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Research of a Three-Dimensional Dynamic System Describing the Process of Assimilation

Temur Chilachava[a*] and George Pochkhua [a]

[a]*Sokhumi State University, 61 Politkovskaya Str., 0186 Tbilisi, Georgia*

The work proposes a nonlinear mathematical model describing the process of two-level assimilation, taking into account the quadratic terms of self-limiting the growth of populations of three sides talking in languages significantly different in dissemination.

Two-level assimilation process considered when in one large region the population speaking the most common language assimilates both the population speaking the second fairly common language and the population speaking the third less common language (small range of language dissemination). In turn, the population speaking the second language, which is quite common, assimilates the population speaking the less common third language. Thus, the population speaking the third less common language is in a situation of bilateral assimilation.

The work assumes that the process of assimilation develops due to numerous direct or remote (electronic communication) mutual meetings between representatives of the population, who consider one of these three languages to be their native language.

In the special case, when the coefficients of the mathematical model are constant, the first integral of a nonlinear three-dimensional dynamic system is found, which in the phase space of solutions is a hyperbolic paraboloid. Using the first integral, the three-dimensional dynamic system is reduced to two-dimensional and, taking into account the Bendixon's criterion, the theorem about the existence of closed integral trajectory solutions in the first quarter of the phase plane has been proved. Thus, conditions were found for model coefficients at which there is no complete assimilation of a third side speaking a less common language.

Keywords: Mathematical model, Two-level assimilation, First integral, Hyperbolic paraboloid, Bendixon's criterion.

AMS Subject Classification: 93A30, 00A71, 97M10, 97M70.

1. Introduction

In science, as well as in life it is important to understand it. It is always simple but difficult to achieve.

Mathematical modeling was formatted like art, based on complex calculations and analyses mostly using well-known calculation methods. But very soon, due to the computer technology development, mathematical physics, oscillation theory, control theory and other issues, mathematical modeling become a comprehensive science, that studies mathematical models independently of its specific content.

Without exaggeration, it can be said, that modern technology (aviation, ground transport, missiles and computers, nuclear power and machine building, tool making and communications), that defines our way of life, is based on the fundamental sciences and is a creation of mathematical modeling.

*Corresponding author. Email: temo_chilachava@yahoo.com

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The goals and objectives of mathematical modeling today are very broad and multifaceted, which can be summarized as qualitative and quantitative study of what is around us: every object of nature, technique and society. At the same time qualitative study means achieving an understanding of the essence of the object to be studied, its properties, behaviours, events and their defining reasons.

The general scheme of mathematical modeling is mostly established, and its realization is based on fundamental science, methods of studying mathematical models (analytical, qualitative and numerical), modern computational techniques.

Mathematical modeling of physical processes involves the model adequacy, which is validated by Newton's non-relative five laws of classical mechanics: mass conservation law; law of conservation of impulse; the law of conservation the momentum of impulse; the first law of thermodynamics, i.e. energy conservation law; the second law of thermodynamics, i.e. entropy conservation law [1–6].

Creation of mathematical models is more original in social sphere, because, they are more difficult to substantiate.

We created a new direction of mathematical modeling, i.e. "Mathematical Modeling of Information Warfare" [7–9]. In these models two antagonistic sides waging with each other information warfare and also the third peacekeeping side trying to extinguish information warfare reconsidered. Conditions on model parameters at which the third side will be able to force the conflicted sides to completion of information warfare are found.

We proposed to create new nonlinear mathematical models of economic cooperation between two politically (not military opposition) mutually warring sides (two countries or a country and its legal region) which consider economic or other type of cooperation between different parts of population aimed to the peaceful resolution of conflicts [10–13]. Taking into consideration the important tendencies in the world, it is important to study demographic and assimilation social processes through mathematical modeling.

It is known that in the world the social process of assimilation of languages (people) is hidden which, as a rule, considers expansion of an area of the dominating languages (state languages of economically powerful states) at the expense of less widespread languages proceeds (state languages economically of rather weak states).

In many cases, nations (languages) are in a state of unilateral, bilateral or tripartite assimilation, when more numerous and economically stronger nations (languages) try to assimilate them by various means. Therefore, it is important to consider and study new mathematical models of one, two or three-level assimilation of nations, which are described by nonlinear two-dimensional, three-dimensional or four-dimensional dynamic systems. With the help of mathematical models, it is possible to determine the possibility of coexistence of very different languages in terms of distribution territory and quantity in the long term. In short, to answer the question, small nations (states) will survive, or the dominant language (English) will lead them to complete linguistic globalization.

In [14], we consider a nonlinear continuous mathematical model of linguistic globalization. Two categories of the world's population are considered: a category that hinders and a category conducive to the dominant position of the English language. With a positive demographic factor of the population, which prevents globalization and the negative demographic factor of the population contributing to globalization, it is shown that the dynamic system describing this process allows for the existence of two topologically not equivalent phase portraits (a stable node, a limit cycle). Under certain restrictions on the parameters of the model, the theorem on the absence of periodic trajectories of the dynamical system is proved and an asymptotically stable equilibrium position (limit cycle) is found. Thus, it is established that complete linguistic globalization is impossible if the demographic factor of the category of the world population contributing to the dominance of the English language is non-positive. Full linguistic globalization is possible only if the demographic factor is positive for the category of the world's population, which contributes to the dominance of the English language and a certain restriction on the parameters of the model associated with the coefficient of assimilation.

In [15] a nonlinear mathematical model of process of three level assimilation which is described by four-dimensional dynamic systems is offered. In case of constancy of coefficients special points of the dynamic system are found. The conditions on constant coefficients for which it is possible to find special points with all four coordinates non-negative are determined. Introducing some dependence among coefficients of the system, two first integrals are derived, and the four-dimensional system is reduced to a two-dimensional one. The sign-variable divergence theorem of a two-dimensional vector field in some one-coherent area of the first quadrant of the phase plane is proved. According to Bendixon's criterion it is possible to have a closed integral curve completely lying in this area.

From this point of view, today for less widespread languages, but from a historical retrospective, in certain cases, of protection of world treasures (classic languages), stay, within mathematical model, those conditions under which there will be no disappearance of the major languages is important, i.e. there will be no full assimilation of people talking in these languages.

2. General mathematical model of two-level assimilation. System of the equations

Consider the social process of two-level assimilation, when in one large region the population speaking the most common language assimilates both the population speaking the second fairly common language and the population speaking the third less common language (small range of language distribution). In turn, the population speaking the second language, which is quite common, assimilates the population speaking the less common third language. Thus, the population speaking the third less common language is in a situation of bilateral assimilation.

We assume that the process of assimilation develops due to numerous direct or remote (electronic communication) mutual meetings between representatives of the population, who consider one of these three languages to be their native language.

The social process of two-level assimilation, which takes into account the quadratic terms of self-limiting population growth, is described by the following nonlinear dynamic system

$$\begin{cases} \frac{du}{dt} = \alpha_1(t)\, u - \delta_1(t)\, u^2 + \beta_1(t)\, uv + \beta_2(t)\, uw, \\ \frac{dv}{dt} = \alpha_2(t)\, v - \delta_2(t)\, v^2 - \beta_3(t)\, uv + \beta_4(t)\, vw, \\ \frac{dw}{dt} = \alpha_3(t)\, w - \delta_3(t)\, w^2 - \beta_5(t)\, uw - \beta_6(t)\, vw, \end{cases} \tag{1.1}$$

with the initial conditions

$$u\,(0) = u_0,\ \ v\,(0) = v_0,\ \ w\,(0) = w_0, \tag{1.2}$$

$$u,\ v,\ w\ \in C^1\,[0, T]\,,$$

$$\alpha_i\,(t)\,, \beta_j\,(t)\,, \delta_i\,(t) \in C[0, T],\ \ i = \overline{1, 3},\ \ j = \overline{1, 6},$$

where

$[0, T]$ - the period of consideration of this model (for various cases, the period can reach several decades),

$u(t)$ - at a given time t the number of people living in the same region (possibly the continent) who consider their native language in this region to be the most common language (dominant language, English),

$v(t)$ - at a given time t the number of people living in the same region who consider their native language to be another common language in this region,

$w(t)$ - at a given time t the number of people living in a small part (in a small area) of the same region who only in this part of the region consider the common language to be their native language,

$\beta_1(t), \beta_3(t)$ - assimilation rates of the population speaking the second sufficiently spoken language by the population speaking the most spoken language,

$\beta_2(t), \beta_5(t)$ - assimilation rates of the population speaking a third less spoken language by the population speaking the most spoken language,

$\beta_4(t), \beta_6(t)$ - assimilation rates of a population speaking a third less spoken language by a population speaking a second sufficiently spoken language,

$\alpha_1(t), \alpha_2(t), \alpha_3(t)$ - natural change rates of populations speaking the first, second and third languages respectively (variable demographic factors),

$\delta_1(t), \delta_2(t), \delta_3(t)$ - self-limiting factors of population growth speaking the first, second and third languages respectively.

Scenario of development of two-level assimilation process is given in figure 1.

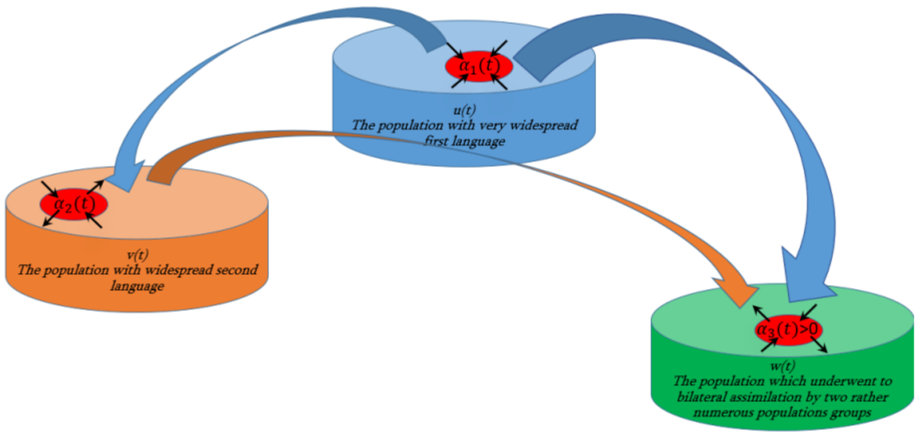

Figure 1.

It is natural to assume that assimilation coefficients and growth self-constraints are positive continuous functions at the time of model consideration

$$\beta_i\left(t\right) > 0,\ i = \overline{1,6},\ t \in \left[0,T\right], \tag{1.3}$$

$$\delta_j(t) > 0,\ j = \overline{1,3}.$$

The non-triviality of the two-level assimilation process (when the assimilation result is not initially predicted) leads to the inequality

$$\alpha_3(t) > 0,\ t \in \left[0,T\right]. \tag{1.4}$$

3. The first integral of a nonlinear system of differential equations. Second order surfaces in phase space

Consider the particular case where all model coefficients are constant

$$\beta_i\left(t\right) = \beta_i = const > 0,\ i = \overline{1,6},\ t \in [0,T],$$

$$\alpha_1\left(t\right) = \alpha_1 = const,\ \alpha_2\left(t\right) = \alpha_2 = const, \alpha_3\left(t\right) = \alpha_3 = const > 0, \tag{2.1}$$

$$\delta_1(t) = \delta_1 = const > 0,\ \delta_2(t) = \delta_2 = const > 0,\ \delta_3(t) = \delta_3 = const > 0.$$

Taking into account (2.1) the system of equations (1.1) becomes

$$\begin{cases} \dfrac{du}{dt} = \alpha_1 u - \delta_1 u^2 + \beta_1 uv + \beta_2 uw, \\ \dfrac{dv}{dt} = \alpha_2 v - \delta_2 v^2 - \beta_3 uv + \beta_4 vw, \\ \dfrac{dw}{dt} = \alpha_3 w - \delta_3 w^2 - \beta_5 uw - \beta_6 vw. \end{cases} \tag{2.2}$$

Find the first integral of the system of nonlinear differential equations (2.2) to lower the order of the system, i.e. from a three-dimensional system go to a two-dimensional one.

The dynamic system (2.2) will be written in the following form

$$\begin{cases} \dfrac{1}{u}\dfrac{du}{dt} = \alpha_1 - \delta_1 u + \beta_1 v + \beta_2 w, \\ \dfrac{1}{v}\dfrac{dv}{dt} = \alpha_2 - \delta_2 v - \beta_3 u + \beta_4 w, \\ \dfrac{1}{w}\dfrac{dw}{dt} = \alpha_3 - \delta_3 w - \beta_5 u - \beta_6 v. \end{cases} \tag{2.3}$$

Multiply the second equation of the system (2.3) by (-1) and get the sum of all three equations

$$\frac{1}{u}\frac{du}{dt}-\frac{1}{v}\frac{dv}{dt}+\frac{1}{w}\frac{dw}{dt} = (\alpha_1-\alpha_2+\alpha_3)-(\delta_1+\beta_5-\beta_3)u+(\beta_1-\beta_6+\delta_2)v+(\beta_2-\delta_3-\beta_4)w. \tag{2.4}$$

Demand the following conditions (system) for model (2.2) constants

$$\begin{cases} \alpha_1-\alpha_2+\alpha_3=0, \\ \delta_1+\beta_5-\beta_3=0, \\ \beta_1-\beta_6+\delta_2=0, \\ \beta_2-\delta_3-\beta_4=0, \end{cases} \tag{2.5}$$

performance that is the equivalent system

$$\begin{cases} \alpha_3=\alpha_2-\alpha_1>0, \\ \delta_1=\beta_3-\beta_5>0, \\ \delta_2=\beta_6-\beta_1>0, \\ \delta_3=\beta_2-\beta_4>0. \end{cases} \tag{2.6}$$

It should be noted that (2.6) the system is not contradictory and it must satisfy (2.1) inequalities. Given (2.5), the first integral of the dynamic system (2.2) is obtained from (2.4) and (1.2)

$$uw=p_1v,\ p_1=\frac{u_0w_0}{v_0}. \tag{2.7}$$

The first integral (2.7) in the three-dimensional $(O,u(t),v(t),w(t))$ phase space is a hyperbolic paraboloid.

Subject to (2.7), a two-dimensional nonlinear dynamic system is obtained from (2.2) a three-dimensional dynamic system

$$\begin{cases} \dfrac{du}{dt}=\alpha_1u-\delta_1u^2+\beta_1uv+\beta_2p_1v, \\ \dfrac{dv}{dt}=\alpha_2v-\delta_2v^2-\beta_3uv+\beta_4\dfrac{p_1v^2}{u}, \end{cases} \tag{2.8}$$

which meets the initial conditions

$$u(0)=u_0, v(0)=v_0.$$

Now find the non-zero special points of the system (2.8). Accordingly, the condition must be fulfilled

$$\begin{cases} \alpha_1 u - \delta_1 u^2 + \beta_1 uv + \beta_2 p_1 v = 0, \\ \alpha_2 v - \delta_2 uv - \beta_3 u^2 + \beta_4 p_1 v = 0. \end{cases} \tag{2.9}$$

Consider a particular case

$$\begin{cases} \alpha_1 + \alpha_2 = 0, \\ \beta_1 = 2\delta_2. \end{cases} \tag{2.10}$$

Then from (2.6), (2.10) we get

$$\alpha_3 = 2\alpha_2 > 0,\ \alpha_1 = -\alpha_2 < 0,\ sign\alpha_3 = sign\alpha_2 = -sign\alpha_1 \tag{2.11}$$

and from (2.9)–(2.11) we get the system

$$\begin{cases} v = \frac{2\delta_1 + \beta_3}{2p_1\beta_4} u^2 = ku^2, \\ k\beta_1 u^2 - (\delta_1 - \beta_2 p_1 k)u + \alpha_1 = 0, \\ k\delta_2 u^2 + (\beta_3 - \beta_4 p_1 k)u + \alpha_1 = 0, \end{cases} \tag{2.12}$$

whose positive solution will be

$$D = (\delta_1 - \beta_2 p_1 k)^2 - 4\alpha_1 k\beta_1 > 0,\ u_* = \frac{(\delta_1 - \beta_2 p_1 k) + \sqrt{D}}{2\beta_1 k} > 0,$$

$$D_1 = (\beta_4 p_1 k - \beta_3)^2 - 2\alpha_1 k\beta_1 > 0,\ u_{**} = \frac{(\beta_4 p_1 k - \beta_3) + \sqrt{D_1}}{\beta_1 k},$$

$$u_{**} = u_*, \tag{2.13}$$

$$\frac{(\delta_1 - \beta_2 p_1 k)\sqrt{(\delta_1 - \beta_2 p_1 k)^2 - 4\alpha_1 k\beta_1}}{2\beta_1 k}$$

$$= \frac{(\beta_4 p_1 k - \beta_3) + \sqrt{(\beta_4 p_1 k - \beta_3)^2 - 2\alpha_1 k\beta_1}}{\beta_1 k},$$

$$v_* = ku_*^2.$$

From (2.13) to (2.11), the non-zero coordinates of the special point $M_*(u_*, v_*)$ in the first quarter of the phase plane $(O, u(t), v(t))$ are written as follows

$$u_* = \frac{(\delta_1 - \beta_2 p_1 k) + \sqrt{D}}{2\beta_1 k}, \; v_* = k u_*^2, \; k \equiv \frac{2\delta_1 + \beta_3}{2p_1\beta_4}. \tag{2.14}$$

Introduce the notes

$$\begin{cases} F_1(u,v) = \alpha_1 u - \delta_1 u^2 + \beta_1 uv + \beta_2 p_1 v, \\ F_2(u,v) = \alpha_2 v - \delta_2 v^2 - \beta_3 uv + \beta_4 \frac{p_1 v^2}{u}. \end{cases} \tag{2.15}$$

Then (2.8) the system of equations is overwritten in the vector form

$$\frac{d\overrightarrow{\Lambda}}{dt} = \overrightarrow{F}, \; \overrightarrow{F}(F_1, F_2), \; \overrightarrow{\Lambda}(u,v), \tag{2.16}$$

initial conditions

$$u(0) = u_0, \; v(0) = v_0.$$

Theorem. *The task (2.16) in some one-coherent area $D \subset (O, u(t), v(t))$ the first quarter of the phase plane $(O, u(t), v(t))$ has the solution in the form of the closed trajectory which completely lies in this area.*

Proof: From (2.15), taking into account (2.16), you can get

$$\frac{\partial F_1(u,v)}{\partial u} = \alpha_1 - 2\delta_1 u + \beta_1 v, \; \frac{\partial F_2(u,v)}{\partial v} = \alpha_2 - 2\delta_2 v - \beta_3 u + 2p_1\beta_4 \frac{v}{u}. \tag{2.17}$$

Calculate the divergence of the vector field $\overrightarrow{F}(F_1, F_2)$ according to (2.17)

$$div\overrightarrow{F} = \frac{\partial F_1(u,v)}{\partial u} + \frac{\partial F_2(u,v)}{\partial v} = \alpha_1 - 2\delta_1 u + \beta_1 v + \alpha_2 - 2\delta_2 v - \beta_3 u + 2p_1\beta_4 \frac{v}{u}$$

or

$$div\overrightarrow{F} = (\alpha_1 + \alpha_2) - (2\delta_1 + \beta_3)\, u + (\beta_1 - 2\delta_2)\, v + 2p_1\beta_4 \frac{v}{u}. \tag{2.18}$$

Given (2.10), (2.18) will take the form

$$div\overrightarrow{F} \equiv G_1(u,v) = -\,(2\delta_1 + \beta_3)\, u + 2p_1\beta_4 \frac{v}{u}. \tag{2.19}$$

Consider a curve on the phase plane $(O, u(t), v(t))$ where the vector field divergence becomes equal to zero.

According to (2.19) this curve will be a parabola

$$G_1(u,v) = 0, \; v = ku^2. \tag{2.20}$$

Thus, the vector field divergence in the first quarter with physical content is equal to zero (2.20) on the right-hand side of the parabola, by dropping the origin of the coordinates. In case of fulfillment of (2.13) a special point $M_*(u_*, v_*)$ is also placed on this parabola.

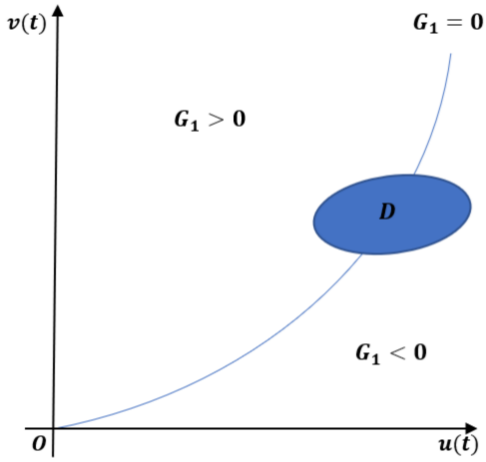

Figure 2.

It is clear, that $G_1(u, v)$ divergence (2.19) of the vector field $\overrightarrow{F}(F_1, F_2)$, in some one-coherent area $D \subset (O, u(t), v(t))$, making physical sense to the first quarter of the phase plane $(O, u(t), v(t))$ changes its sign, which contains part of the right branch of the parabola with zero divergence (Figure 2). According to the Bendixon criterion, there is a closed integral trajectory that lies entirely in this area [16, 17]. □

4. Conclusion

Thus according to (2.19), (2.20) in the phase plane (O, u, v) there exists a one-coherent area where the divergence of the vector field $G_1(u, v)$ changes its sign and, according to the Bendixon criterion, in this area there exists a closed integral curve, where $(u(t) \neq 0, v(t) \neq 0)$.

In this case, according to (2.7), the values of the function $w(t)$ do not vanish anywhere, which indicates that under these conditions there is no complete assimilation of the third side.

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