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Geometry of the Limiting Solution of a Strongly Competing System

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We report on known results on the geometry of the limiting solutions of a reaction-diffusion system in any number of competing species k as the competition rate μ tends to infinity. The case $k = 8$ is studied in detail. We provide numerical simulations of solutions of system for $k = 4, 6, 8$ and large competition rate. Thanks to FreeFEM++ software, we obtain nodal partitions showing the predicted limiting configurations.

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1. Introduction

A model for the description of population dynamics when many species interact in a highly competitive way in a bounded domain is given by the system of k differential equations (see [3] and the references therein)

$$\begin{cases} -\Delta u_i(x) = -\mu u_i(x) \displaystyle\sum_{\substack{j=1 \\ j\neq i}}^{k} u_j(x) & \text{in } D\,, \\ u_i(x) \geq 0 & \text{in } D\,, \\ u_i(x) = \phi_i(x) & \text{on } \partial D\,. \end{cases} \qquad i = 1, ..., k\,. \tag{1}$$

Here $D \subseteq \mathbf{R}^2$ is an open bounded, simply connected domain with a smooth boundary ∂D. The boundary datum $\Phi = (\phi_1, ..., \phi_k)$ satisfies the following assumptions:

- $\phi_i \in W^{1,\infty}(\partial D)$, $\phi_i \geq 0$, $i = 1, ..., k$;
- $\phi_i \cdot \phi_j = 0$ a.e. in ∂D for $i \neq j$;
- the sets $\{\phi_i > 0\}$ are nonempty, open connected arcs and the function $\sum_{i=1}^{k} \phi_i$ vanishes exactly in k points of ∂D.

Such boundary datum $\Phi = (\phi_1, ..., \phi_k)$ will be called *admissible*.

System (1) governs the steady state of k competing species, coexisting in the same area. Each component u_i expresses the population density of the i-th species; the real parameter $\mu > 0$ represents the interaction between two different species.

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The existence of solutions $U^{(\mu)} = (u_{1,\mu}, ..., u_{k,\mu})$ of system (1) for any positive μ in the class

$$\mathcal{U} = \{U = (u_1, ..., u_k) \in (H^1(D))^k : u_i = \phi_i \text{ on } \partial D, \quad u_i \geq 0 \text{ in } D\}.$$

is proved in [3]. The uniqueness is proved in [9].

Large interaction induces the spatial segregation of the species in the limiting configuration as $\mu \to \infty$. Indeed, in [3] it is proved that there exists $U = (u_1, ..., u_k) \in \mathcal{U}$ such that, up to subsequences, $u_{i,\mu} \longrightarrow u_i$ in $H^1(D)$, $i = 1, ..., k$, and the limiting configurations belong to the class

$$\mathcal{S} = \left\{ \begin{array}{ll} U = (u_1, ..., u_k) \in \mathcal{U} : & u_i \cdot u_j = 0 \text{ for } i \neq j \text{ a.e. in } D \\ & -\Delta u_i \leq 0 \text{ in } D \\ & -\Delta\big(u_i - \sum_{j \neq i} u_j\big) \geq 0 \text{ in } D \end{array} \right\}.$$

Moreover, the class $\mathcal{S}$ consists of one element, the limiting configuration (see [3] for $k = 2$, [5] for $k = 3$ and [9] in the case of arbitrary space dimension and arbitrary number of species; see also [1] for a different proof). Regularity properties of $U \in \mathcal{S}$ were achieved in [2], [3], [4], [5] and [8]. Hence, the study of $\mathcal{S}$ helps to understand the segregated states of k species, induced by strong competition.

The description of the qualitative properties of the limiting configurations in the planar case was considered in [3] for $k = 2$, in [5] for $k = 3$, in [6] for $k = 4$ and in [7] for any number of species. In particular, in the case of even k, in [7] connections between the limiting configuration and the solution of a Dirichlet problem for the Laplace equation are studied.

In this paper we report on known results on the geometry of the limiting configurations in any number of species and, in particular, in the case of even k. The case $k = 8$ is studied in detail (section 2). We provide numerical simulations of solutions of the system (1) for $k = 4, 6, 8$ and large μ. Thanks to FreeFEM++ software we obtain nodal partitions showing the predicted limiting configurations (section 3).

2. Main results

Suppose that D is a simply connected domain in $\mathbf{R}^2$. Due to the conformal invariance of the problem, without loss of generality we can assume that D is the ball

$$D = \{p = (p_1, p_2) \in \mathbf{R}^2 : |p| < 1\}.$$

Let $U \in \mathcal{S}$. The sets

$$\omega_i = \{p \in D : u_i(p) > 0\}, \quad i = 1, ..., k,$$

are the *nodal regions*. We define *multiplicity of a point* $p \in \overline{D}$ (with respect to U) the number

$$m(p) = \#\{i : |\omega_i \cap B_r(p)| > 0 \quad \forall r > 0\}$$

where $B_r(p) = \{q \in \mathbf{R}^2 : |q - p| < r\}$ and the *interfaces* between two densities u_i and u_j, $i \neq j$,

$$\Gamma_{ij} = \partial\omega_i \cap \partial\omega_j \cap \{p \in D : m(p) = 2\}\,.$$

Let $U \in \mathcal{S}$, we define the set of points of multiplicity greater than or equal to $h \in \mathbf{N}$

$$\mathcal{Z}_h(U) = \{p \in \overline{D} : m(p) \geq h\}.$$

The set $\mathcal{Z}_h(U)$ consists of a finite number of isolated points (cfr. [4, Theorems 9.11 and 9.13]). Moreover $\mathcal{Z}_3(U)$ is nonempty, does not contain points of multiplicity higher than k and $1 \leq \#\{\mathcal{Z}_3(U)\} \leq k-2$ (cfr. [7]).

Let $k \geq 3$ and $U \in \mathcal{S}$. We proved in [7] the following relation

$$k - 2 = \sum_{p \in \mathcal{Z}_3(U)} (m(p) - 2). \tag{2}$$

Formula (2) can be used to classify the possible limiting configurations. For example, from relation (2) we infer that

- if $k = 3$, only one limiting configuration is possible: $\mathcal{Z}_3(U)$ consists of one point with multiplicity 3;
- if $k = 4$, only two configurations are possible:
  i) $\mathcal{Z}_3(U)$ consists of one point with multiplicity 4 (example 3.1),
  ii) $\mathcal{Z}_3(U)$ consists of two points with multiplicity 3 (example 3.2);
- if $k = 5$, three configurations are possible:
  i) $\mathcal{Z}_3(U)$ consists of one point with multiplicity 5,
  ii) $\mathcal{Z}_3(U)$ consists of two points q_1, q_2 such that $m(q_1) = 4, m(q_2) = 3$,
  iii) $\mathcal{Z}_3(U)$ consists of three points $q_i, i = 1, 2, 3$, such that $m(q_1) = m(q_2) = m(q_3) = 3$;
- if $k = 6$, five possible configurations are possible. This case was studied in detail in [7]. In particular, there are two cases in which the set $\mathcal{Z}_3(U)$ contains only points with even multiplicity (see examples 3.3 and 3.4).

If $k = 2s$, $s \geq 2$, we proved in [7] that $U \in \mathcal{S}$ can be strictly connected to the solution ψ_a of the boundary value problem

$$\begin{cases} -\Delta\psi_a = 0 & \text{in } D \\ \psi_a = \displaystyle\sum_{j=1}^{2s} (-1)^j \phi_j & \text{on } \partial D\,. \end{cases} \tag{3}$$

We are interested in the critical points of the function ψ_a.

Proposition 2.1: *([7]) Let $\Phi = (\phi_1, ..., \phi_{2s})$ be an admissible boundary datum. The harmonic function ψ_a which solves* (3) *possesses at most $s - 1$ critical points p in D such that $\psi_a(p) = 0$.*

The existence of critical points for the solution of (3) is not guaranteed (see [6, Remark 3.4]). However, a critical point for ψ_a at level zero can be strictly connected to multiple points of U. This feature is shown in the next Propositions.

Proposition 2.2: *([7]) Let $\Phi = (\phi_1, ..., \phi_{2s})$ be an admissible boundary datum and suppose that the harmonic function ψ_a, solution to* (3)*, has $q_1, ..., q_{s-1}$ critical points in D such that $\psi_a(q_i) = 0$, $i = 1, ..., s-1$. Then $q_1, ..., q_{s-1}$ are 4-points for the function $U = |\psi_a| \in \mathcal{S}$.*

Proposition 2.3: *([7]) Let $\Phi = (\phi_1, ..., \phi_{2s})$ be an admissible datum. If $U \in \mathcal{S}$ possesses a $2s$-point a_U in $\overline{D}$ then $U = |\psi_a|$, where ψ_a is the solution of* (3)*. If $a_U \in D$ then a_U is a critical point for ψ_a at zero level.*

The next theorem gives necessary and sufficient conditions on the admissible datum Φ such that $U \in \mathcal{S}$ generates a $2s-$point configuration with the $2s-$point $p \in D$.

Theorem 2.4: *([7]) Let $\Phi = (\phi_1, ..., \phi_{2s})$ be an admissible boundary datum and let ψ_a be the solution to* (3)*. The point $p \in D$ is a $2s-$point for the function $U = |\psi_a|$ if and only if $\Phi^a = \sum_{j=1}^{2s}(-1)^j \phi_j$ satisfies the conditions*

$$\int_{\partial D} \Phi^a \left(\frac{\zeta + p}{\overline{p}\zeta + 1}\right) \zeta_1^{j-h} \zeta_2^h ds_\zeta = 0, \qquad h = 0, ..., j; j = 0, ..., s-1 \tag{4}$$

with $\zeta = (\zeta_1, \zeta_2)$.

If $k = 4$, conditions (4) reduce to (cfr. [6])

$$\int_{\partial D} \Phi^a \left(\frac{\zeta + p}{\overline{p}\zeta + 1}\right) ds_\zeta = \int_{\partial D} \Phi^a \left(\frac{\zeta + p}{\overline{p}\zeta + 1}\right) \zeta_r \, ds_\zeta = 0, \qquad r = 1, 2. \tag{5}$$

If $k = 6$, conditions (4) can be formulated as (cfr. [7])

$$\int_{\partial D} \Phi^a \left(\frac{\zeta + p}{\overline{p}\zeta + 1}\right) ds_\zeta = \int_{\partial D} \Phi^a \left(\frac{\zeta + p}{\overline{p}\zeta + 1}\right) \zeta_1 \zeta_2 ds_\zeta = 0, \tag{6}$$

$$\int_{\partial D} \Phi^a \left(\frac{\zeta + p}{\overline{p}\zeta + 1}\right) \zeta_r ds_\zeta = \int_{\partial D} \Phi^a \left(\frac{\zeta + p}{\overline{p}\zeta + 1}\right) \zeta_r^2 ds_\zeta = 0, \quad r = 1, 2. \tag{7}$$

For $k = 4$ and $k = 6$, conditions (5) and conditions (6)-(7) can be written in an equivalent way, as shown in the next Propositions.

Proposition 2.5: *([6]) Let $k = 4$. Let $\Phi = (\phi_1, ..., \phi_4)$ be an admissible datum and let ψ_a be the solution to* (3)*. Conditions* (5) *are equivalent to $\psi_a(p) = 0$, $\nabla\psi_a(p) = (0, 0)$, where $\nabla\psi_a = \{\partial_{x_1}\psi_a, \partial_{x_2}\psi_a\}$.*

Proposition 2.6: *([7]) Let $k = 6$. Let $\Phi = (\phi_1, ..., \phi_6)$ be an admissible datum and let ψ_a be the solution to* (3)*. Conditions* (6)*-*(7) *are equivalent to*

$$\psi_a(p) = 0, \quad \nabla\psi_a(p) = (0, 0), \quad H\psi_a(p) = \mathbf{0}$$

where $H\psi_a = \left(\partial^2_{x_i x_j}\psi_a\right)_{i,j=1,2}$ denotes the hessian matrix of the function ψ_a.

Propositions 2.5 and 2.6 can be extended to the case of 8 species.

Proposition 2.7: *Let $k = 8$. Let $\Phi = (\phi_1, ..., \phi_8)$ be an admissible datum and let ψ_a be the solution to* (3). *Conditions* (4) *i.e.*

$$\int_{\partial D} \Phi^a \left(\frac{\zeta + p}{\overline{p}\zeta + 1} \right) \zeta_1^{j-h} \zeta_2^h ds_\zeta = 0, \qquad h = 0, ..., j; j = 0, 1, 2, 3$$

are equivalent to

$$\partial_x^\alpha \psi_a(p) = 0, \quad 0 \leq |\alpha| \leq 3 \tag{8}$$

where $\alpha = (\alpha_1, \alpha_2)$ denotes a multi-index and $\partial_x^\alpha = \partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2}$.

Proof: We introduce the transformation

$$x = R_p(\zeta) = \frac{\zeta + p}{\overline{p}\zeta + 1}.$$

The conformal map R_p maps the unit disk $\overline{D}$ into itself with $R_p(\partial D) = \partial D$ and $R_p(0) = p$. Set

$$\widetilde{\psi}_a(\zeta) = \psi_a(R_p(\zeta)), \qquad \widetilde{\Phi}^a(\zeta) = \Phi^a(R_p(\zeta)).$$

The function $\widetilde{\psi}_a$ solves the Dirichlet problem

$$\begin{cases} -\Delta \widetilde{\psi}_a = 0 & \text{in } D, \\ \widetilde{\psi}_a = \widetilde{\Phi}^a & \text{on } \partial D. \end{cases} \tag{9}$$

Here we identificate the complex numbers $x = x_1 + ix_2$ and $\zeta = \zeta_1 + i\zeta_2$ with the points (x_1, x_2) and $(\zeta_1, \zeta_2) \in \mathbf{R}^2$, respectively. Direct calculation leads to

$$\psi_a(p) = \psi_a(R_p(0)) = \widetilde{\psi}_a(0);$$

$$\partial_{x_r} \psi_a(p) = (1 - |p|^2)^{-1} \partial_{\zeta_r} \widetilde{\psi}_a(0), \quad r = 1, 2;$$

$$\partial^2_{x_1 x_1} \psi_a(p) = (1 - |p|^2)^{-2} \left(\partial^2_{\zeta_1 \zeta_1} \widetilde{\psi}_a(0) + 2(p_1 \partial_{\zeta_1} \widetilde{\psi}_a(0) - p_2 \partial_{\zeta_2} \widetilde{\psi}_a(0)) \right);$$

$$\partial^2_{x_1 x_2} \psi_a(p) = (1 - |p|^2)^{-2} \left(\partial^2_{\zeta_1 \zeta_2} \widetilde{\psi}_a(0) - 2(p_2 \partial_{\zeta_1} \widetilde{\psi}_a(0) + p_1 \partial_{\zeta_2} \widetilde{\psi}_a(0)) \right);$$

$$\partial^2_{x_2 x_2} \psi_a(p) = (1 - |p|^2)^{-2} \left(\partial^2_{\zeta_2 \zeta_2} \widetilde{\psi}_a(0) + 2(p_1 \partial_{\zeta_1} \widetilde{\psi}_a(0) - p_2 \partial_{\zeta_2} \widetilde{\psi}_a(0)) \right);$$

$$\partial^3_{x_1x_1x_1}\psi_a(p) = (1-|p|^2)^{-3}\Big(\partial^3_{\zeta_1\zeta_1\zeta_1}\widetilde{\psi}_a(0) + 6p_1\partial^2_{\zeta_1\zeta_1}\widetilde{\psi}_a(0)$$
$$-6p_2\partial^2_{\zeta_1\zeta_2}\widetilde{\psi}_a(0) + 6(p_1^2-p_2^2)\partial_{\zeta_1}\widetilde{\psi}_a(0) - 12p_1p_2\partial_{\zeta_2}\widetilde{\psi}_a(0)\Big);$$

$$\partial^3_{x_1x_1x_2}\psi_a(p) = (1-|p|^2)^{-3}\Big(\partial^3_{\zeta_1\zeta_1\zeta_2}\widetilde{\psi}_a(0) + 4p_2\partial^2_{\zeta_1\zeta_1}\widetilde{\psi}_a(0)$$
$$+6p_1\partial^2_{\zeta_1\zeta_2}\widetilde{\psi}_a(0) - 2p_2\partial^2_{\zeta_2\zeta_2}\widetilde{\psi}_a(0) + 6(p_1^2-p_2^2)\partial_{\zeta_2}\widetilde{\psi}_a(0) + 12p_1p_2\partial_{\zeta_1}\widetilde{\psi}_a(0)\Big);$$

$$\partial^3_{x_1x_2x_2}\psi_a(p) = (1-|p|^2)^{-3}\Big(\partial^3_{\zeta_1\zeta_2\zeta_2}\widetilde{\psi}_a(0) + 4p_1\partial^2_{\zeta_2\zeta_2}\widetilde{\psi}_a(0)$$
$$+6p_2\partial^2_{\zeta_1\zeta_2}\widetilde{\psi}_a(0) - 2p_1\partial^2_{\zeta_1\zeta_1}\widetilde{\psi}_a(0) + 6(p_1^2+p_2^2)\partial_{\zeta_1}\widetilde{\psi}_a(0) + 12p_1p_2\partial_{\zeta_2}\widetilde{\psi}_a(0)\Big);$$

$$\partial^3_{x_2x_2x_2}\psi_a(p) = (1-|p|^2)^{-3}\Big(\partial^3_{\zeta_2\zeta_2\zeta_2}\widetilde{\psi}_a(0) + 6p_2\partial^2_{\zeta_2\zeta_2}\widetilde{\psi}_a(0)$$
$$-6p_1\partial^2_{\zeta_1\zeta_2}\widetilde{\psi}_a(0) - 6(p_1^2-p_2^2)\partial_{\zeta_2}\widetilde{\psi}_a(0) - 12p_1p_2\partial_{\zeta_1}\widetilde{\psi}_a(0)\Big)$$

where $p=(p_1,p_2)$.

On the other hand, for the Poisson integral formula, the solution $\widetilde{\psi}_a$ of system (9) admits the following representation

$$\widetilde{\psi}_a(\zeta) = \frac{1-|\zeta|^2}{2\pi}\int_{\partial D}\frac{\widetilde{\Phi}^a(\eta)}{|\zeta-\eta|^2}ds_\eta.$$

Hence,

$$\widetilde{\psi}_a(0) = \frac{1}{2\pi}\int_{\partial D}\frac{\widetilde{\Phi}^a(\eta)}{|\eta|^2}ds_\eta = \frac{1}{2\pi}\int_{\partial D}\widetilde{\Phi}^a(\eta)ds_\eta;$$

$$\partial_{\zeta_j}\widetilde{\psi}_a(0) = \frac{1}{\pi}\int_{\partial D}\widetilde{\Phi}^a(\eta)\eta_j ds_\eta, \quad j=1,2;$$

$$\partial^2_{\zeta_j\zeta_j}\widetilde{\psi}_a(0) = -\frac{2}{\pi}\int_{\partial D}\widetilde{\Phi}^a(\eta)ds_\eta + \frac{4}{\pi}\int_{\partial D}\widetilde{\Phi}^a(\eta)\eta_j^2 ds_\eta, \qquad j=1,2;$$

$$\partial^2_{\zeta_1\zeta_2}\widetilde{\psi}_a(0) = \frac{4}{\pi}\int_{\partial D}\widetilde{\Phi}^a(\eta)\eta_1\eta_2 ds_\eta;$$

$$\partial^3_{\zeta_j\zeta_j\zeta_j}\widetilde{\psi}_a(0) = \frac{24}{\pi}\int_{\partial D}\widetilde{\Phi}^a(\eta)\eta_j^3 ds_\eta - \frac{18}{\pi}\int_{\partial D}\widetilde{\Phi}^a(\eta)\eta_j ds_\eta, \qquad j = 1,2;$$

$$\partial_{\zeta_i\zeta_i\zeta_j}\widetilde{\psi}_a(0) = \frac{24}{\pi}\int_{\partial D}\widetilde{\Phi}^a(\eta)\eta_i^2\eta_j ds_\eta - \frac{6}{\pi}\int_{\partial D}\widetilde{\Phi}^a(\eta)\eta_j ds_\eta, \qquad i,j = 1,2, i \neq j.$$

The equivalence between (4) and (8) easily follows. □

From formula (2) we infer that, for $k = 8$, eleven configurations are possible and the set $\mathcal{Z}_3(U)$ contains only points with even multiplicity in three cases. Indeed,

Proposition 2.8: *Let $k = 8$ and $U \in \mathcal{S}$. Then*

- *if $\mathcal{Z}_3(U) = \{q\}$ then $m(q) = 8$;*
- *if $\mathcal{Z}_3(U) = \{q_1, q_2\}$ then $m(q_1)+m(q_2) = 10$. Three situations can occur: $m(q_1) = 7, m(q_2) = 3$ or $m(q_1) = 6, m(q_2) = 4$ or $m(q_1) = m(q_2) = 5$;*
- *if $\mathcal{Z}_3(U) = \{q_1, q_2, q_3\}$ then $m(q_1) + m(q_2) + m(q_3) = 12$. Three situations can occur: $m(q_1) = 3, m(q_2) = 4, m(q_3) = 5$ or $m(q_1) = m(q_2) = 3, m(q_3) = 6$ or $m(q_1) = m(q_2) = m(q_3) = 4$;*
- *if $\mathcal{Z}_3(U) = \{q_1, q_2, q_3, q_4\}$ then $m(q_1) + m(q_2) + m(q_3) + m(q_4) = 14$. Two situations can occur: $m(q_i) = 3, i = 1,2,3, m(q_4) = 5$ or $m(q_1) = m(q_2) = 3, m(q_4) = m(q_5) = 4$;*
- *if $\mathcal{Z}_3(U) = \{q_i, i = 1, ..., 5\}$ then $\sum_{i=1}^5 m(q_i) = 16$. We infer that $m(q_i) = 3, i = 1,2,3,4, m(q_5) = 4$;*
- *if $\mathcal{Z}_3(U) = \{q_i, i = 1, ..., 6\}$ then $\sum_{i=1}^6 m(q_i) = 18$. We infer that $m(q_i) = 3, i = 1, ..., 6$.*

Proof: The set $\mathcal{Z}_3(U)$ is not empty, consists of at most 6 points, say $q_1, ..., q_\ell$, $1 \leq \ell \leq 6$, and, from (2) ,

$$6 + 2\ell = \sum_{i=1}^{\ell} m(q_i), \qquad m(q_i) \geq 3\,.$$

The classification follows. □

In [7] we proved that, for $k = 6$, if ψ_a has two critical points at level zero then they are 4-points for $U = |\psi_a|$. This result can be extended to the case of $2s$ species (Proposition 2.2). In the next Proposition we formulate the result in the case of 8 species and give a detailed proof.

Proposition 2.9: *Let $k = 8$. Let $\Phi = (\phi_1, ..., \phi_8)$ be an admissible datum and let ψ_a be the solution to (3) with boundary datum $\Phi^a = \sum_{j=1}^8 (-1)^j \phi_j$. If ψ_a has 3 critical points $q_i \in D$ such that $\psi_a(q_i) = 0$, $i = 1,2,3$ then $q_i, i = 1,2,3$ are 4-points for the function $U = |\psi_a| \in \mathcal{S}$.*

Proof: By standard theory of harmonic functions the zero set of ψ_a around a critical point q_i at level 0 is made by (at least) 4 half-lines, meeting with equal angles. We infer that locally around q_i the function ψ_a defines k_{q_i} nodal components with $k_{q_i} \geq 4$ and k_{q_i} is even because ψ_a has alternate positive or negative sign on adjacent sets. If ψ_a has three critical points at level zero then, by Proposition 2.8, we infer that $k_{q_i} = 4$, $i = 1, 2, 3$. The function $U = |\psi_a|$ is nonnegative, satisfies the boundary datum Φ, has exactly 8 nodal regions and generates an element of $\mathcal{S}$. $\square$

3. Numerical examples

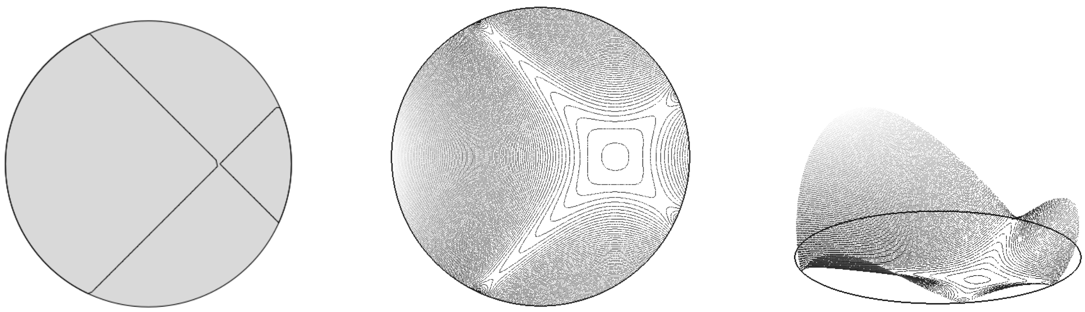
Example 3.1 Let $k = 4$. The nodal regions of the harmonic function

$$\Psi(x_1, x_2) = 4(x_1^2 - x_2^2) - 4x_1 + 1 \tag{10}$$

are represented in figure 1a). On the boundary of the unit disk, Ψ vanishes in four points: $((1+\sqrt{7})/4, \pm(1-\sqrt{7})/4)$, $((1-\sqrt{7})/4, \pm(1+\sqrt{7})/4)$. Denote by γ_i the arcs of positivity of Ψ on ∂D and by ω_i the sets of positivity of Ψ between γ_i and the lines where Ψ vanishes. If we assume $\phi_i = |\Psi|$ on γ_i, $i = 1, ..., 4$, then $\Phi = (\phi_1, ..., \phi_4)$ is an admissible datum. Defining $u_i = |\Psi|$ in ω_i, then $U = (u_1, ..., u_4) \in \mathcal{S}$. We have

$$\Psi(1/2, 0) = 0, \quad \operatorname{grad}\Psi(1/2, 0) = (0, 0).$$

Hence, for Proposition 2.5 and Theorem 2.4, the point $(1/2, 0)$ is a 4-point for $U = |\Psi|$. In figures 1b)-1c) we show some level lines of the approximate solution of (1) for large value of μ ($\mu = 10^5$) obtained with the FreeFEM++ software, showing a point with multiplicity 4.



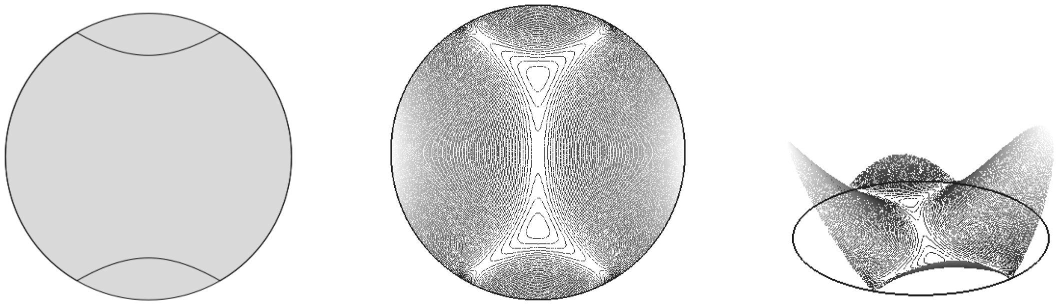
a) b) c)

Figure 1. a): Nodal regions of Ψ in (10). b) and c): Level lines of the approximate solutions of (1) with $k = 4$ differential equations and for large μ, showing a point with multiplicity 4.

Example 3.2 Let $k = 4$. The nodal regions of the harmonic function

$$\Psi(x_1, x_2) = 4(x_1^2 - x_2^2) + 2 \tag{11}$$

are represented in figure 2a). On the boundary of the unit disk, Ψ vanishes in four points: $(1/2, \pm\sqrt{3}/2)$, $(-1/2, \pm\sqrt{3}/2)$. We define the arcs γ_i as in example 3.1. Assuming $\phi_i = |\Psi|$ on γ_i, $i = 1, ..., 4$, we have that $\Phi = (\phi_1, ..., \phi_4)$ is an admissible datum. The gradient of Ψ vanishes at the origin but $\Psi(0, 0) \neq 0$. We infer that the limiting configuration exhibits two 3-points. In figure 1b)-1c) we show some level lines of the approximate solution of (1) for $\mu = 10^5$ obtained with the FreeFEM++ software, showing two points with multiplicity 3.



a) b) c)

Figure 2. a): Nodal regions of Ψ in (11). b) and c): Level lines of the approximate solutions of (1) with $k = 4$ differential equations and for large μ, showing two points with multiplicity 3.

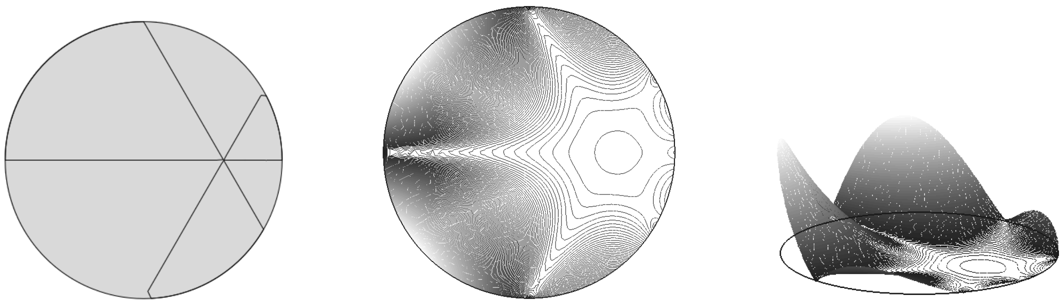
Example 3.3 Let $k = 6$. Consider the harmonic function

$$\Psi(x_1, x_2) = x_2 \left(\sqrt{3}x_1 - x_2 - 1\right) \left(\sqrt{3}x_1 + x_2 - 1\right) \tag{12}$$

which nodal regions are represented in the figure 3a). On the boundary of the unit disk the function Ψ vanishes in six points: $(0, \pm 1)$, $(\pm 1, 0)$, $\left(\pm\sqrt{3}/2, 1/2\right)$. Denote by γ_i the arcs of positivity of Ψ on ∂D and by ω_i the sets of positivity of Ψ between γ_i and the lines where Ψ vanishes. If we assume $\phi_i = |\Psi|$ on γ_i , $i = 1, ..., 6$ then $\Phi = (\phi_1, ..., \phi_6)$ is an admissible datum. We have

$$\Psi(1/\sqrt{3}, 0) = 0, \quad \text{grad}\, \Psi(1/\sqrt{3}, 0) = (0, 0), \qquad H\Psi(1/\sqrt{3}, 0) = \mathbf{0}\,.$$

For Proposition 2.6 and Theorem 2.4 we infer that $(1/\sqrt{3}, 0)$ is a point of multiplicity 6 for $U = |\Psi| \in \mathcal{S}$. Thanks to FreeFEM++ software we can obtain nodal partitions, showing a 6-point solution of (1) for large value of μ (see figures 3b)-3c)).



a) b) c)

Figure 3. a): Nodal regions of Ψ in (12). b) and c): Level lines of the approximate solutions of (1) with $k = 6$ differential equations and for large μ, showing a point with multiplicity 6.

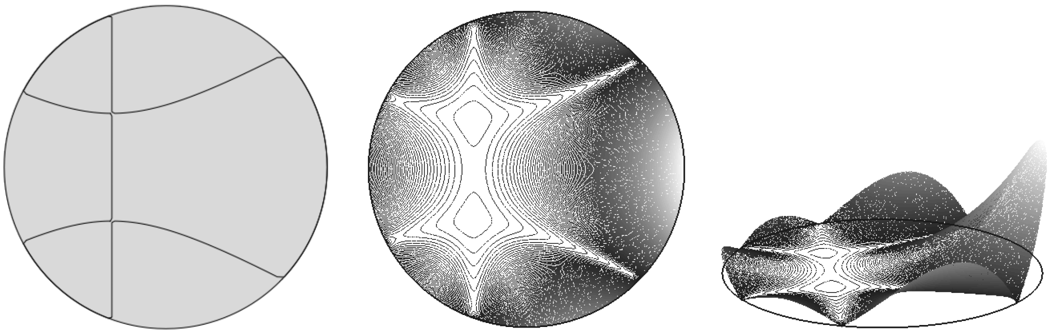
Example 3.4 Let $k = 6$. Consider the harmonic function

$$\Psi(x_1, x_2) = (3x_1 + 1) \left(9x_1^2 + 6x_1 - 27x_2^2 + 4\right) \tag{13}$$

whose nodal lines are represented in figure 4a). On the boundary of the unit disk the function Ψ vanishes in six points: $\left(-1/3, \pm 2\sqrt{2}/3\right)$, $\left(\left(\pm\sqrt{93}-1\right)/12, \pm\sqrt{\left(25-\sqrt{93}\right)/(6\sqrt{2})}\right)$. If we assume, as in the example 3.3, $\phi_i = |\Psi|$ on $\gamma_i\,, i = 1, ..., 6$ then $\Phi = (\phi_1, ..., \phi_6)$ is an admissible datum. We have

$$\Psi(-1/3, \pm 1/3) = 0, \quad \operatorname{grad}\Psi(-1/3, \pm 1/3) = (0, 0).$$

Since $(-1/3, \pm 1/3)$ are two critical points at level zero, for Proposition 2.2 they are points of multiplicity 4 for $U = |\Psi| \in \mathcal{S}$. Figures 4b)-4c) show some level lines of the approximate solution of (1) for large value of μ, showing the predicted limiting configuration.



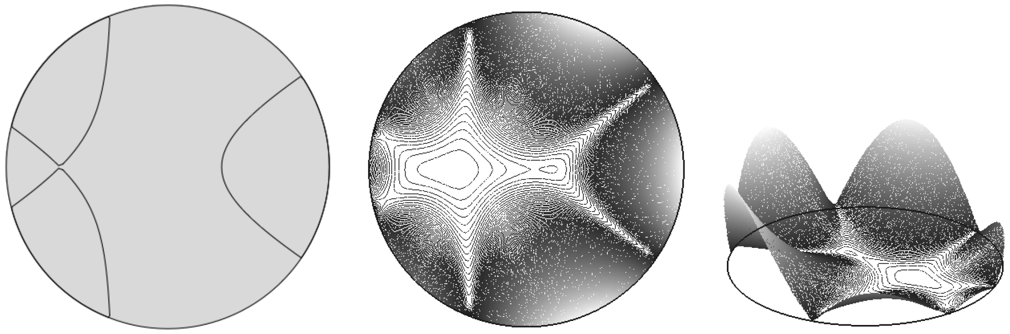
a) b) c)

Figure 4. a): Nodal regions of Ψ in (13). b) and c): Level lines of the approximate solutions of (1) with $k = 6$ differential equations and for large μ, showing two points with multiplicity 4.

Example 3.5 Let $k = 6$. The harmonic function

$$\Psi(x_1, x_2) = 27x_1^3 + 27x_1^2 - 81x_1x_2^2 - 27x_2^2 - 4 \tag{14}$$

vanishes in six points on ∂D. The nodal regions are represented in figure 5a). Defining the arc γ_i as in the example 3.3 and assuming $\phi_i = |\Psi|$ on $\gamma_i\,, i = 1, ..., 6$, we consider the admissible datum $\Phi = (\phi_1, ..., \phi_6)$. The gradient of Ψ vanishes in two points: $p_1 = (0, 0), p_2 = (-2/3, 0)$ with $\Psi(p_1) = -4, \Psi(p_2) = 0$. We can exclude the existence of two points with multiplicity 4 because $\Psi(p_1) \neq 0$. The figures 5b)-5c) show a configuration with two multiple points, a 5-point and a 3-point respectively.



(a) b) c)

Figure 5. a): Nodal regions of Ψ in (14). b) and c): Level lines of the approximate solutions of (1) with $k = 6$ differential equations and for large μ, showing a point with multiplicity 5 and a point with multiplicity 3 for the limiting configuration.

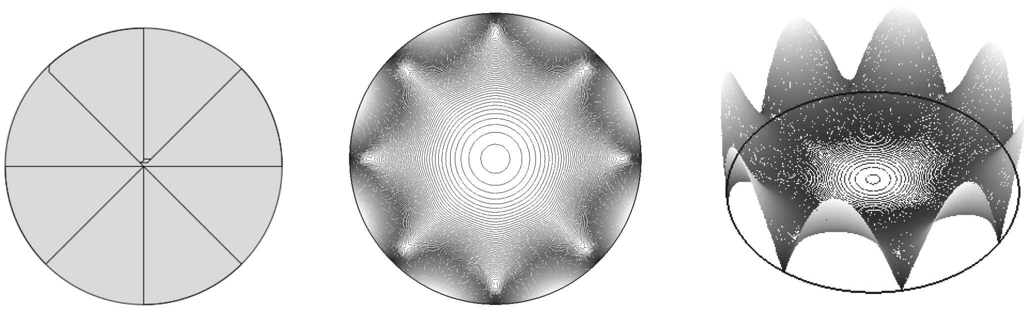
Example 3.6 Let $k = 8$. The harmonic function

$$\Psi(x_1, x_2) = x_1 x_2 (x_1^2 - x_2^2) \tag{15}$$

whose nodal regions are represented in figure 6a), vanishes in eight points on ∂D. Denote by γ_i the arcs of positivity of Ψ on ∂D and by ω_i the sets of positivity of Ψ between γ_i and the lines where Ψ vanishes. If we assume $\phi_i = |\Psi|$ on γ_i, $i = 1, ..., 8$ then $\Phi = (\phi_1, ..., \phi_8)$ is an admissible datum. We have

$$\Psi(0,0) = \partial_{x_i}\Psi(0,0) = \partial^2_{x_i x_j}\Psi(0,0) = \partial^3_{x_i x_j x_k}\Psi(0,0) = 0, i, j, k = 1, 2.$$

For Proposition 2.7 and Theorem 2.4 we infer that $(0,0)$ is a point with multiplicity 8 for $U = |\Psi| \in \mathcal{S}$. Figures 6b)-6c) exhibit a nodal partition showing an 8-point solution of (1) for large value of μ.



(a) (b) (c)

Figure 6. a): Nodal regions of Ψ in (15). b) and c): Level lines of the approximate solutions of (1) with $k = 8$ differential equations and for large μ, showing a configuration with a point with multiplicity 8.

Example 3.7 Let $k = 8$. The harmonic function

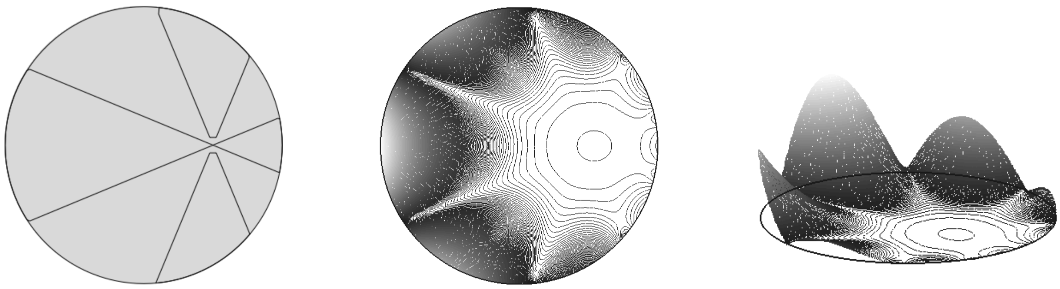
$$\Psi(x_1, x_2) = -24(1 - 2x_1)^2 x_2^2 + (1 - 2x_1)^4 + 16x_2^4 \tag{16}$$

which nodal regions are represented in figure 7a), vanishes in eight points on ∂D.

We define the admissible datum as in the example 3.6. We have

$$\Psi(1/2,0) = \partial_{x_i}\Psi(1/2,0) = \partial^2_{x_i x_j}\Psi(1/2,0) = \partial^3_{x_i x_j x_k}\Psi(1/2,0) = 0, i, j, k = 1, 2.$$

For Proposition 2.7 and Theorem 2.4 we infer that $(1/2, 0)$ is a point with multiplicity 8 for $U = |\Psi| \in \mathcal{S}$. This is confirmed in figures 7b) and 7c).



(a) (b) (c)

Figure 7. a): Nodal regions of Ψ in (16). b) and c): Level lines of the approximate solutions of (1) with $k = 8$ differential equations and for large μ, showing a configuration with a point with multiplicity 8.

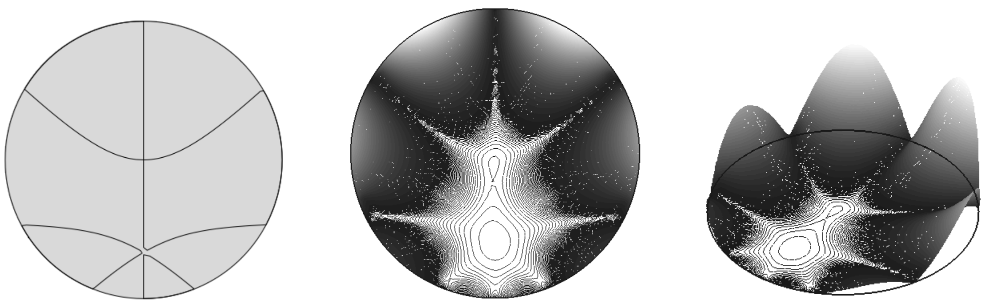
Example 3.8 Let $k = 8$. The harmonic function

$$\Psi(x_1, x_2) = 9x_1^3 x_2 + 4x_1^3 - 9x_1 x_2^3 - 12x_1 x_2^2 - 4x_1 x_2 \quad (17)$$

which nodal regions are represented in figure 8a), vanishes in eight points on ∂D. Its gradient vanishes in two points and

$$\Psi(0,0) = \Psi(-2/3,0) = \partial_{x_i}\Psi(0,0) = \partial_{x_i}\Psi(-2/3,0) = 0, \quad i = 1, 2.$$

We define the admissible datum Φ as in the example 3.6. The function Ψ defines 8 nodal regions. We infer that $U = |\Psi| \in \mathcal{S}$ and $\mathcal{Z}_3(U)$ consists of two points with multiplicity 4 and 6, respectively. Figures 8b)-8c) exhibit a nodal partition showing a solution of (1) for large value of μ with two multiple points, a 4-point and a 6-point, respectively.



(a) (b) (c)

Figure 8. a): Nodal regions of Ψ in (17). b) and c): Level lines of the approximate solutions of (1) with $k = 8$ differential equations and for large μ, showing a point with multiplicity 4 and a point with multiplicity 6 for the limiting configuration.

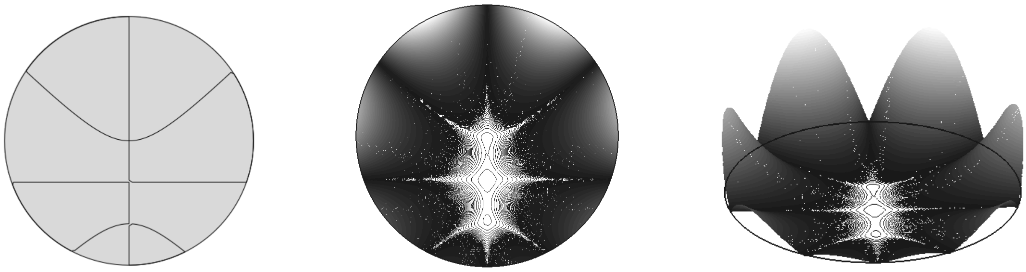
Example 3.9 Let $k = 8$. The harmonic function

$$\Psi(x_1, x_2) = x_1(3x_2 + 1)\left(3x_1^2 - 3x_2^2 - 2x_2\right) \tag{18}$$

which nodal regions are represented in figure 9a), vanishes in eight points on ∂D. Its gradient vanishes in three points and

$$\Psi(0,0) = \Psi(0,-1/3) = \Psi(0,-2/3) = 0$$
$$\partial_{x_i}\Psi(0,0) = \partial_{x_i}\Psi(0,-1/3) = \partial_{x_i}\Psi(0,-2/3) = 0, i = 1,2.$$

We define the admissible datum Φ as in the example 3.6. The function $U = |\Psi| \in \mathcal{S}$ and, for Proposition 2.9, $\mathcal{Z}_3(U)$ consists of three points with multiplicity 4. Figures 9b)-9c) exhibit the nodal partition of the solution of (1) for large value of μ, showing the predicted limiting configuration with three 4-points.



(a) (b) (c)

Figure 9. a): Nodal regions of Ψ in (18). b) and c): Level lines of the approximate solutions of (1) with $k = 8$ differential equations and for large μ, showing three points with multiplicity 4 for the limiting configuration.

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