 \geq 1) ,$$

which will be called a sharp cusped edge, one can not prescribe the displacement vector; while on the cusped edge, where

$$\frac{\partial h}{\partial n} = +\infty (\text{in the case (2) this means } \kappa_2 < 1) ,$$

called a blunt cusped edge, the displacement vector can be prescribed. In the case of the classical bending problem with a cusped edge, where

$$\frac{\partial h}{\partial n} = O(d^{\kappa - 1}) \text{as d} \to 0, \ \kappa = \text{const} > 0 \tag{3}$$

and where d is the distance between an interior reference point of the plate projection and the cusped edge, the edge can not be fixed if $\kappa \geq \frac{1}{3}$, but it can be fixed if $0 < \kappa < \frac{1}{3}$; it can not be freely supported if $\kappa \geq \frac{2}{3}$, and it can be freely supported if $0 < \kappa < \frac{2}{3}$; it can be free or arbitrarily loaded by a shear force and a bending moment if $\kappa > 0$. Note that in the case (2), the condition (3) implies that $d_2 = x_2$ and $\kappa = \kappa_2 = \frac{\kappa_1}{3}$.

For the specific cases of cusped cylindrical and conical shell bending, the above results remain valid as it has been shown by G. Tsiskarishvili and N. Khomasuridse [89]-[92]. These results also remain valid in the case of classical bending of orthotropic cusped plates (see [51]). However, for general cusped shells and also for general anisotropic cusped plates, the corresponding analysis is done.

The problems involving cusped plates lead to correct mathematical formulations of BVPs for even order elliptic equations and systems whose orders degenerate at the boundary (see [47], [52]-[53]).

Applying the functional–analytic method developed by G. Fichera in [28], [29] (see also [21], [22]), in [47] the particular case of Vekua's system for general cusped plates has been investigated.

The classical bending of plates with the stiffness (1) in energetic and in weighted Sobolev spaces has been studied by G. Jaiani in [48], [50]. In the energetic space some restrictions on the lateral load has been relaxed by G. Devdariani in [20]. G. Tsiskarishvili [90] characterized completely the classical axial symmetric bending of specific circular cusped plates without or with a hole.

In the case (2), the basic BVPs have been explicitly solved in [43] and [53] with the help of singular solutions depending only on the polar angle.

If we consider the cylindrical bending of a plate, in particular of a cusped one, with rectangular projection $a \leq x_1 \leq b$, $0 \leq x_2 \leq \ell$, then we actually get the corresponding results also for cusped beams (see [49], [43], [93], [73]-[77], [12], [13], [54], [55]).

In 1999-2001 two contact problems were considered by N. Shavlakadze [86], [87], namely, the contact problem for an unbounded elastic medium composed of two half-planes $x_1 > 0$ and $x_1 < 0$ having different elastic constants and strengthened on the semi-axis $x_2 > 0$ by an inclusion of variable thickness (cusped beam) with constant Young's modulus and Poisson's ratio. It was assumed that the plate is subjected to plane deformation, the flexural rigidity D had the form

$$D = D_0 x_2^{\varkappa}, \ \ D_0, \ \varkappa = \text{const} > 0,$$

and the cusped end $x_2 = 0$ of the beam was free.

At the same time (in the fifties of the twentieth century), I.Vekua [95] introduced a new mathematical model for elastic prismatic shells (i.e., of plates of variable thickness) which was based on expansions of the three–dimensional displacement vector fields and the strain and stress tensors in linear elasticity into orthogonal Fourier-Legendre series with respect to the variable plate thickness. By taking only the first $N + 1$ terms of the expansions, he introduced the so–called N–th approximation. Each of these approximations for $N = 0, 1, ...$ can be considered as an independent mathematical model of plates. In particular, the approximation for $N = 1$ corresponds to the classical Kirchhoff plate model. In the sixties, I. Vekua [96] developed the analogous mathematical model for thin shallow shells. All his results concerning plates and shells are collected in his monograph [97]. Works of I. Babuška, D. Gordeziani, V. Guliaev, I. Khoma, A. Khvoles, T. Meunargia, C. Schwab, T. Vashakmadze, V. Zhgenti, and others (see [2], [31], [33], [61], [62], [69], [84], [85], [94], [100] and the references therein) are devoted to further analysis of I.Vekua's models (rigorous estimation of the modeling error, numerical solutions, etc.) and their generalizations (to non-shallow shells, to the anisotropic case, etc.).

In [56] variational hierarchical two–dimensional models for cusped elastic plates are constructed. With the help of variational methods, existence and uniqueness theorems for the corresponding two–dimensional boundary value problems are proved in appropriate weighted functional spaces. By means of the solutions of these two–dimensional boundary value problems, a sequence of approximate solutions in the corresponding three-dimensional region is constructed. This sequence converges in the Sobolev space H^1 to the solution of the original three-dimensional boundary value problem. The systems of differential equations corresponding to the two-dimensional variational hierarchical models are explicitly given for a general orthogonal system and for Legendre polynomials, in particular.

Recently N.Chinchaladze, R. Gilbert, G. Jaiani, S. Kharibegashvili and D. Natroshvili have studied the well posedness of boundary value problems for elastic cusped prismatic shells in the Nth approximation of I. Vekua's hierarchical models under (all reasonable) boundary conditions at the cusped edge and given displacements at the non-cusped edge and stresses at the upper and lower faces of the shell [19].

For the last decades the direct and inverse problems connected with the interaction between difference vector fields have received much attention in the mathematical and engineering scientific literature and have been intensively investigated. They arise in many physical and mechanical models describing the interaction of two different media where the whole process is characterized by a vector-function of dimension k in one medium and by a vector-function of dimension n in the other (for example, fluid-structure interaction where a streamlined body is an elastic obstacle, scattering of acoustic and electromagnetic waves by an elastic obstacle, interaction between an elastic body and seismic waves, etc.).

A lot of authors have considered and studied in detail the direct problems of interaction between an elastic isotropic body occupying a bounded region $\overline{\Omega}$ with a three-dimensional elastic vector field to be defined, and some isotropic medium (say fluid) occupying the unbounded exterior region, the compliment of Ω with respect to the whole space, where a scalar field is to be defined. The time-harmonic dependent unknown vector and scalar fields are coupled by some kinematic and dynamic conditions on the boundary $\partial\Omega$, which lead to various type of non-classical interface problems of steady oscillations for a piecewise homogeneous isotropic medium. An exhaustive information in this direction concerning theoretical and numerical results can be found in [4], [6], [7], [24], [25], [59], [60], [32], [34] [26], [27], [78], [84].

Some particular cases where the elastic body under consideration is anisotropic have been treated in [57], [58], [79].

Various authors dedicated their works to the solid-fluid (see e.g. [79], [83], [98]-[99], [80]-[82], [9]-[11]), [14]-[18] contact problems. The present work is devoted to the interaction problems when profile of an elastic part is cusped on some part boundary.

Bending problems of cusped plates fall outside of the limits of classical bending theory. The aim of the dissertation is to study the problem of well-possedness of boundary value problems and initial boundary value problems in case of cylindrical bending of shells with two cusped edges and in some cases to solve these problems in explicit forms.

The work consists of the list of notations, introduction, three chapters and bibliography.

The Introduction contains a survey of results related to the subject and a brief presentation of results of the present work.

In Chapter 1 some auxiliary materials are given used in Chapters 2 and 3.

Chapter 2 deals with the problems of cylindrical bending and bending vibration of a plate.

Let us consider the plate whose projection on $x_3 = 0$ occupies the domain Ω

$$\Omega = \{(x_1, x_2, x_3) : \ -\infty < x_1 < \infty, \ \ 0 < x_2 < l, \ \ x_3 = 0\},$$

and where the thickness of the plate are given by the equation

$$2h(x_2) = h_0 x_2^{\alpha/3}(l - x_2)^{\beta/3}, \quad h_0,\, l,\, \alpha,\, \beta = \text{const}, \ \ h_0,\, l > 0, \quad \alpha,\, \beta \geq 0.$$

When $\alpha^2 + \beta^2 > 0$ a plate is called a cusped plate. A profile of the plate under consideration has one of the forms shown in Figures 4-12.

The equation of cylindrical bending of the plate has the form (see, e.g., [88])

$$(D(x_2)w,_{22}(x_2)),_{22} = q(x_2), \qquad 0 < x_2 < l, \tag{4}$$

where $w(x_2)$ is a deflection of the plate, $q(x_2)$ is a load, $D(x_2)$ is a flexural rigidity of the plate, and by $w_{,i}$ we denote $w_{,i} := \dfrac{\partial w}{\partial x_i}$.

In general,

$$D(x_2) := \frac{2Eh^3(x_2)}{3(1-\sigma^2)},$$

where E is a Young's modulus, σ is a Poisson's ratio. Let E =const, σ =const, and

$$D(x_2) = D_0 x_2^{\alpha}(l - x_2)^{\beta}, \quad D_0 = \text{const} > 0.$$

In the case of cylindrical bending of an isotropic plate, the bending moment $M_2(x_2)$ and the intersection force $Q_2(x_2)$ are given by the formulae (see [88])

$$M_2(x_2) := -D(x_2)w_{,22}(x_2), \;\; Q_2(x_2) := M_{2,2}(x_2). \tag{5}$$

Section 2.1 is devoted to the investigation of properties of equation (4) and formulation of all admissible classical bending boundary value problems (BVPs).

If $q(x_2) \in C([0,l])$ then

$$M_2(x_2), \;\; Q_2(x_2) \in C([0,l]),$$

the behaviour of the $w_{,2}(x_2)$ and $w(x_2)$ when $x_2 \to 0_+$ and $x_2 \to l_-$ depends on α and β. As a result of the corresponding analysis we obtain that, e.g., at the point $x_2 = 0$ the following classical bending boundary conditions are admissible

$$1.\; w(0) = w'(0) = 0 \;\text{ iff(if and only if) }\; \alpha < 1; \tag{6}$$

$$2.\; w'(0) = Q_2(0) = 0 \;\text{ iff }\; \alpha < 1; \tag{7}$$

$$3.\; w(0) = M_2(0) = 0 \;\text{ iff }\; \alpha < 3; \tag{8}$$

$$4.\; M_2(0) = Q_2(0) = 0 \;\text{ for any }\; \alpha. \tag{9}$$

Similar conditions we have at the point $x_2 = l$, under the same restrictions on β. All BVPs are solved in the explicit integral forms. Using these integral representations and the difference equation corresponding to (4) by means of MATLAB we get numerical results for the deflection, the bending moment and the intersecting force for different materials (see Figures 13-16).

In Section 2.2 a dynamical problem is investigated for the above cusped plate. The corresponding equation has the following form

$$(D(x_2)w,_{22}(x_2,t)),_{22} = q(x_2,t) - 2\rho h(x_2)\frac{\partial^2 w(x_2,t)}{\partial t^2}, \qquad 0 < x_2 < l, \tag{10}$$

where ρ is a density of the plate.

We solve equation (10) under the following initial conditions (IC)

$$w(x_2, 0) = \varphi_1(x_2), \;\; w,_t(x_2, 0) = \varphi_2(x_2), \;\; x_2 \in [0, l], \tag{11}$$

where $\varphi_1(x_2)$, $\varphi_1(x_2) \in C([0, l])$ are given functions.

In this case the bending moment and the intersecting force are given by the expressions

$$\begin{aligned} M_2(x_2, t) &:= -D(x_2) w,_{22}(x_2, t), \qquad (12)\\ Q_2(x_2, t) &:= M_{2,2}(x_2, t). \qquad (13) \end{aligned}$$

Since of (10) is not degenerate equation with respect to $t = 0$, taking into account (6)-(9), the following initial boundary value problems (IBVPs) are admissible

Problem 11 *Let $0 \le \alpha < 3$, $0 \le \beta < 1$. Find a function $w(x_2, t)$, which satisfies the following smoothness conditions*

$$\begin{aligned} &w(\cdot, t) \in C^4(]0, l[) \cap C([0, l]) \cap C^1(]0, l]), \;\; M_2(\cdot, t) \in C([0, l]), \;\; Q_2(\cdot, t) \in C([0, l]),\\ &w(x_2, \cdot) \in C^1(t \ge 0) \cap C^2(t > 0),\\ &\;\; w(x_2, t) \in C(0 \le x_2 \le l, \; t \ge 0), \end{aligned}$$

equation (10), the BCs

$$w(0, t) = M_2(0, t) = w,_2(l, t) = Q_2(l, t) = 0, \;\; t > 0,$$

and ICs (11), where

$$\varphi_i(x_2) \in C^4(]0, l[) \cap C^1(]0, l]) \cap C([0, l]),$$

$$\begin{aligned} \varphi_i(0) &= -D(x_2)\varphi_i''(x_2)|_{x_2=0_+} = \varphi_i'(l)\\ &= \left(-D(x_2)\varphi_i''(x_2)\right)'|_{x_2=l_-} = 0, i = 1, 2. \end{aligned}$$

Problem 12 *Let $0 \le \alpha$, $\beta < 1$. Find a function $w(x_2, t)$, which satisfies the following smoothness conditions*

$$\begin{aligned} &w(\cdot, t) \in C^4(]0, l[) \cap C^1([0, l]),\\ &w(x_2, \cdot) \in C^1(t \ge 0) \cap C^2(t > 0), \;\; w(x_2, t) \in C(0 \le x_2 \le l, \; t \ge 0), \end{aligned}$$

equation (10), the boundary conditions (BCs)

$$w(0, t) = w,_2(0, t) = w(l, t) = w,_2(l, t) = 0, \;\; t > 0,$$

and ICs (11), where

$$\varphi_i(x_2) \in C^4(]0, l[) \cap C^1([0, l]),$$
$$\varphi_i(0) = \varphi_i'(0) = \varphi_i(l) = \varphi_i'(l) = 0, \;\; i = 1, 2.$$

Problem 13 *Let* $0 \leq \alpha,\ \beta < 1$. *Find a function* $w(x_2,t)$, *which satisfies the following smoothness conditions*

$$\begin{array}{l} w(\cdot,t) \in C^4(]0,l[) \cap C^1([0,l]),\ \ Q_2(\cdot,t) \in C([0,l]), \\ w(x_2,\cdot) \in C^1(t \geq 0) \cap C^2(t > 0),\ \ w(x_2,t) \in C(0 \leq x_2 \leq l,\ t \geq 0), \end{array}$$

equation (10), the BCs

$$w(0,t) = w,_2(0,t) = w,_2(l,t) = Q_2(l,t) = 0,\ \ t > 0,$$

and ICs (11), where

$$\varphi_i(x_2) \in C^4(]0,l[) \cap C^1([0,l]),$$

$$\varphi_i(0) = \varphi_i'(0) = \varphi_i'(l) = (-D(x_2)\varphi_i''(x_2))'\,|_{x_2=l_-} = 0,\ i = 1,2.$$

Problem 14 *Let* $0 \leq \alpha, < 1,\ 0 \leq \beta < 3$. *Find a function* $w(x_2,t)$, *which satisfies the following smoothness conditions*

$$\begin{array}{l} w(\cdot,t) \in C^4(]0,l[) \cap C^1([0,l[) \cap C([0,l]),\ \ M_2(\cdot,t) \in C([0,l]), \\ w(x_2,\cdot) \in C^1(t \geq 0) \cap C^2(t > 0),\ \ w(x_2,t) \in C(0 \leq x_2 \leq l,\ t \geq 0), \end{array}$$

equation (10), the BCs

$$w(0,t) = w,_2(0,t) = w(l,t) = M_2(l,t) = 0,\ \ t > 0,$$

and ICs (11), where

$$\varphi_i(x_2) \in C^4(]0,l[) \cap C^1([0,l[) \cap C([0,l]),$$

$$\varphi_i(0) = \varphi_i'(0) = \varphi_i(l) = (-D(x_2)\varphi_i''(x_2))\,|_{x_2=l_-} = 0,\ i = 1,2.$$

Problem 15 *Let* $0 \leq \alpha < 1,\ \beta \geq 0$. *Find a function* $w(x_2,t)$, *which satisfies the following smoothness conditions*

$$\begin{array}{l} w(\cdot,t) \in C^4(]0,l[) \cap C^1([0,l[),\ \ M_2(\cdot,t) \in C([0,l]),\quad Q_2(\cdot,t) \in C([0,l]), \\ w(x_2,\cdot) \in C^1(t \geq 0) \cap C^2(t > 0),\ \ w(x_2,t) \in C(0 \leq x_2 < l,\ t \geq 0), \end{array}$$

equation (10), the BCs

$$w(0,t) = w,_2(0,t) = M_2(l,t) = Q_2(l,t) = 0,\ \ t > 0,$$

and ICs (11), where

$$\varphi_i(x_2) \in C^4(]0,l[) \cap C^1([0,l[),$$

$$\begin{array}{rcl} \varphi_i(0) & = & \varphi_i'(0) = (-D(x_2)\varphi_i''(x_2))\,|_{x_2=l_-} \\ & = & (-D(x_2)\varphi_i''(x_2))'\,|_{x_2=l_-} = 0,\ i = 1,2. \end{array}$$

Problem 16 *Let $0 \le \alpha,\ \beta < 1$. Find a function $w(x_2,t)$, which satisfies the following smoothness conditions*

$$\begin{array}{l} w(\cdot,t) \in C^4(]0,l[) \cap C^1([0,l]),\ \ Q_2(\cdot,t) \in C([0,l]), \\ w(x_2,\cdot) \in C^1(t \ge 0) \cap C^2(t>0),\ \ w(x_2,t) \in C(0 \le x_2 \le l,\ t \ge 0), \end{array}$$

equation (10), the BCs

$$w,_2(0,t) = Q_2(0,t) = w(l,t) = w,_2(l,t) = 0,\ \ t>0,$$

and ICs (11), where

$$\varphi_i(x_2) \in C^4(]0,l[) \cap C^1([0,l]),$$

$$\varphi_i'(0) = (-D(x_2)\varphi_i''(x_2))'\,|_{x_2=0_+} = \varphi_i(l) = \varphi_i'(l) = 0,\ i=1,2.$$

Problem 17 *Let $0 \le \alpha < 1,\ 0 \le \beta < 3$. Find a function $w(x_2,t)$, which satisfies the following smoothness conditions*

$$\begin{array}{l} w(\cdot,t) \in C^4(]0,l[) \cap C^1([0,l[) \cap C([0,l]),\ \ M_2(\cdot,t) \in C([0,l]),\ \ Q_2(\cdot,t) \in C([0,l]), \\ w(x_2,\cdot) \in C^1(t \ge 0) \cap C^2(t>0),\ \ w(x_2,t) \in C(0 \le x_2 \le l,\ t \ge 0), \end{array}$$

equation (10), the BCs

$$w,_2(0,t) = Q_2(0,t) = w(l,t) = M_2(l,t) = 0,\ \ t>0,$$

and ICs (11), where

$$\varphi_i(x_2) \in C^4(]0,l[) \cap C^1([0,l[) \cap C([0,l]),$$

$$\begin{aligned} \varphi_i'(0) &= (-D(x_2)\varphi_i''(x_2))'\,|_{x_2=0_+} = \varphi_i(l) \\ &= (-D(x_2)\varphi_i''(x_2))\,|_{x_2=l_-} = 0,\ i=1,2. \end{aligned}$$

Problem 18 *Let $0 \le \alpha < 3,\ 0 \le \beta < 1$. Find a function $w(x_2,t)$, which satisfies the following smoothness conditions*

$$\begin{array}{l} w(\cdot,t) \in C^4(]0,l[) \cap C^1(]0,l]) \cap C([0,l]),\ \ M_2(\cdot,t) \in C([0,l]),\ \ Q_2(\cdot,t) \in C([0,l]), \\ w(x_2,\cdot) \in C^1(t \ge 0) \cap C^2(t>0),\ \ w(x_2,t) \in C(0 \le x_2 \le l,\ t \ge 0), \end{array}$$

equation (10), the BCs

$$w(0,t) = M_2(0,t) = w(l,t) = w,_2(l,t) = 0,\ \ t>0,$$

and ICs (11), where

$$\varphi_i(x_2) \in C^4(]0,l[) \cap C^1(]0,l]) \cap C([0,l]),$$

$$\varphi_i(0) = (-D(x_2)\varphi_i''(x_2))\,|_{x_2=0_+} = \varphi_i(l) = \varphi_i'(l) = 0,\ i=1,2.$$

Problem 19 *Let $0 \leq \alpha,\ \beta < 3$. Find a function $w(x_2,t)$, which the satisfies following smoothness conditions*

$$\begin{aligned}&w(\cdot,t) \in C^4(]0,l[) \cap C([0,l]),\ \ M_2(\cdot,t) \in C([0,l]),\\ &w(x_2,\cdot) \in C^1(t \geq 0) \cap C^2(t>0),\ \ w(x_2,t) \in C(0 \leq x_2 \leq l,\ t \geq 0),\end{aligned}$$

equation (10), the BCs

$$w(0,t) = M_2(0,t) = w(l,t) = M_2(l,t) = 0,\ \ t>0,$$

and ICs (11), where

$$\varphi_i(x_2) \in C^4(]0,l[) \cap C([0,l]),$$

$$\begin{aligned}\varphi_i(0) &= (-D(x_2)\varphi_i''(x_2))\,|_{x_2=0+} = \varphi_i(l)\\ &= (-D(x_2)\varphi_i''(x_2))\,|_{x_2=l-} = 0,\ i=1,2.\end{aligned}$$

Problem 20 *Let $\alpha \geq 0,\ 0<\beta<1$. Find a function $w(x_2,t)$, which satisfies the following smoothness conditions*

$$\begin{aligned}&w(\cdot,t) \in C^4(]0,l[) \cap C^1(]0,l]),\ \ M_2(\cdot,t) \in C([0,l]),\ \ Q_2(\cdot,t) \in C([0,l]),\\ &w(x_2,\cdot) \in C^1(t \geq 0) \cap C^2(t>0),\ \ w(x_2,t) \in C(0 < x_2 \leq l,\ t \geq 0),\end{aligned}$$

equation (10), the BCs

$$M_2(0,t) = Q_2(0,t) = w(l,t) = w,_2(l,t) = 0,\ \ t>0,$$

and ICs (11), where

$$\varphi_i(x_2) \in C^4(]0,l[) \cap C^1(]0,l]),$$

$$\begin{aligned}(-D(x_2)\varphi_i''(x_2)) &= (-D(x_2)\varphi_i''(x_2))'\,|_{x_2=0+}\\ &= \varphi_i(l) = \varphi_i'(l) = 0,\ i=1,2.\end{aligned}$$

Let $q \equiv 0$. Using the Fourier method, we look for $w(x_2,t)$ in the following form

$$w(x_2,t) = X(x_2)T(t),$$

where $T(t)$ and $X(x_2)$ are satisfying the following equations

$$T''(t) + \lambda T(t) = 0,$$

and

$$X(x_2) = \lambda \int_0^l g(\xi)K(x_2,\xi)X(\xi)d\xi,\ \ g(x_2) := 2\rho h(x_2), \tag{14}$$

where $K(x_2,\xi) \in C([0,l] \times [0,l])$ is constructed explicitly and it depends on the coefficients of equation (10) and the type of boundary conditions in Problems 11-20.

We denote by λ_n and X_n the corresponding eigenvalues and eigenfunctions of (14).

The following propositions hold.

Proposition 2.2 *$K(x_2, \xi)$ is symmetric with respect to x_2 and ξ.*

Proposition 2.3 *Number of eigenvalues λ_n of (14) is not finite.*

Proposition 2.4 *All λ_n are positive.*

The solution of equation (10) under the initial conditions (11) and one of the boundary conditions (see Problems 11-20) can be written as follows [15]

$$w(x_2, t) = \sum_{n=1}^{\infty} X_n(x_2)\left(b_1^n \sin(\sqrt{\lambda_n} t) + b_2^n \cos(\sqrt{\lambda_n} t)\right), \tag{15}$$

where

$$b_1^n = \frac{1}{\sqrt{\lambda_n}} \int_0^l g(x_2) X_n(x_2) \varphi_2(x_2) dx_2, \;\; b_2^n = \int_0^l g(x_2) X_n(x_2) \varphi_1(x_2) dx_2. \tag{16}$$

Let us consider one of the IBVP. For the sake of simplicity we consider Problem 11.

Further, if we suppose that $\psi_i(x_2) := \frac{(D\varphi_i'')''}{\sqrt{g(x_2)}} \in C([0,l])$ $(i = 1, 2)$, we can prove the following theorems [15]

Theorem 2.5 *The series (15) converges absolutely and uniformly on $[0, l]$. Moreover, the series*

$$w_{,t}(x_2, t) = \sum_{n=1}^{\infty} X_n(x_2) \sqrt{\lambda_n} \left(b_1^n \cos(\sqrt{\lambda_n} t) - b_2^n \sin(\sqrt{\lambda_n} t)\right)$$

and

$$w_{,tt}(x_2, t) = -\sum_{n=1}^{\infty} X_n(x_2) \lambda_n \left(b_1^n \sin(\sqrt{\lambda_n} t) + b_2^n \cos(\sqrt{\lambda_n} t)\right)$$

converge absolutely and uniformly on any $[a, b] \subset]0, l[$ if the functions

$$\Psi_i(x_2) := \frac{\psi_i(x_2)}{\sqrt{g(x_2)}} \mathrm{for} i = 1, 2 \mathrm{satisfy BCs given in Problem 11} \tag{17}$$

and the functions

$$\chi_i(x_2)\sqrt{g(x_2)} := \left(D(x_2)\Psi_i''(x_2)\right)'', \;\; i = 1, 2, \;\; \text{are integrable on} \;\;]0, l[\tag{18}$$

(For this, e.g., it is sufficient that $\frac{d^j}{dx_2^j}\varphi_i(x_2) = O(x_2^{\gamma_{ij}})$, $\gamma_{ij} = \text{const} > 7 - j - \frac{5\alpha}{3}$, $x_2 \to 0_+$, $\frac{d^j}{dx_2^j}\varphi_i(x_2) = O((l - x_2)^{\delta_{ij}})$, $\delta_{ij} = \text{const} > 7 - j - \frac{5\beta}{3}$, $x_2 \to l_-$, $i = 1, 2$; $j = \overline{2, 8}$).

Theorem 2.6 *The series*

$$\frac{\partial^i}{\partial x_2^i} w(x_2,t) = \sum_{n=1}^{\infty} \frac{d^i}{dx_2^i} X_n(x_2) \left(b_1^n \sin(\sqrt{\lambda_n} t) + b_2^n \cos(\sqrt{\lambda_n} t) \right), \quad i = 1,2,3,4,$$

are convergent absolutely and uniformly on any $[a,b] \subset]0,l[$, *while the series*

$$\frac{\partial^{i-1}}{\partial x_2^{i-1}} (D(x_2) w_{,x_2x_2}(x_2,t)) = \sum_{n=1}^{\infty} \frac{d^{i-1}}{dx_2^{i-1}} (D(x_2) X_n''(x_2)) \left(b_1^n \sin(\sqrt{\lambda_n} t) + \right.$$

$$\left. + b_2^n \cos(\sqrt{\lambda_n} t) \right), \quad i = 1,2$$

are convergent absolutely and uniformly on $[0,l]$.

Thus, (15) is the solution of the Problem 11 for $q(x_2,t) \equiv 0$.

Let us consider the case when $q(x_2,t) \not\equiv 0$, $\varphi_i = 0$, and let $\frac{q}{\sqrt{g}}(\cdot,t) \in L_2(0,l)$. Then $q(x_2,t)$ can be represented as a convergent series in $L_2(0,l)$:

$$q(x_2,t) = \sum_{n=1}^{\infty} g(x_2) X_n(x_2) q_n(t), \quad q_n(t) := \int_0^l q(x_2,t) X_n(x_2) dx_2.$$

Further, we look for the solution in the form $w(x_2,t) = \sum\limits_{n=1}^{\infty} w_n(x_2,t)$, where $w_n(x_2,t)$ is a solution of the equation (10) under the homogeneous initial conditions and under the boundary conditions given in Problem 11 with $q(x_2,t)$ replaced by $g(x_2)X_n(x_2)q_n(t)$. Now, using the method of separation of variables we can write

$$w_n(x_2,t) = X_n(x_2) T_{1n}(t),$$

where

$$T_{1n}''(t) + \lambda_n T_{1n}(t) = q_n(t).$$

Therefore, $w(x_2,t)$ can be expressed as follows

$$w(x_2,t) = \sum_{n=1}^{\infty} \frac{1}{\sqrt{\lambda_n}} X_n \int_0^t \sin(\sqrt{\lambda_n}(t-\tau)) q_n(\tau) d\tau. \tag{19}$$

Similarly to Theorems 2.5 and 2.6, if the following conditions are fulfilled

$$\tau(x_2,t) := \frac{1}{\sqrt{g(x_2)}} \left(D(x_2) \left(\frac{q(x_2,t)}{g(x_2)} \right)_{,x_2x_2} \right)_{,x_2x_2} \in C[0,l],$$

$$\text{and } \frac{\tau(x_2,t)}{\sqrt{g(x_2)}} \text{ satisfies the BCs given in Problem 11} \tag{20}$$

(For this, e.g., it is sufficient that $\frac{\partial^j}{\partial x_2^j}q(x_2,t)=O(x_2^{\gamma_j})$ $x_2\to 0_+$, $\gamma_j>7-j-\frac{2\alpha}{3}$, $\frac{\partial^j}{\partial x_2^j}q(x_2,t)=O((l-x_2)^{\delta_j})$ $x_2\to l_-$, $\gamma_j>7-j-\frac{2\beta}{3}$, $j=\overline{0,8}$) we get the absolute and uniform convergence of the series (19) and

$$\frac{\partial^i}{\partial x_2^i}(D(x_2)w_{,x_2x_2}(x_2,t))=\sum_{n=1}^{\infty}\frac{d^i}{dx_2^i}(D(x_2)X_n'')T_{1n}(t),\quad i=0,1,$$

on $[0,l]$, and absolute and uniform convergence of

$$\frac{\partial^i}{\partial x_2^i}w_{x_2}(x_2,t)=\sum_{n=1}^{\infty}\frac{d^i}{dx_2^i}X_n(x_2)T_{1n}(t),\quad i=1,...,4,$$

$$\frac{\partial^i}{\partial t^i}w(x_2,t)=\sum_{n=1}^{\infty}X_n(x_2)\frac{d^i}{dt^i}T_{1n}(t),\quad i=1,2,$$

on any $[a,b]\subset]0,l[$.

Now, let $q(x_2,t)\not\equiv 0$, $\varphi_i(x_2)\not\equiv 0$. If conditions (20), (17), and (18) are satisfied then the solution of Problem 11 can be expressed as follows

$$w(x_2,t)=\sum_{n=1}^{\infty}w_n(x_2,t),$$

where

$$w_n(x_2,t)=X_n(x_2)(T_{1n}(t)+T_n(t)),$$

$w^1(x_2,t):=X_nT_{1n}(t)$ is given by the formula (19) and $w^2(x_2,t):=X_nT_n(t)$ is given by the formula (15).

Remark 1 *Similarly are solved IBVPs corresponding to the Problems 12-20.*

We can avoid the restrictions (20) if we consider harmonic vibration. In this case

$$w(x_2,t)=e^{i\omega t}w_0(x_2),\quad q(x_2,t)=e^{i\omega t}q_0(x_2),$$

where $\omega=const$ is an oscillation frequency, $q_0(x_2)\in C([0,l])$ is a given function. E.g., in the case of Problem 11, for $w_0(x_2)$ we get the following problem

$$\begin{aligned}(D(x_2)w_0''(x_2))'' &= q_0(x_2)+2\omega^2\rho h(x_2)w_0(x_2),\\ w_0(0)=M_2(0) &= w'(l)=Q_2(l)=0,\quad 0\le\alpha<2,\ 0\le\beta<1,\\ w_0(x_2) &\in C^4(]0,l[)\cap C([0,l])\cap C^1(]0,l]).\end{aligned}\tag{21}$$

This problem is equivalent to the integral equation

$$w_0(x_2)-\omega^2\int_0^l K(x_2,\xi)\,g(\xi)\,w_0(\xi)d\xi=F(x_2),\tag{22}$$

where

$$F(x_2) := \int\limits_0^l K(x_2, \xi)\, q_0(\xi) d\xi,$$

$K(x_2, \xi)$ has the same form as in integral equation (14).

If $\omega^2 \neq \lambda_n$, the unique solution of (22) can be written as follows (see, e.g., [66])

$$\begin{aligned} w_1(x_2) &= F(x_2)\sqrt{g(x_2)} \\ &+ \omega^2 \sum_{n=1}^{\infty} \left[\frac{1}{\lambda_n - \omega^2} \int\limits_0^l F(\xi)\, \sqrt{g(x_2)}\, Y_n(\xi) d\xi \right] Y_n(x_2), \end{aligned} \tag{23}$$

It is shown that the series in the right hand side of (23) is absolutely and uniformly convergent on $[0, l]$, because of $q_0 \in C([0, l])$.

Using the difference equation corresponding to (21), by means of MATLAB we get numerical and graphical results for harmonic vibration problems.

Chapter 3 is dedicated to the interface problem of the interaction of a plate with two cusped edges and a flow of a fluid.

We assume that the flow is independent of x_1, parallel to the plane $0x_2x_3$, i.e. $v_1 \equiv 0$, and generates a bending of the plate. Let at infinity, for pressure we have

$$p(x_2, x_3, t) \to p_\infty(t), \quad \text{when} \quad |x| \to \infty, \tag{24}$$

and let for the velocity components conditions at infinity be either

$$v_2(x_2, x_3, t) = O(1), \quad v_3(x_2, x_3, t) \to v_{3\infty}(t), \tag{25}$$

or

$$v_j(x_2, x_3, t) = O(1), \quad j = 2, 3, \tag{26}$$

where $v := (v_2, v_3)$ is a velocity vector of the fluid, $p(x_2, x_3, t)$ is a pressure, and $v_{3\infty}(t)$, $p_\infty(t)$ are given functions.

Let us introduce the following notations

$$\begin{aligned} &I := \{[0, l] \times 0\}, \\ &\Omega^f := \left\{x_1, x_2, x_3 : x_1 = 0,\ x := (x_2, x_3) \in \mathbb{R}^2 \backslash I\right\}. \end{aligned}$$

If the middle plane of the plate lies in the plane $0x_1x_2$ and the flow of moving fluid involves bending of the plate then transmission conditions could have the form:

$$\sigma^f_{N3}\left(x_1, x_2, \overset{(+)}{h}(x_1, x_2), t\right) - \sigma^f_{N3}\left(x_1, x_2, \overset{(-)}{h}(x_1, x_2), t\right) = q(x_1, x_2, t), \tag{27}$$

$$v_3\Bigg(x_1 - \overset{(+)}{h}(x_1, x_2) w_{,1}(x_1, x_2, t), x_2 - \overset{(+)}{h}(x_1, x_2) w_{,2}(x_1, x_2, t), \overset{(+)}{h}(x_1, x_2)$$

$$
\begin{aligned}
&+w(x_1,x_2,t),t\Big) = v_3\Big(x_1 - \overset{(-)}{h}(x_1,x_2)w_{,1}(x_1,x_2,t), x_2 \\
&-\overset{(-)}{h}(x_1,x_2)w_{,2}(x_1,x_2,t), \overset{(-)}{h}(x_1,x_2)+w(x_1,x_2,t),t\Big) = \frac{\partial w(x_1,x_2,t)}{\partial t},
\end{aligned} \tag{28}
$$

(the first of the last pair of equalities is valid since deflection of plate w is independent of x_3).

After corresponding analysis we arrive at the conclusion that, for the normal component of the velocity vector and the pressure in the case of an ideal fluid, we have the following transmission conditions (compare with [65], [99], [83])

$$
v_3(x_2,0,t) = \frac{\partial w(x_2,t)}{\partial t}, \quad x_2 \in]0,l[, \quad t \geq 0. \tag{29}
$$

$$
\begin{aligned}
&- \quad p(x_2, \overset{(-)}{h}(x_2),t)\cos(\overrightarrow{n}(x_2, \overset{(-)}{h}(x_2)),x_3) \\
&- \quad p(x_2, \overset{(+)}{h}(x_2),t)\cos(\overrightarrow{n}(x_2, \overset{(+)}{h}(x_2)),x_3) = q(x_2,t), \quad x_2 \in]0,l[.
\end{aligned} \tag{30}
$$

In the case of a viscous fluid we add to (29) the transmission condition for the tangential component of the velocity vector

$$
v_2(x_2,0,t) = 0, \quad x_2 \in]0,l[, \quad t \geq 0. \tag{31}
$$

In Section 3.1 the solution of the interaction problem in the case of an ideal fluid is given [2].

For the potential motion of the flow there exists a complex function $\Phi = -\psi + i\varphi$ such that

$$
\begin{aligned}
&\frac{\partial \varphi(x_2,x_3,t)}{\partial x_2} = \frac{\partial \psi(x_2,x_3,t)}{\partial x_3} = v_2(x_2,x_3,t), \\
&\frac{\partial \varphi(x_2,x_3,t)}{\partial x_3} = -\frac{\partial \psi(x_2,x_3,t)}{\partial x_2} = v_3(x_2,x_3,t).
\end{aligned} \tag{32}
$$

The pressure is given by the formula

$$
p(x_2,x_3,t) = \rho^f\left[\frac{v_\infty^2}{2} + \frac{p_\infty}{\rho^f} + \frac{\partial \varphi_\infty}{\partial t} - \frac{\partial \varphi}{\partial t} - \frac{1}{2}(v_2^2+v_3^2)\right]. \tag{33}
$$

We calculate $w(x_2,t)$ from the equation (10).

Problem 21 *Find a function $w(\cdot,t) \in C^4(]0,l[)$ (and additional smoothness conditions indicated in Problems 11-20), also the functions $v_2(x_2,x_3,t) \in C^2(\Omega^f) \cup C^1(t > 0)$, $v_3(x_2,x_3,t) \in C^2(\Omega^f) \cup C^1(t > 0)$ and $p(x_2,x_3,t) \in C(\Omega^f) \cup C(t > 0)$ which satisfy the system of equations (10), (32), (33), transmission conditions (29), (30), conditions at infinity (24), (25) and one of the BCs given in Problems 11-20.*

For $\Phi_{,2}(x_2, x_3, t) = v_3 + iv_2$, in view of (25) and (29), we get the following expression [72]

$$\Phi_{,2} = -\frac{1}{\pi i\sqrt{(x_2+ix_3)(x_2+ix_3-l)}}\int_0^l \frac{\sqrt{(\xi_2+ix_3)(\xi_2+ix_3-l)}}{(\xi_2-x_2)-ix_3} w_{,t}(\xi_2,t)d\xi_2$$
$$+v_{3\infty}\frac{x_2+ix_3-l/2}{\sqrt{(x_2+ix_3)(x_2+ix_3-l)}}. \tag{34}$$

Let

$$w(x_2,t) = e^{i\omega t}w_0(x_2),\ \ q(x_2,t) = e^{i\omega t}q_0(x_2),$$
$$p(x_2,x_3,t) = e^{i\omega t}p_0(x_2,x_3),$$

$$u_2(x_2,x_3,t) = e^{i\omega t}u_2^0(x_2,x_3),\ \ u_3(x_2,x_3,t) = e^{i\omega t}u_3^0(x_2,x_3),$$
$$\varphi(x_2,x_3,t) = ie^{i\omega t}\varphi_0(x_2,x_3),\ \ \psi(x_2,x_3,t) = ie^{i\omega t}\psi_0(x_2,x_3),$$

$$v_2(x_2,x_3,t) = ie^{i\omega t}v_2^0(x_2,x_3),\ \ v_3(x_2,x_3,t) = ie^{i\omega t}v_3^0(x_2,x_3),$$

$$p_\infty(t) = e^{i\omega t}p_\infty^0,\ \ v_{3\infty}(t) = ie^{i\omega t}v_{3\infty}^0,\ \ p_\infty^0,\ v_{3\infty}^0 = \text{const},$$

where $\omega = \text{const} > 0$ is an oscillation frequency, $v_2 = u_{2,t}$ ($v_3 = u_{3,t}$).

After separating real and imaginary parts of (34), we obtain the expressions for v_2 and v_3. By means of the latter, in view of (32), we can calculate φ and then substitute it into (33). Then substituting the obtained expression for $p(x_2, x_3, t)$ into (30), we get the expression for $q(x_2)$. Therefore, all the mechanical quantities in the fluid part and the lateral load are calculated by means of deflection. In the case of harmonic vibration for deflection we get the second order Fredholm type linear integral equation [2]

$$w_0(x_2) - \omega^2\int_0^l K_1(x_2,\xi)w_0(\xi)d\xi = f_1(x_2), \tag{35}$$

where $K_1(x_2, \xi_2) \in C([0, l] \times [0, l])$ and $f_1(x_2) \in C([0, l])$ are defined explicitly. They depend on the coefficients of the equation (21), on the type of the boundary conditions in Problems 11-20, and the conditions at infinity (24)-(25).

The following proposition is valid

Proposition 3.2 *Problem of the harmonic vibration corresponding to the Problem 17 has a unique solution when*

$$\omega^2 < \frac{1}{Ml},$$

where

$$M := \max_{x_2,\xi\in[0,l]}\{|K_1(x_2,\xi)|\}.$$

Remark 2 *If the plate thickness is sufficiently small, we can assume that:*
1. the fluid occupies $\mathbb{R}^2 \backslash I$*;*
2. the plate occupies I *(its geometry depending on the thickness is taken into account in the coefficient of the bending equation);*
3. $\overset{(\pm)}{h}$ *can be neglected. Since the normals of* I *are* $(0,0,1)$ *and* $(0,0,-1)$*, (30) can be rewritten as follows*

$$-p(x_2, 0_+, t) + p(x_2, 0_-, t) = q(x_2, t), \quad x_2 \in]0, l[.$$

In Section 3.2 an interaction problem for the case of an incompressible viscous fluid is solved [3]. We consider the case when the motion of the fluid is sufficiently slow, i.e., v_j and $v_{j,k}$ $(j,k = 2,3)$ are so small that linearized Navier-Stokes equations can be applied

$$\begin{aligned} \frac{\partial v_2}{\partial t} &= -\frac{1}{\rho^f}\frac{\partial p}{\partial x_2} + \nu \Delta v_2, \\ \frac{\partial v_3}{\partial t} &= -\frac{1}{\rho^f}\frac{\partial p}{\partial x_3} + \nu \Delta v_3, \end{aligned} \tag{36}$$

where $\nu = \mu/\rho^f$ is a coefficient of viscosity, $\Delta = \dfrac{\partial^2}{\partial x_2^2} + \dfrac{\partial^2}{\partial x_3^2}$.

We add to the (36) the following equation

$$\operatorname{div} v(x_2, x_3, t) = 0, \quad (x_2, x_3) \in \Omega^f, \quad t \geq 0.$$

Let us consider the problem of harmonic vibration.

Problem 22 *Find a function* $w_0(x_2)$ *on* I*, which satisfies the equation (21), one of the boundary conditions given in the Problems 11-20, and also find functions* $u_i^0(x_2, x_3)$*,* $p_0(x_2, x_3)$*,* $q_0(x_2)$ *on* Ω^f*, which satisfy the following system of equations*

$$\begin{aligned} &\Delta p_0(x_2, x_3) = 0, \\ &-\omega^2 u_j^0 = -\frac{1}{\rho^f}\frac{\partial p_0}{\partial x_j} + \nu i \omega \Delta u_j^0, \quad j = 2, 3, \end{aligned}$$

smoothness conditions

$$\begin{aligned} &u_i^0 \in C^2(\Omega^f) \cap C(\mathbb{R}^2) \cap C(t > 0), \quad i = 2, 3; \\ &p_0 \in C^2(\Omega^f); \\ &q_{0,2}(\cdot, t) \in C^\gamma([0, l]), \quad 0 < \gamma \leq 1, \end{aligned}$$

following conditions at infinity

$$p_0\big|_{|x| \to \infty} = O(1), \quad u_j^0\big|_{|x| \to \infty} = O(1), \quad j = 2, 3,$$

and transmissions conditions as follows

$$-p_0(x_2, 0_+) + p_0(x_2, 0_-) = q_0(x_2), \quad x_2 \in]0, l[,$$
$$u_3^0(x_2, 0) = w_0(x_2), \quad u_2^0 = 0, \quad x_2 \in]0, l[,$$

where $q(x_2, t) = e^{i\omega t} q_0(x_2)$.

In this case all the mechanical quantities in question are calculated by means of the lateral load q_0, for which we obtain a second order supersingular integral equation

$$\int_0^l \frac{q_0(\xi_2)}{(\xi_2 - x_2)^2} d\xi_2 + 2\pi\omega^2\rho^f \int_0^l K_2(x_2, \xi_2) q_0(\xi_2) d\xi_2 = f_2(x_2),$$

where the supersingular integral is defined in H'adamard's finite part sense, $K_2(x_2, \xi_2) \in C([0, l] \times [0, l])$ and $f_2(x_2) \in C([0, l])$ are quite definite functions. If $q_{0,2}(x_2) \in C^\gamma([0, l])$ $0 < \gamma \leq 1$, this equation is solved by a method developed by Boikov, I.V., Dobrynin, N.F., Domnin, L. in [9] (see also Chapter 1, section 1.3).

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