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Material Conservation and Balance Laws in Linear Elasticity with Applications

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Abstract. Classical mechanics, or, as we like to call it, *Mechanics in Physical Space*, is the body of knowledge which is concerned with equilibrium and motion of objects which possess mass and which are placed in the Euclidian space. The recognition that materials, on some scale, cannot regarded as perfect continua, but rather contain a variety of defects, which can move within the body through several mechanisms can lead to the construction of a whole edifice of knowledge called *Mechanics in Material Space.* A far-reaching duality exists between Newtonian (physical) and Eshelbyan (material) mechanics. Some examples of those dualities are given in the introduction. The main focus of this Lecture Notes is on the establishment of material conservation and balance laws within the tree-dimensional theory of elasticity and its applications. The mathematical basics as Noether's theorem and the Neutral-Action method are introduced, and specialized to the one-dimensional bar theory. The ensuing conservation laws are applied to a hole/dislocation-interaction problem, and possible applications in fracture mechanics are discussed.

Key words and phrases: configurational mechanics, Eshelby tensor, Noether's theorem, Neutral-Action method, fracture mechanics, defect interaction

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1 Introduction

The subject of Newtonian Mechanics, or, as we would like to call it, *Mechanics in Physical Space*, is the description and the in-advance estimate of motions and deformations of material bodies. Also the physical forces connected with those movements are of interest. For example, we might be interested in the trajectory of a body under given forces and initial conditions, in the deflection of a bridge, in the vibration of a machine, etc. The description by physical conservation and balance laws are well established thanks to the ingenious advances of Galileo, Newton, Euler, Lagrange, Hamilton and others.

Dual to Newtonian Mechanics, a whole edifice of an Eshelbyan Mechanics has been constructed during recent years [1] - [4]. The subject of this *Mechanics in Material Space* (or Configurational Mechanics) is the description of the motion of defects within the surrounding material. Defects might be missing atoms in a lattice, a dislocation, a hole, a crack, etc.; motion might be dislocation movement or climbing, self-similar expansion of a hole, crack extension, etc.; and mechanisms might be diffusion, melting or accretion, fracture etc.

In his pioneering work, Eshelby [5] advanced the notion of a force on an elastic singularity (i.e., a defect). This force, which has later been called also material force, configurational force, thermodynamic force or driving force, is calculated from the change of the total elastic energy due to a (infinitesimal) material translation, i.e., a change of configuration. Other notions frequently used in mechanics like trajectories, stability, reciprocity, etc. can be similarly adopted within the Mechanics of Material Space. It may be intriguing, therefore, to juxtapose the Mechanics in Physical Space and Material Space exemplarily.

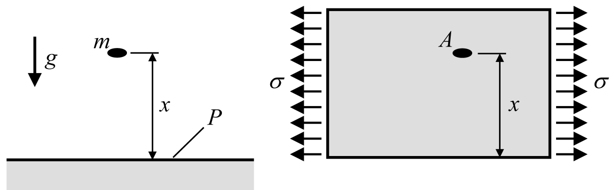

Figure 1: Mass under gravity (left) and defect in a body under load (right)

In order to calculate the total energy of both systems depicted in Fig. 1, we need three ingredients. A characteristic of the object under consideration, an applied field and some distance. In physical space, this is the mass m of the mass point, which could be called (quite pompously) an inhomogeneity in the otherwise empty physical space; the gravity field g (without the gravity field, the mass point would not have any potential); and the height above some arbitrarily fixed reference plane P. In material space, the object A, i.e., a defect, may be characterized by some parameters a_i ,e.g., by the Burgers vector of a dislocation, the stiffness difference with respect to the surrounding material of an inclusion, a crack length, etc., the applied load characterized by σ (without the applied field, the defect would not experience any driving force); and the distance from some boundary. Thus

$$\Pi = mgx, \qquad \Pi = \Pi\left(a_i, \sigma, x\right), \tag{1.1}$$

and the physical force F as the well as the material force J are calculated from the negative gradient with respect to x

$$F = -\frac{\partial \Pi}{\partial x} = -mg, \qquad J = -\frac{\partial \Pi}{\partial x}. \tag{1.2}$$

A change of x in Newtonian Mechanics is the change of the placement of the mass point within the physical space, whereas the change of x in Eshelbyan Mechanics is the change of the configuration of the defect within its surrounding material.

In anticipation of the following, we could already discuss conservation laws, path-independent integrals and balance laws in physical and material space. Let us consider a body B of volume Vsurrounded by a surface S of area A with unit outward normal vector n as depicted in Fig. 2.

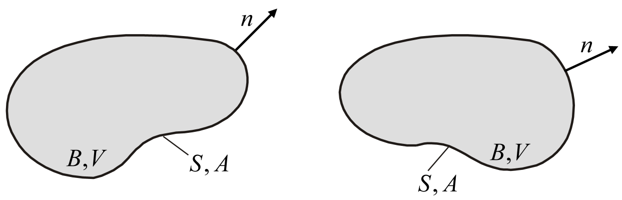

Figure 2: Body in the absence of body forces, i.e., physical homogeneity (left) and body in the absence of defects, i.e., material homogeneity (right)

In the absence of body forces, i.e., physical homogeneity, the local physical homogeneous equilibrium equations written in terms of the Chauchy-stress tensor σ_{ij} are satisfied and establish a conservation law given in equation (1.3 left). Whereas in the absence of defects, i.e., material homogeneity, the local material homogeneous equilibrium equations written in terms of the Eshelby-stress tensor b_{ij} are given in equation (1.3 right) and deliver also a divergence-free relation

$$\sigma_{ij,i} = 0, \qquad b_{ij,i} = 0. \tag{1.3}$$

The Eshelby-stress tensor will be specified later.

Integration over the body B and implying the divergence theorem leads to

$$F_j = \int_S \sigma_{ij} n_i dA = 0, \qquad J_j = \int_S b_{ij} n_i dA = 0. \tag{1.4}$$

In physical space, it follows merely that the external tractions have to be self-equilibrated, in material space, it can be concluded from the material tractions along the boundary of the body that the body is homogeneous.

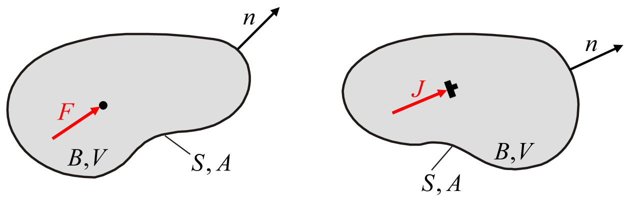

Figure 3: Body loaded by a point force F (left) and body with point defect (right)

The situation is changed as soon as the body contains either a physical or a material inhomogeneity as depicted in Fig. 3.
The surface integrals do not vanish anymore. Instead they deliver now the resulting physical force F acting on the body and the resulting material force J acting on the defect

$$F_j = \int_S \sigma_{ij} n_i dA \neq 0, \qquad J_j = \int_S b_{ij} n_i dA \neq 0.. \tag{1.5}$$

Both integrals are path independent: As long as the integration contour surrounds one and the same (physical or material) defect, the values of the integrals are equal.

Let us also introduce the notion of physical and material translations. To this extent, we consider in Fig. 4 a linear spring with spring constant c extended by an amount u due to an applied force F. The potential of internal energy Π^i and of outer forces Π^a are given by

$$\Pi^i = \frac{1}{2} c u^2, \qquad \Pi^a = -F u, \tag{1.6}$$

respectively. Keeping the value of F fixed, we apply a virtual (physical) translation δu. The total potential energy of the system $\Pi = \Pi^i + \Pi^a$ is changed. According to the virtual work theorem [6], this first variation $\delta \Pi$ has to vanish, i.e.,

$$\delta_u \Pi = \frac{\partial \Pi}{\partial u} \delta u = (cu - F)\, \delta u = 0. \tag{1.7}$$

Since this must hold for arbitrary variations δu, we can calculate the displacement u as

$$u = \frac{F}{c} \tag{1.8}$$

Using (1.8), the total potential Π is modified to

$$\Pi = \Pi^i + \Pi^a = \frac{1}{2} c u^2 - F u = \frac{1}{2} c u^2 - c u^2 = -\frac{1}{2} c u^2 = -\Pi^i \tag{1.9}$$

(Clapeyron's theorem, cf., e.g., [7]).

Now, instead, we apply a material translation δx, i.e., we change the configuration of the system (under the applied load) moving the fixed support of

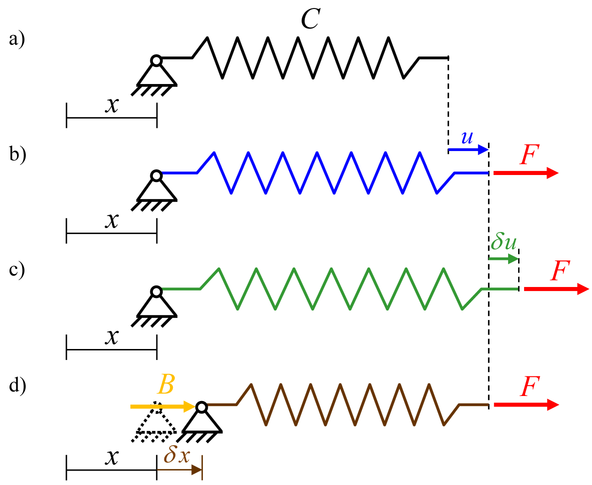

Figure 4: Linear spring, spring constant c (a), extended by force F by an amount u (b), virtual (physical) displacement δu under fixed force F (c), material translation δx and material force B.

the spring. The work of the material force B has also been taken into account leading with $\frac{du}{dx} = u^{/}$ to

$$\begin{aligned}
\delta_x \Pi &= -\delta_x \Pi^i + B\delta x \\
&= -\frac{1}{2}c\left[(u(x+\delta x))^2 - (u(x))^2\right] + B\delta x \\
&= -\frac{1}{2}c\left[\left(u(x) + \frac{du}{dx}\delta x + ...\right)^2 - (u(x))^2\right] + B\delta x \\
&= -\frac{1}{2}c\left[u^2(x) + 2u(x)u^{/}\delta x + O\left((\delta x)^2\right) - \left(u^2(x)\right)\right] + B\delta x \\
&= -cuu^{/}\delta x + B\delta x + O\left((\delta x)^2\right) \\
&= -\left(Fu^{/} - B\right)\delta x \overset{!}{=} 0.
\end{aligned} \tag{1.10}$$

The magnitude of the material force B turns out to be

$$B = Fu^{/}. \tag{1.11}$$

Note that the material force B is acting also in Fig. 4 b and c, but contributes neither to the energy nor to the virtual work since its application point is not moved.

A similar example is given in Fig. 5. We consider a beam of span ℓ with bending stiffness EI (Fig. 5). Due to the applied load F at $\ell/2$ the beam experiences a transverse displacement $w(x)$.
The virtual work theorem (see, e.g., [8]) yields

$$w(\ell/2) = \frac{F\ell^3}{48EI}, \tag{1.12}$$

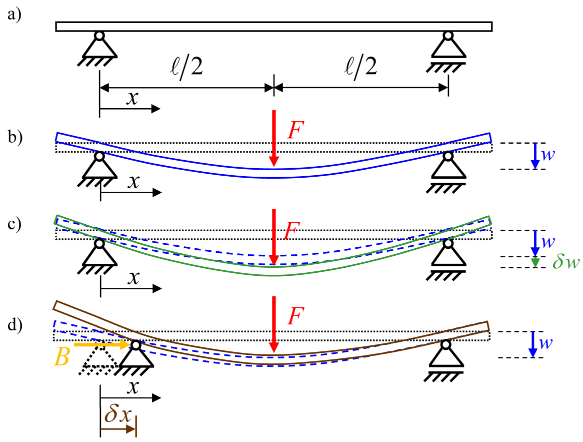

Figure 5: Beam of span ℓ and bending stiffness EI (a), bent by a force F applied at $\ell/2$ resulting in a transverse displacement $w(x)$ (b), virtual displacement δw (c) and material displacement (d)

whereas the change in total energy due to the material translation delivers with the supporting force $A = F/2$ and the slope $w^{/} = \frac{dw}{dx}$ (see [9]-[11])

$$B = -Aw^{/}(0) = -\frac{F^2\ell^2}{32EI}. \tag{1.13}$$

Note that material forces are also present at the load application point and at the right support but do not contribute to our considerations here.
Further dualities between the mechanics in physical and material space will be discussed later.

2 Three-dimensional linear theory of elastostatics and one-dimensional bar theory

The aim of this section is to recollect the basic equations of the linear theory of elastostatics in cartesian coordinates and to introduce the notation. Details may be found in any textbook on linear elasticity. In addition, the equations of the one-dimensional theory of bars in tension/compression are assembled.

The equations of equilibrium connect the divergence of the symmetric Chauchy-stress tensor σ_{ij} with the applied body forces p_j

$$\sigma_{ij,i} + p_j = 0, \sigma_{ij} = \sigma_{ji}. \tag{2.1}$$

The summation convention is applied for repeated indices and a comma denotes partial differentiation with respect to the coordinate indicated. Within the geometrical linearized theory, the components of the strain tensor ε_{ij} are calculated from the symmetric part of the displacement gradient $u_{i,j}$

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}), \quad \varepsilon_{ij} = \varepsilon_{ji}. \tag{2.2}$$

The generalized Hooke's law combines stresses with strains by the fourth-rank elasticity tensor $E_{ijk\ell}$

$$\sigma_{ij} = E_{ijk\ell}\varepsilon_{k\ell}. \tag{2.3}$$

For isotropic material, the components of $E_{ijk\ell}$ depend on two material constants only, e.g., on the Lamé constants λ and μ, and may be given with the Kronecker symbol of unity δ_{ij} as

$$E_{ijk\ell} = \lambda\delta_{ij}\delta_{k\ell} + 2\mu\delta_{ik}\delta_{j\ell}. \tag{2.4}$$

With three equations of equilibrium (2.1), six kinematic relations (2.2) and the six equations of the material law (2.3), we have 15 equations for the 15 unknowns, i.e., six stresses σ_{ij}, six strains ε_{ij} and three displacements u_i. By replacing the stresses σ_{ij} via (2.3) and (2.4) by the strains and introducing (2.2) we arrive at the Navier-Lamé equations for isotropic, homogeneous materials

$$\mu u_{i,jj} + (\lambda + \mu)u_{j,ji} + p_i = 0, \tag{2.5}$$

i.e., three equations for the three unknown displacements u_i. The loads p_i and Lamé constants λ and μ are assumed to be given. Boundary conditions have to be specified in order to arrive at unique solutions for specific boundary value problems. At each print of the boundary $S = S_u \cup S_t$ of the body under consideration, either displacements $\hat{u}_i$ or tractions $\hat{t}_j = \sigma_{ij}n_i$ are prescribed

$$\begin{aligned} u_i\,|_{S_u} &= \hat{u}_i \\ t_j\,\big|_{S_t} &= \sigma_{ij}n_i\,|_{S_t} = \hat{t}_j\ . \end{aligned} \tag{2.6}$$

In the following, we will need the strain-energy density W and the potential of external forces V defined by

$$\begin{aligned} W &= \frac{1}{2}\sigma_{ij}\varepsilon_{ij} = \frac{1}{2}E_{ijk\ell}\varepsilon_{k\ell}\varepsilon_{ij}, \\ V &= -p_i u_i, \end{aligned} \tag{2.7}$$

from which constitutive equations are derived

$$\begin{aligned} \frac{\partial W}{\partial \varepsilon_{ij}} &= \frac{\partial W}{\partial u_{i,j}} = \sigma_{ji}, \\ \frac{\partial V}{\partial u_i} &= -p_i. \end{aligned} \tag{2.8}$$

Note that (2.8.a) and (2.3) are equivalent.

The Lagrange function is defined as difference between the kinetic-energy density T and the sum of $W + V$. In elastostatics the kinematic energy has no contribution. Thus we have merely

$$L = -(W + V). \tag{2.9}$$

Finally, the action integral is the integral of the Lagrangian over the body B with volume V as

$$A = \int_B L\, dV. \tag{2.10}$$

The transition from the three-dimensional equations of the linear theory of elasticity to the one-dimensional equations of linear bar theory is straight forward

$$\begin{array}{lcl} \sigma_{ij} & \to & \sigma_{11} A = N, \\ p_i & \to & p_1 A = n, \\ u_i & \to & u_1 = u, \\ \varepsilon_{ij} & \to & \varepsilon_{11} = \varepsilon, \\ ()_{,j} & \to & ()_{,1} = ()'. \end{array} \tag{2.11}$$

N is the normal force, n is the load in axial direction per unit of length and A is the cross-sectional area.

Thus the equations far bars in tension/compression become

$$\begin{array}{rl} \text{equilibrium:} & N' + n = 0, \\ \text{kinematics} & \varepsilon = u', \\ \text{Material law} & N = EA\varepsilon, \\ \text{Navier-Lamé equation} & EAu'' + n = 0, \\ \text{displacement boundary} & u\,|_{S_u} = \hat{u}, \\ \text{traction boundary} & N\,|_{S_t} = \hat{N}, \\ \text{strain-energy per unit length} & \hat{W} = \frac{1}{2} N\varepsilon = \frac{1}{2} EA\varepsilon^2 = \frac{1}{2} EAu'^2, \quad (2.12) \\ \text{potential of external forces} & \hat{V} = -nu, \quad \frac{\partial \hat{V}}{\partial u} = -n, \\ \text{constitutive equation} & \frac{\partial \hat{W}}{\partial \varepsilon} = \frac{\partial \hat{W}}{\partial u'} = N, \\ \text{Lagrangian per unit of length} & \hat{L} = -\frac{1}{2} EAu'^2 + nu, \\ \text{action integral} & A = \int_0^\ell \hat{L}\, dx. \end{array}$$

The product of Young's modulus E and cross-sectional area A is called axial stiffness EA.

3 Euler-Lagrange equations

As soon as a Lagrange function for a problem of linear elasticity is postulated, the equilibrium equations and even the Navier-Lamé equations are predetermined by the principle of minimal energy (stationary of the action integral)

$$\delta A = 0, \tag{3.1}$$

i.e., the first variation of the action integral vanishes. The Lagrange function (2.9) is due (2.7) and (2.2) a function of the displacements u_i and the displacement gradients $u_{i,j}$. It may also explicitly depend on the independent variables x_i, if the material is inhomogeneous. This will not be considered in the following.

Thus

$$L = L(u_i, u_{j,k}). \tag{3.2}$$

Without knowledge of the specific form of L, i.e., its specific dependence on u_i and $u_{j,k}$, the variation δA of A can be calculated by varying u_i and $u_{i,k}$ under the integral sign according to

$$\begin{aligned} u_i &\rightarrow u_i + \delta u_i, \\ u_{i,j} &\rightarrow \delta u_{i,j}, \end{aligned} \tag{3.3}$$

but doing nothing to x_i. The variations δu_i and $\delta u_{i,j}$ have to be kinematical admissible, i.e., they vanish along the boundary S, fulfill the kinematic constraint

$$\begin{aligned} \delta u_i\,|_s = 0, \\ \delta(u_{i,j}) = (\delta u_i)_{,j}. \end{aligned} \tag{3.4}$$

and they are assumed to be small in the sense that a Taylor series

$$f(u_i + \delta u_i) = f(u_i) + \frac{df}{du_k}\delta u_k + \frac{1}{2!}\frac{\partial^2 f}{\partial u_k \partial u_\ell}\delta u_k \delta u_\ell + ... \tag{3.5}$$

may be truncated after the linear term, i.e., terms of the order δu_i^2, $\delta u_i \delta u_{k,\ell}$, $(\delta u_{k,\ell})^2$, and higher order products, abbreviated by $O(\delta^2)$, are neglected. A one-dimensional sketch is given in Fig. 6

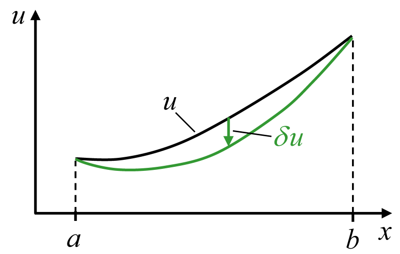

Figure 6: Function u and its variation δu, δu is fixed at a and b during the variation

The variation of the action integral results with (3.4), (3.5), integration by parts and application of the divergence theorem in

$$\begin{aligned} \delta A &= \int_B L(u_i + \delta u_i, u_{i,j} + \delta u_{i,j})dV - \int_B L(u_i, u_{i,j})dV \\ &= \int_B L(u_i, u_{i,j}) + \frac{\partial L}{\partial u_i}\delta u_i + \frac{\partial L}{\partial u_{i,j}}\delta u_{i,j} - L(u_i, u_{i,j})dV + O(\delta^2) \\ &= -\int_B \frac{\partial L}{\partial u_i} + \left(\frac{\partial L}{\partial u_{i,j}}\delta u_i\right)_{,j} - \left(\frac{\partial L}{\partial u_{i,j}}\right)_{,j} \delta u_i dV \\ &= \int_B \left(\frac{\partial L}{\partial u_i} - \left(\frac{\partial L}{\partial u_{i,j}}\right)_{,j}\right) \delta u_i dV + \int_S \frac{\partial L}{\partial u_{i,j}}\delta u_i n_j dA. \end{aligned} \tag{3.6}$$

Since δu_i vanishes by definition along S, the surface term vanishes, and since the variation δu_i is arbitrary within B, the first variation of the action integral vanishes if the Euler-Lagrange equations of the variational problem ($\delta A = 0$) are satisfied

$$\frac{\partial L}{\partial u_i} - \left(\frac{\partial L}{\partial u_{i,j}} \right)_{,j} = 0. \tag{3.7}$$

The reader interested in details of the variational calculus may be referred, e.g., to [12].

The operator

$$E_i\,() = \left(\frac{\partial}{\partial u_i} - \left(\frac{\partial}{\partial u_{i,j}} \right)_{,j} \right) () \tag{3.8}$$

is referred to as the Euler operator, which is always acting on the Lagrangian L. Thus

$$E_i\,(L) = 0 \tag{3.9}$$

may be written for short as Euler-Lagrange equation (3.7).

By use of (2.9) and (2.8) it is easily shown that (3.9) and (2.1) are equivalent. By additional use of (2.2) - (2.4), (3.9) resembles also the Navier-Lamé equations (2.5).

In the one-dimensional setting (2.11), the Euler operator is given as

$$E\,() = \left(\frac{\partial}{\partial u} - \left(\frac{\partial}{\partial u'} \right)' \right) (),$$

and $E\left(L(u, u')\right)$delivers with (2.12 g and h) the equilibrium equation (2.12 a), or, with (2.12 b and c) the Navier-Lamé equation (2.12 d) for the bar problem.

4 Noether's theorem

We return to the action integral (2.10) and do something which is usually not done within a standard course of study in engineering science or in applied mechanics, namely, we subject the Lagrangian L to an infinitesimal transformation of *both* the independent *and* dependent variables, i.e., we pass from the usual quantities x_i, u_i to starred quantities x_i^*, u_i^* according to the prescription

$$\begin{aligned} x_i \to x_i^* &= x_i + \varepsilon \xi_i\,(x_j, u_k)\,, \\ u_i \to u_i^* &= u_i + \varepsilon \varphi_i\,(x_j, u_k)\,. \end{aligned} \tag{4.1}$$

Here, the single constant parameter ε is supposed to be small in the sense, that the ensuing terms without ε and linear in ε will be retained, while terms with ε^2 and higher powers will be omitted. The functions ξ_i and φ_i with the arguments indicated are completely arbitrary.

The transformed action integral shall be called A^* and reads

$$A^* = \int_{B^*} L\left(x_i^*, u_j^*, u_{k,\ell}^*\right) dV^*. \tag{4.2}$$

Note that due to the transformation (4.1), the integration domain is also changed from B to B^*. We next wish to express all the starred quantities in terms of the original unstarred ones. This means in particular, that the transformed domain B^* will be expressed in terms of the original domain B and the differential volume element can be expressed by dV. Using the transformation prescription (4.1 a) we find

$$\begin{aligned} dV^* &= dx_1^*\, dx_2^*\, dx_3^* \\ &= (1+\varepsilon\,\xi_{1,1})dx_1(1+\varepsilon\,\xi_{2,2})dx_2(1+\varepsilon\,\xi_{3,3})dx_3 \\ &= dx_1dx_2dx_3 + \varepsilon\left(\xi_{1,1}+\xi_{2,2}+\xi_{3,3}dx_1dx_2dx_3 + O(\varepsilon^2)\right) \\ &= (1+\varepsilon\,\xi_{i,i})dV. \end{aligned} \tag{4.3}$$

The term u_{k,l^*}^* transforms accordingly to

$$\begin{aligned} \frac{du_k^*}{\partial x_\ell^*} &= \frac{\partial\left(u_k+\varepsilon\varphi_k\right)}{\partial x_m}\frac{\partial x_m}{\partial x_\ell^*} \\ &= (u_{k,m}+\varepsilon\varphi_{k,m})\frac{\partial}{\partial x_l^*}(x_m^* - \varepsilon\xi_m) \\ &= (u_{k,m}+\varepsilon\varphi_{k,m})\left(\delta_{ml} - \varepsilon\frac{\partial\xi_m}{\partial x_l} + O(\varepsilon^2)\right) \\ &= u_{k,l} + \varepsilon(\varphi_{k,l} - u_{km}\xi_{ml}) + O(\varepsilon^2). \end{aligned} \tag{4.4}$$

Now we develop the action integral (4.2)

$$\begin{aligned} A^* = \textstyle\int_B L\left(x_i+\varepsilon\xi_i, u_j+\varepsilon\varphi_j, u_{k,\ell}+\varepsilon(\varphi_{k,\ell}-u_{k,m}\xi_{m,\ell})\right) \\ (1+\varepsilon\xi_{i,i})dV \end{aligned} \tag{4.5}$$

into a Taylor series and obtain after some lengthy algebra, which may be pursued in detail in [3],

$$\begin{aligned} A^* = \textstyle\int_B & L\left(x_i, u_j, u_{k,\ell}\right) \\ & +\varepsilon\left[\xi_i\frac{\partial}{\partial x_i} + \varphi_j\frac{\partial}{\partial u_j} + \left(\frac{d\varphi_k}{dx_\ell} - u_{k,m}\frac{d\xi_m}{dx_\ell}\right)\frac{\partial}{\partial u_{k,\ell}}\right. \\ & \left. +\xi_{i,i}\right] L + O(\varepsilon^2)dV. \end{aligned} \tag{4.6}$$

Note that the total differential operator d/dx_ℓ is

$$\frac{d}{dx_\ell} = \frac{\partial}{\partial x_\ell} + \frac{\partial}{\partial u_j}u_{j,\ell} + \frac{\partial}{\partial u_{k,m}}u_{k,m\ell}. \tag{4.7}$$

As a side remark, the operator $\xi_i \partial/\partial x_i + \varphi_j \partial/\partial u_j$ is referred to as the infinitesimal generator w of a Lie group in the space x_i, u_j

$$w = \xi_i \frac{\partial}{\partial x_i} + \varphi_j \frac{\partial}{\partial u_j}, \tag{4.8}$$

and the operator

$$pr^{(1)} w = w + \left(\frac{d\varphi_k}{dx_\ell} - u_{k,m} \frac{d\xi_m}{dx_\ell} \right) \frac{\partial}{\partial u_{k,\ell}} \tag{4.9}$$

is referred to as the first prolongation $pr^{(1)} w$ of the group into the jet bundle space $x_i, u_j, u_{k,\ell}$. These designations belong to the theory of continuous Lie groups, whose knowledge is not essential for us and which we simply use here. But to explore this background, the reader is referred to, e.g., [13].

Thus we can write

$$A^* = A + \varepsilon \int_B \left(pr^{(1)} w + \xi_{i,i} \right) L \, dV. \tag{4.10}$$

The first term in the integrand above describes the change of L to L^* in the domain B and is a differential operator, while the second term is a factor which describes the change of domain from B to B^*.

By use of (4.7), application of the product role and rearrangements (details may be found again in [3]), (4.10) may be written as

$$A^* = A + \varepsilon \int_B P_{i,i} + Q_j E_j(L) dV \tag{4.11}$$

where the characteristics Q_j and the current P_i are given, respectively, as

$$\begin{aligned} Q_j &= \varphi_j - \xi_i u_{j,i}, \\ P_i &= \varphi_j \frac{\partial L}{\partial u_{j,i}} + \xi_j \left(L\delta_{ij} - \frac{\partial L}{\partial u_{i,k}} u_{k,j} \right). \end{aligned} \tag{4.12}$$

$E_j(L)$ are the Euler-Lagrange equations as defined in (3.7), (3.9).

We now seek for transformations ξ_i and φ_j which leave the action integral invariant, i.e.,

$$\delta^* A = A^* - A = 0. \tag{4.13}$$

The transformation functions are determined from the invariance condition (4.10)

$$\left(pr^{(1)} w + \xi_{i,i} \right) L = 0, \tag{4.14}$$

which leads with (4.8) and (4.9) to an overdetermined system of partial differential equations for these functions. Once ξ_i and φ_j are known, the characteristics Q_j (4.12a) and the currents P_i (4.12b) are determined.

Along solutions of a boundary value problem, all Euler-Lagrange equations should vanish

$$E_j(L) = \frac{\partial L}{\partial u_j} - \frac{d}{dx_i}\left(\frac{\partial L}{\partial u_{j,i}}\right) = 0, \tag{4.15}$$

and $\delta^* A = 0$ delivers with (4.11) a **conservation law**. With the divergence theorem we have

$$\int_B P_{i,i} dV = \int_S P_i n_i dA = 0, \tag{4.16}$$

provided that B is simply connected and does not contain singularities. The local form of (4.16) reads

$$P_{i,i} = \frac{dP_i}{dx_i} = \mathbf{div}\ P = 0. \tag{4.17}$$

In combination with (4.12), this is, in essence, Emmy Noether's theorem [14]. In order to give a first physical interpretation, we introduce the definition (2.9) and the constitutive law (2.8a) into (4.12b) yielding

$$\begin{aligned} -P_i &= \varphi_j \sigma_{ij} + \xi_j \left((W+V)\delta_{ij} - \sigma_{ik} u_{k,j}\right) \\ &= \varphi_j \sigma_{ij} + \xi_j b_{ij}. \end{aligned} \tag{4.18}$$

The tensor σ_{ij} is the physical momentum tensor or Cauchy-stress tensor, and

$$b_{ij} = (W+V)\delta_{ij} - \sigma_{ik} u_{k,i} \tag{4.19}$$

is the material momentum tensor or Eshelby-stress tensor, which both have been used already in the introduction.

If we recall the transformations (4.1) and take as transformations constant (physical) transformations (cf. the virtual translations δu in the introduction), i.e., $\varphi_i = c_j$ const. and $\xi_j = 0$, the physical momentum is conserved in the absence of body forces ($p_i = 0$)

$$P_{i,i} = 0 \quad \Rightarrow \quad \sigma_{ij,i} = 0. \tag{4.20}$$

On the other hand, if we take constant coordinate transformations, or material translations (cf. δx in the introduction), i.e., $\xi_j = c_j$ =const. and $\varphi_j = 0$, the material momentum is conserved in the absence of body forces ($V = 0$)

$$P_{i,i} = 0 \quad \Rightarrow \quad b_{ij,i} = 0 \tag{4.21}$$

which is easily verified by insertion.

Both conservation laws give rise to path-independent integrals [3]. In general, φ_j and ξ_j have to be determined from the condition (4.14).

5 Neutral-Action method

If a Lagrangian function is not available, and the system is given only by some set of partial differential equations

$$\Delta_i(x_j, u_k, u_{k,\ell}) = 0, \tag{5.1}$$

the Neutral-Action (NA) method [15] might be used to advantage. Firstly, we need the notion of a "null Lagrangian". If a Lagrange function $\tilde{L}$ is expressible as a divergence of a vector-valued function $g_i(x_j, u_k, u_{k,\ell})$ then it follows [13]

$$\tilde{L} = g_{i,i} \Leftrightarrow E_j(\tilde{L}) \equiv 0 \Leftrightarrow \delta\tilde{A} = 0, \tag{5.2}$$

i.e., the action integral $\tilde{A} = \int \tilde{L} dV$ is insensitive (or behaves neutrally with respect) to a (classical) variation δ of only the dependent variables u_i, and we arrive at a so-called trivial variational principle, which is valid independent of whether u_k are solutions of the governing differential equations or not.

Now, instead of the characteristics Q_i being specified by the transformation functions ξ_j and ϕ_i (cf. (4.12a), we determine Q_i, employing the symbol $-f_i$ instead (in order to avoid confusion), such that the product $f_i\Delta_i$ forms a null Lagrangian

$$f_i\Delta_i = P_{i,i}. \tag{5.3}$$

The functions f_i, therefore, have to be determined from

$$E_j(f_i\Delta_i) = 0. \tag{5.4}$$

As soon as suitable characteristics f_i are found from (5.4), the conserved currents P_i follow from (5.3), and due to (5.1), conservation laws in the form (4.16) are established.

It may be mentioned that the NA method to construct conservation laws might be applied also to systems possessing a Lagrangian. In that case, it leads to the same result as long as a unrestricted version of Noether's theorem [16] is employed together with the Bessel-Hagen extension [17].

6 Conservation laws of linear elasticity

We adopt the Navier-Lamé equations for a three-dimensional body made of a homogeneous isotropic material (Lamé constants λ and μ) in the absence of body forces

$$\Delta_i = \mu u_{i,jj} + (\lambda + \mu) u_{j,ji} = 0 \tag{6.1}$$

(cf. eq. (2.5)). We restrict the characteristics f_i to depend on the independent variables x_j, the dependent variables, i.e., the displacements u_k and the displacement gradients $u_{\ell,m}$

$$f_i = f_i(x_j, u_k, u_{\ell,m}). \tag{6.2}$$

Application of the Neutral-Action method leads to equations to determine f_k as

$$E_k(f_i\Delta_i) \equiv 0 \Rightarrow \mu f_{k,jj} + (\lambda + \mu) f_{j,jk}. \tag{6.3}$$

It turns out that the f_k are governed by the same differential equations as the displacements u_i (6.1) are, which is not surprising since the Navier-Lamé operator (6.1) is self-adjoint. If we consider two boundary-value problems $\{1\}$ and $\{2\}$ for the same body B with the solutions

$$u_i = \overset{\{1\}}{u_i}, f_i = \overset{\{2\}}{u_i}, \tag{6.4}$$

then from (5.3), Betti-Maxwell's reciprocity relations in physical space are recovered as

$$\int_{\partial B} \overset{\{1\}}{\sigma_{ji}} \, n_j \, \overset{\{2\}}{u_i} \, dA = \int_{\partial B} \overset{\{2\}}{\sigma_{ji}} \, n_j \, \overset{\{1\}}{u_i} \, dA \tag{6.5}$$

(cf. [3]) with the Cauchy stress tensors $\overset{\{1\}}{\sigma_{ji}}$ and $\overset{\{2\}}{\sigma_{ji}}$ of problem $\{1\}$ and $\{2\}$, respectively.

In order to reach further conclusions, the characteristics (6.2) are inserted into (6.3) and the differentiations have to be carried out in detail. We arrive at equations involving second- and third-order derivatives of the displacement fields. Since the characteristics f_i depend on derivatives up to the first order only, the coefficients of higher derivatives have to vanish. The results indicate (cf. [18], [19]) that f_i are linear in the displacements and the displacement gradients

$$f_i = f^1_{imn}(x_j) u_{m,n} + f^2_{im}(x_j) u_m + f^3_i(x_j). \tag{6.6}$$

Preceding further along this line of reasoning, the functional dependence of f_i can be restricted to

$$f^1_{ijk} = a(x_\ell)\varepsilon_{ijk} + b_k(x_\ell)\delta_{ij} + c_m(x_\ell)\left[(\lambda + 2\mu)\delta_{jk}\delta_{im} + \mu\delta_{ik}\delta_{jm}\right] \tag{6.7}$$

The scalar- and vector-valued quantities a and b_k, c_m, respectively, are functions of the independent variable x_ℓ and will be further restricted by comparing equal coefficients of terms involving different orders of derivatives of u_i (ε_{ijk} is the completely screw-symmetric permutation tensor).

The conservation laws resulting from $a\,(x_\ell)$ will be dealt with elsewhere and will not be considered further in what follows. We also discharge the quantities $f^3_i(x_j)$. They lead to physical conservation laws, which have been thoroughly discussed in [18]. Considering, for the moment, only the b_k-term, equation (6.6) reads as follows

$$f_i = b_k(x_\ell) u_{i,k} + f^2_{ik} u_k. \tag{6.8}$$

Comparing (6.8) with (4.12a) ($f_i = -\,Q_i$) we can identify

$$\begin{aligned} \xi_k(x_\ell, u_m) &= b_k(x_\ell), \\ \phi_i(x_\ell, u_m) &= -f^2_{ik}(x_\ell) u_k, \end{aligned} \tag{6.9}$$

i.e., b_k describe material transformations (see (4.1a)). From the one-dimensional theory of elasticity, i.e., tension and compression of bars, we know (cf. [3]) that these functions involve constant linear and quadratic terms in x_ℓ. Guided by this knowledge, we investigate the functional dependence of $b_k = b_k(x_\ell)$ further, and it can be shown (cf. [18]) that the characteristic (6.6) has the following appearance

$$
\begin{aligned}
f^1_{ijk} &= \delta_{ij} b_k \\
&= \delta_{ij}(\beta^{(0)}_k + \varepsilon_{nmk} x_m \beta^{(1)}_n + x_k \beta + (2x_k x_m - \delta_{km} x_n x_n)\beta^{(2)}_m), \\
f^2_{ij} &= \varepsilon_{ijk}\beta^{(1)}_k + \frac{n-2}{2}\delta_{ij}\beta + (n-2)\delta_{ij} x_m \beta^{(2)}_m.
\end{aligned}
\tag{6.10}
$$

The integer n designates the dimensionality of the problem, whether we treat a three-dimensional ($n = 3$), a two-dimensional (plane strain, $n = 2$) or a one-dimensional (tension and compression of a bar, $n = 1$) body.

The terms $\beta^{(0)}_k, \beta^{(1)}_n, \beta^{(2)}_m$ and β are vector-valued and scalar-valued constants, respectively.

Before we precede to the corresponding conservation laws, let us interpret the material translations in geometrical terms. Obviously, $\beta^{(0)}_k$ describe material translations (see Fig. 7a and the comments before eq. (4.21))

$$
\begin{aligned}
x_k &\mapsto x^*_k = x_k + \varepsilon \beta^{(0)}_k, \\
u_k &\mapsto u^*_k = u_k.
\end{aligned}
\tag{6.11}
$$

For reasons of clarity we sketch the transformations in the (x_1, x_2)-plane only ($\beta^{(0)}_3 = 0$).
Obviously again, $\beta^{(1)}_n$ describe material rotations

$$
\begin{aligned}
x_k &\mapsto x^*_k = x_k + \varepsilon\varepsilon_{nmk} x_m \beta^{(1)}_n, \\
u_k &\mapsto u^*_k = u_k + \varepsilon\varepsilon_{nmk} u_m \beta^{(1)}_n
\end{aligned}
\tag{6.12}
$$

whilst the displacement field is co-rotated. In the (x_1, x_2)-plane, $\beta^{(1)}_n$ has only one possible component $\beta^{(1)}_3 = \omega$, see Fig. 7b.
The constant β describes scaling (see Fig. 7c)

$$
\begin{aligned}
x_k &\mapsto x^*_k = x_k + \varepsilon\beta x_k, \\
u_k &\mapsto u^*_k = u_k + \varepsilon\tfrac{n-2}{2}\beta u_k.
\end{aligned}
\tag{6.13}
$$

Due to this transformation the body under consideration is expanded (or shrunk) self-similarly and translated (cf. Fig. 7c). The corresponding displacement transformation depends on the dimension of the problem. For plane strain ($n = 2$), u_k is not changed.

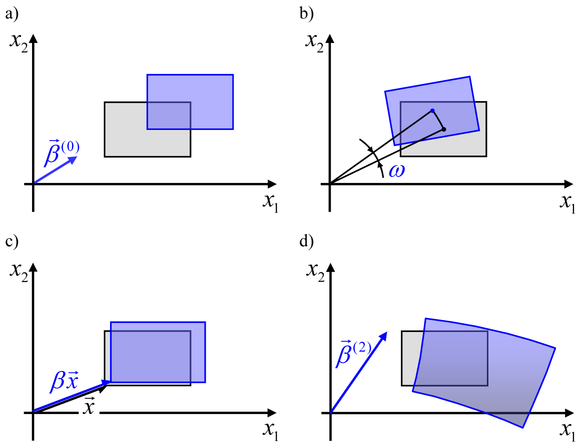

Figure 7: Material transformations: (a) translation, (b) rotation, (c) scaling, (d) inversion.

The transformation described by $\beta_m^{(2)}$ is rather strange. If we introduce the length of the vector $\vec{x} = x_i\vec{e}_i$ by $|\vec{x}| = (x_n x_n)^{1/2}$ and introduce unit vectors by $n_k = x_k / |\vec{x}|$, the transformation reads as

$$\begin{aligned} x_k \mapsto x_k^* &= x_k + \varepsilon\,|\vec{x}|^2\,(2n_k n_m - \delta_{km})\beta_m^{(2)}, \\ u_k \mapsto u_k^* &= u_k + \varepsilon(n-2)u_k x_m \beta_m^{(2)}. \end{aligned} \tag{6.14}$$

The matrix $W_{km} = 2n_k n_m - \delta_{km}$ is proper orthogonal

$$W^{-1} = W^T, \det W = +1, \tag{6.15}$$

and rotates the vector $\vec{\beta}^{(2)} = \beta_m^{(2)}\vec{e}_m$ around the position vector $\vec{x}$ by an angle of π (cf. [20], [21]). In addition, the vector is scaled by $|\vec{x}|^2$. This transformation is called "inversion" (cf. [22]). Fig. 7d shows a qualitative sketch for $\beta_1^{(2)} = 1$ and $\beta_2^{(2)} = 3$ in the (x_1, x_2) plane.
The four transformations $\beta_j^{(0)}$, $\beta_n^{(1)}$, β and $\beta_m^{(2)}$ lead to the four conservation and balance laws [18], respectively

$$\begin{aligned} &\text{Translation} && \beta_j^{(0)} \neq 0: && b_{ij,i} = \left[W\delta_{ij} - \sigma_{ik}u_{k,j}\right]_{,i} = 0, \\ &\text{Rotation} && \beta_n^{(1)} \neq 0: && \varepsilon_{nkj}\left[x_k b_{ij} + u_k\sigma_{ij}\right]_{,i} = 0, \\ &\text{Scaling} && \beta \neq 0: && \left[x_j b_{ij} + \tfrac{2-n}{2}u_j\sigma_{ij}\right]_{,i} = 0, \\ &\text{Inversion} && \beta_m^{(2)} \neq 0: && \big[(2x_m x_k - x_\ell x_\ell \delta_{mk})b_{ik} \\ &&&&& + \left(2x_k u_m + (2-n)x_m u_k - 2x_\ell u_\ell \delta_{mk}\right)\sigma_{ik} \\ &&&&& + n\mu\,\left(u_m u_i + \tfrac{1}{2}\,u_\ell u_\ell \delta_{mi}\right)\big]_{,i} \\ &&&&& = (n\lambda + (4+n)\mu)\,u_{\ell,\ell}u_m. \end{aligned} \tag{6.16}$$

The term b_{ij} is, again, the Eshelby-stress tensor involving the strain-energy density W. On integration over the volume V of a body B and application of the divergence theorem, equation (6.16a) gives rise to Rice's J-integral [23], which describes the energy-release rate due to the translation of a material inhomogeneity within the body. In a similar way, (6.16b) and (6.16c) resemble the $L-$ and $M-$ integrals introduced in [24], but discussed much earlier in [11]. The integrals L and M indicate the energy-release rates due to a rotation and self-similar expansion of the inhomogeneity, respectively.

Inversion does not give rise to a conservation law but rather a (more or less) trivial balance law. The right-hand side of (1.16d) vanishes either under the unphysical condition $(n = 3): \ 3\lambda + 7\mu = 0$, i.e., Poisson's ratio $\nu = 7/8$ or for the special case of an isochoric deformation, i.e., $u_{k,k} = 0$.

We turn back now to equation (6.7) and realize that, first of all, the transformations involving $c_m(x_\ell)$ follow from Noether's theorem only, if we admit an unrestricted or extended form of the transformation (4.1), cf. [16].

Secondly, the transformation coefficients c_m are scaled with the material constants λ and μ of the elastic body under consideration. Finally, the transformations c_k leading to conservation or balance laws have a similar form as the transformations b_k have, cf. (6.10), and so have the governing conservation laws (cf. [18])

$$\begin{aligned}
&\text{Translation} \quad \gamma_i^{(0)} \neq 0 : c_{ji,j} = 0,\\
&\text{Rotation} \quad \gamma_i^{(1)} \neq 0 : \varepsilon_{ik\ell}\left[(\lambda+\mu)x_k c_{j\ell} + \mu(\lambda+3\mu)u_k\sigma_{j\ell}\right.\\
&\qquad\qquad \left. + 2\mu^3 u_k\left(u_{m,m}\delta_{j\ell} - u_{m,\ell}\delta_{jm}\right)\right]_{,j} = 0,\\
&\text{Scaling} \quad \gamma \neq 0 : \left[x_i c_{ji} + \mu u_i\left(\sigma_{ij} + \mu(u_{k,k}\delta_{ij} - u_{j,i})\right)\right.\\
&\qquad\qquad \left. + \mu^2\left(u_j u_{i,i} - u_i u_{j,i}\right)\right]_{,j}\\
&\qquad\qquad = \tfrac{1}{2}\left(n\lambda + (n+4)\mu\right)(\lambda+2\mu)u_{i,i}u_{j,j}
\end{aligned} \tag{6.17}$$

$$\begin{aligned}
&\text{Inversion} \quad \gamma_\ell^2 \neq 0 : \quad \Big[\left(x_k x_\ell - \tfrac{1}{2}x_n x_n\delta_{k\ell}\right)c_{mk} + \mu(x_k u_\ell - x_\ell u_k - x_n u_n\delta_{k\ell})\sigma_{mk}\\
&\quad + 2\mu^2(x_m u_{k,k}u_\ell + x_\ell u_{k,m}u_k - x_n u_{k,k}u_n\delta_{m\ell}) + 2\mu(\lambda+\mu)x_\ell u_{k,k}u_m\\
&\quad + \frac{2\mu^2(\lambda+2\mu)}{\lambda+\mu}u_m u_\ell + \frac{2\mu^3}{\lambda+\mu}x_k(u_{k,m}u_\ell - u_{\ell,m}u_k)\Big]_{,m}\\
&\quad = (n\lambda + (n+4)\mu)(\lambda+2\mu)u_{k,k}\left[\frac{1}{2}x_\ell u_{m,m} + \frac{\mu}{\lambda+\mu}u_\ell\right].
\end{aligned}$$

with f^1_{ijk} and f^2_{ij} given in this case as

$$\begin{aligned}
f^1_{ijk} &= c_m\left[(\lambda+2\mu)\delta_{jk}\delta_{im}+\mu\delta_{ik}\delta_{jm}\right], \\
c_m &= \gamma^{(0)}_m+\varepsilon_{knm}x_n\gamma^{(1)}_k+x_m\gamma+\left(x_mx_\ell-\frac{1}{2}x_nx_n\delta_{m\ell}\right)\gamma^{(2)}_\ell, \\
c_m &= \gamma^{(0)}_m+\varepsilon_{knm}x_n\gamma^{(1)}_k+x_m\gamma+\left(x_mx_l-\frac{1}{2}x_nx_n\delta_{ml}\right)\gamma^{(2)}_l, \\
f^2_{ij} &= \frac{\mu(\lambda+3\mu)}{(\lambda+\mu)}\varepsilon_{ijk}\gamma^{(1)}_k+\mu\frac{n-2}{2}\delta_{ij}\gamma+\mu\frac{n-2}{2}\delta_{ij}x_m\gamma^{(2)}_m \qquad (6.18)
\end{aligned}$$

(Note that some minor flaws have been corrected and some terms have been specified in comparison to [19]).

The tensor c_{ij} is given in displacement gradients as

$$\begin{aligned}
c_{ij} = {} & \tfrac{1}{2}(\lambda+2\mu)(\lambda+\mu)u_{k,k}u_{\ell,\ell}\delta_{ij}+\mu^2u_{j,k}(u_{k,i}-u_{i,k}) \\
& +\mu(\lambda+2\mu)u_{k,k}u_{j,i}
\end{aligned} \qquad (6.19)$$

and coincides with Q_{ji} in Olver's paper [22]. It will be applied to a crack in Section 9. Rotation leads again to a conservation law (6.17b), whilst scaling (6.17c) and inversion (6.17d) yield rather balance laws, the right-hand side being proportional to the same factor $3\lambda+7\mu$ as discussed above.

7 Conservation laws for bars in tension / compression

For later use, we complete the list of correspondence (2.11) between the three-dimensional theory of elasticity and the one-dimensional bar theory by

$$\begin{aligned}
&\mu \to \tfrac{1}{2}EA, \\
&\lambda \to 0.
\end{aligned} \qquad (7.1)$$

The material momentum tensor (Eshelby-stress tensor) b_{ij} specializes to the material force B, which has already been mentioned in the introduction. In the absence of a load in axial direction per unit of length $n=0$, i.e., $\hat{V}=0$, we find

$$b_{ij}\to B=\hat{W}-Nu^{/}. \qquad (7.2)$$

The conservation laws (6.16) transform with (2.11). (2.12) and (7.1) to

$$\begin{aligned}
&\text{Translation} && B^{/}=0, \\
&\text{Scaling} && \left(xB+\tfrac{1}{2}Nu\right)^{/}=0, \\
&\text{Inversion} && \left(x^2+xNu+\tfrac{3}{4}EAu^2\right)^{/}=\tfrac{5}{2}EAu^{/}u.
\end{aligned} \qquad (7.3)$$

Material rotation cannot be applied within a one-dimensional theory without leaving a bar axis. The right-hand side of (7.3c) can be transformed to a "one dimensional divergence" as

$$\frac{5}{2}EAu'u = \frac{5}{4}(EAu^2)', \tag{7.4}$$

and can be turned to the left-hand side, giving raise to a conservation rather than a balance law. With (2.12), B from (7.2) can be rewritten, and abbreviations may be introduced as

$$\begin{aligned} B &= -\tfrac{1}{2}EAu'^2, \\ H &= -\tfrac{1}{2}Nu = -\tfrac{1}{2}EAu'u, \\ R &= -\tfrac{1}{2}EAu^2. \end{aligned} \tag{7.5}$$

Thus equations (7.3) can be transformed to

$$\begin{aligned} B' &= 0, \\ (Bx - H)' &= 0, \\ (Bx^2 - 2Hx + R)' &= 0. \end{aligned} \tag{7.6}$$

The conservation and balance laws (6.17) convert all to conservation laws, but they do not provide any new information, since

$$c_{ij} \to C = -\frac{3}{2}EA\,B, \tag{7.7}$$

and the equations (6.17) transform to (7.6) merely multiplied by $-3/2EA$.

It may be further mentioned that also crack interaction problems might be easily assessed on the basis of these considerations ([25]-[27]).

The physical interpretation of the material force B and the first- and second -order material virials H and R, respectively, are given in [3]. The application of the conservation laws (7.6) to bars with cracks yield remarkable simple formulae to estimate the associate stress-intensity factors. The theory can easily extended to beams in bending and shafts in torsion. Details, further references and several examples of application may be found in [3].

8 Dislocation / hole interaction

As an example of a defect interaction problem let us consider an infinite plane (x_1, x_2) with a circular hole (radius r_0) placed into the eigenstress field of a dislocation (Burgers-vector b). As depicted in Fig. 8, the dislocation ($b_1 = 0,\ b_2 = b$) is fixed at the origin of a cartesian or polar coordinate system, and the mid-point of the circular hole is placed at an arbitrary position $\xi_1,\ \xi_2$ or

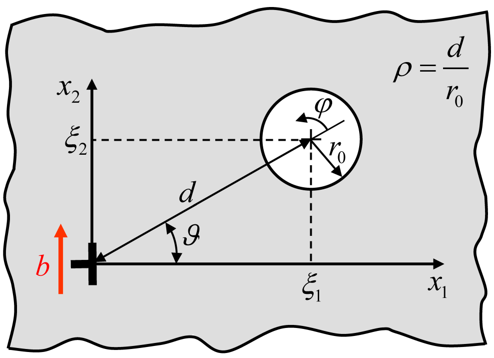

Figure 8: Dislocation / hole interaction

d, ϑ, respectively ($d = \sqrt{\xi_1^2 + \xi_1^2}) > r_0$).The dimensionless distance is denoted by $\rho = d/r_0$, $L > 1$.
The initial configuration is changed by material transformations, and so is the total energy of the system. The translation of the hole (relatively to the dislocation) in x_1-direction by an amount λ_1 is shown in Fig. 9a.

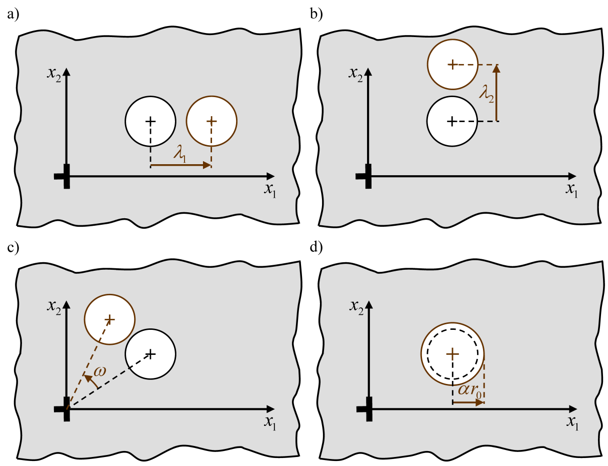

Figure 9: Material transformations of the hole relatively to the dislocation: material translation in x_1 and x_2 directions λ_1 (a) and λ_2 (b), respectively, material rotation with respect to the origin ω (c), self-similar expansion of the hole $r_0 \to \alpha r_0$ (d).

The material transformations are considered infinitesimal small but sketched for reasons of visuality on a finite scale.

As mentioned in section 6, the negative rate of change of the total energy is calculated by the J-integral

$$-\frac{\partial \Pi}{\partial \lambda_1} = J_1 = \oint_{\Gamma} b_{j1}\, n_j ds. \tag{8.1}$$

The line integral is performed along a contour Γ with are length s and unit outward normal vector n. Since the J-Integral is path independent, any contour Γ may be chosen which surrounds the hole (but not the inclusion) completely.

Accordingly, a material translation in x_2-direction by an amount λ_2 (see Fig. 9b) yields

$$-\frac{\partial \Pi}{\partial \lambda_2} = J_2 = \oint_\Gamma b_{j2}\ n_j ds. \tag{8.2}$$

Material rotation in the plane is possible only around the x_3-axis (see Fig. 9c). The vector-valued L-Integral has only the component L_3 and is given as

$$-\frac{\partial \Pi}{\partial \omega_3} = L_3 =: L^{\perp} = \oint_\Gamma \varepsilon_{3kj}\left[x_k b_{ij} + u_k \sigma_{ij}\right] n_i ds. \tag{8.3}$$

In the following, we will consider a rotation ω around the origin of the coordinate system. In contrary, we will consider the self-similar expansion of the hole, $r_0 \to \alpha r_0$ with respect to the center of the hole (Fig. 9d). The energy change is described by $(n = 2)$

$$-\frac{\partial \Pi}{\partial (\alpha - 1)} = M =: M^{\circ} = \oint_\Gamma x_j\ b_{ij} n_i ds. \tag{8.4}$$

The M-Integral is a material virial, i.e., a moment of the kind $r \bullet F$ instead of an angular moment $r \times F$. Material inversion will not be considered. Like in strength-of-materials courses we can sketch free-body diagrams also for material forces/moments as depicted in Fig. 10

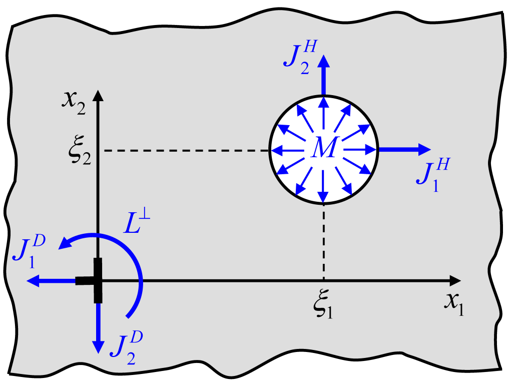

Figure 10: Free body diagram for material quantities

Material equilibrium requires

$$\begin{aligned} \rightarrow &: \quad J_1^H - J_1^D = 0, \\ \uparrow &: \quad J_2^H - J_2^D = 0, \\ \widetilde{\mathbf{N}} &: \quad L^{\perp} + \xi_1 J_2^H - \xi_2 J_1^H = 0, \\ \leftrightarrow\!\updownarrow &: \quad M^{\circ} + \xi_1 J_1^D + \xi_2 J_2^D = 0. \end{aligned} \tag{8.5}$$

The integrals obtain a quite simple form, if we choose as integration path the contour along the rim of the hole. The only nonvanishing stess component is $\sigma_{\varphi\varphi}$, and the (complementary) strain-energy density W is given by

$$W = \frac{1}{2E^*}\sigma^2_{\varphi\varphi}(\varphi). \tag{8.6}$$

The constant E^* is given with Young's modulus E and Poisson's ratio ν by

$$E^* \begin{cases} \dfrac{E}{1-\nu^2} & \text{for plane strain} \\ E & \text{for plain stress} \end{cases} . \tag{8.7}$$

The integrals reduce to

$$\begin{aligned} J_1 &= \frac{r_0}{2E^*}\int_0^{2\pi} \sigma^2_{\varphi\varphi}(\varphi)\cos\varphi\, d\varphi, \\ J_2 &= \frac{r_0}{2E^*}\int_0^{2\pi} \sigma^2_{\varphi\varphi}(\varphi)\sin\varphi\, d\varphi, \\ M^\circ &= \frac{r_0^2}{2E^*}\int_0^{2\pi} \sigma^2_{\varphi\varphi}(\varphi)\, d\varphi, \\ L^? &= \frac{r_0 d}{2E^*}\int_0^{2\pi} \sigma^2_{\varphi\varphi}(\varphi)\sin\varphi\, d\varphi. \end{aligned} \tag{8.8}$$

In [28] a simple formula is derived to obtain the hoop stresses at the boundary of a stress-free circular hole from the stress distribution that would exist along the boundary of the hole in its absence.

We adopt the Volterra solution for the eigen-stress field of a dislocation placed at the origin of the coordinate system of an infinite plane, which may be found in textbooks (cf. e.g., [3]). Let $\sigma^{(0)}_{rr}(r_0,\varphi)$ and $\sigma^{(0)}_{\varphi\varphi}(r_0,\varphi)$ be the stresses calculated from the Volterra solution along the curve coinciding with the boundary of the prospective hole (r,φ polar coordinates wich respect to the center of the hole) and $I_1^{(0)}$ the first stress invariant at the center of the prospective hole, i.e.,

$$I_1^{(0)} = \left(\sigma^{(0)}_{11} + \sigma^{(0)}_{22}\right)\Big|_{\substack{x_1=\xi_1 \\ x_2=\xi_2}} . \tag{8.9}$$

The hoop stress $\sigma_{\varphi\varphi}(\varphi)$ at the boundary of the (now present) hole due to the applied load (in this case the eigen-stress field of the dislocation) follows from

$$\sigma_{\varphi\varphi}(\varphi) = I_1^{(0)} + 2\left(\sigma^{(0)}_{\varphi\varphi}(r_0,\varphi) - \sigma^{(0)}_{rr}(r_0,\varphi)\right). \tag{8.10}$$

For the dislocation / hole interaction problem, we find (cf., e.g., [3], [28], [29])

$$\begin{aligned} \sigma_{\varphi\varphi}(\varphi) = \frac{bE^*}{2\pi r_0\rho}\Bigg\{ & \frac{\xi_1}{r_0}\left[\frac{1}{\rho} - 4\sin^2\varphi\frac{1+\rho\cos\varphi}{\left[\rho^2+1+2\rho\cos\varphi\right]^2}\right] \\ & + \frac{\xi_2}{r_0}\sin\varphi\left[\left(\frac{\rho^2-1}{\rho^2+1+2\rho\cos\varphi}\right)^2 - 1\right]\Bigg\}. \end{aligned} \tag{8.11}$$

With (8.8) and (8.11), the integrals can be readily evaluated as [30]

$$
\begin{aligned}
J_1 &= -\frac{b^2E^*}{4\pi r_0}\frac{\cos\vartheta}{\rho^3(\rho^2-1)}\left[\rho^2+2\sin^2\vartheta(\rho^2-1)\right], \\
J_2 &= -\frac{b^2E^*}{4\pi r_0}\frac{\sin\vartheta}{\rho^3(\rho^2-1)}\left[1+2\sin^2\vartheta(\rho^2-1)\right],
\end{aligned}
\tag{8.12}
$$

or by transformation in r, ϑ-direction (see Fig. 11)

$$
\begin{aligned}
J_r &= -\frac{b^2E^*}{4\pi r_0\rho}\left[\frac{1}{\rho^2-1}+\frac{\sin^2\vartheta}{\rho^2}\right], \\
J_\vartheta &= +\frac{b^2E^*}{8\pi r_0\rho}\frac{\sin 2\vartheta}{\rho^2},
\end{aligned}
\tag{8.13}
$$

and further

$$
\begin{aligned}
L^\perp &= -\frac{b^2E^*}{8\pi}\frac{\sin 2\vartheta}{\rho^2}, \\
M^\circ &= +\frac{b^2E^*}{4\pi}\left[\frac{1}{\rho^2-1}+\frac{\sin^2\vartheta}{\rho^2}\right].
\end{aligned}
\tag{8.14}
$$

The material equilibrium conditions (8.5) can easily be verified.

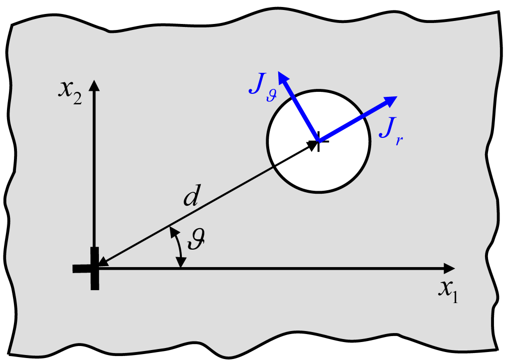

Figure 11: Interaction forces between circular hole and dislocation, polar directions

The closed-form analytical equations (8.12) - (8.14) can be used to study material reciprocity relations for defect interactions [30]. Material reciprocity relations as counterpart to the (physical) Betti-Maxwell reciprocity relations have been introduced quite recently [31] - [33].

From (8.13), it can be seen that the force between the dislocation and the hole is always a force of attraction ($J_r < 0$). Thus, if the hole or the dislocation could move, the distance between them would decrease.

We would like to consider next the shape of trajectories envisaging the possibility that the cavity, by means of some mechanism (e.g., diffusion) could move towards the dislocation. Lacking equations of motion (of the type of Newton's second law for mass points, where, in general, the force is not tangential to the trajectory), it is usually assumed that the material force is tangential to the trajectory, i.e., path of motion. The trajectories of possible motion of the cavity have thus to be determined from the differential equation

$$\frac{dx_2}{dx_1} = \frac{J_2}{J_1}, \tag{8.15}$$

or in polar coordinates

$$\frac{d\rho}{d\varphi} = \rho\frac{J_r}{J_\varphi}. \tag{8.16}$$

If the distance between dislocation and hole is large, i.e., $\rho > 1$, the differential equation can readily be integrated, leading to

$$\rho = \rho_0 \frac{\sin\varphi_0}{\cos^2\varphi_0}\frac{\cos^2\varphi}{\sin\varphi}. \tag{8.17}$$

Each trajectory is specified by the choice of φ_0 and ρ_0. Some of the trajectories, with the restriction $\rho \gg I$, are sketched in Figure 12

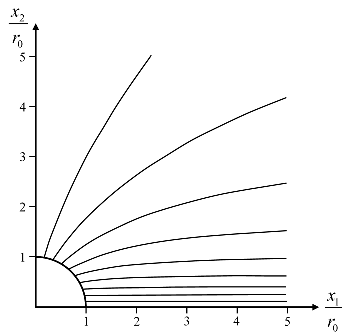

Figure 12: Trajectories of motion of a cavity in a stress field due to a dislocation

Concluding this Section, we consider the stability of material equilibrium of a circular hole in the stress field of two symmetric dislocations as depicted in Figure 13.

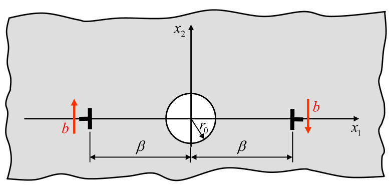

Figure 13: Interaction of a circular hole with two dislocations

Since the theory is linear, the stress fields of the two dislocations may be superimposed. The results for J_1, J_2 or J_r, J_φ , however, do not follow from superposition, because they are quadratic forms in $\sigma_{\varphi\varphi}$ (8.8) and an interaction term will occur. Due to the symmetry of the problem it can be concluded, however, that $\xi_1 = \xi_2 = 0$ corresponds to an equilibrium position. The material force in the x_2-direction is zero and the material forces towards the dislocations in the x_1-direction are equal but opposite. If the hole is shifted now by a small amount along the x_1-axis to the right, say, the attracting force

of the right dislocation becomes larger than that of the left. The hole will thus move further to the right. Therefore, the equilibrium position is unstable with respect to x_1.
If the hole is shifted, on the other hand, by a small amount in the x_2-direction, the x_1-components of the forces exerted by the two dislocations are still equal and opposite. The forces, however, are now inclined with respect to the x_1-axis, both components in x_2-direction add and drive the hole back into the original position. Thus, the equilibrium position is stable with respect to x_2. The total potential energy given as function of the position of the hole $\Pi(\xi_1, \xi_2)$ (influence surface), therefore, possesses a saddle point at $\xi_1 = \xi_2 = 0$ and this equilibrium position is overall unstable.

The corresponding physical stability problem is sketched in Fig. 14.

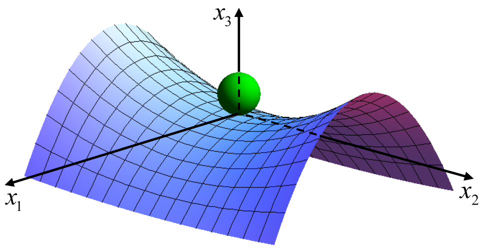

Figure 14: Mass point m on a saddle-shaped surface in a field of gravity (acting in negative x_3-direction)

9 Application in fracture mechanics

For later use, we modify the tensor c_{ij} (6.19) by a linear combination with b_{ij} (6.16a) and a trivial conservation law t_{ij}, i.e., a conservation law which is satisfied independently of whether or not the displacement field u_i satisfies the Navier-Lamé equations (6.1)

$$\begin{aligned} t_{ij} &= \varepsilon_{i\ell n}\varepsilon_{jkm}u_{k,\ell}u_{m,n} \\ t_{ij,i} &\equiv 0. \end{aligned} \tag{9.1}$$

The resulting tensor is called d_{ij} and is defined as

$$d_{ij} = \frac{\lambda + \mu}{2(\lambda + 2\mu)}\left(b_{ij} + \frac{1}{\mu}c_{ij} + \mu t_{ij}\right). \tag{9.2}$$

This tensor has been derived in a different way in [34]. Replacing the Lamé constants λ and μ by Young's modulus E and Poisson's ratio ν via

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}, \mu = \frac{E}{2(1+\nu)} \tag{9.3}$$

and writing b_{ij} and d_{ij} in terms of displacement gradients

$$b_{ij} = \frac{E}{4(1+\nu)(1-2\nu)} \{\delta_{ij} [2\nu u_{\ell,\ell} u_{m,m} + (1-2\nu)\ u_{\ell,m}(u_{\ell,m} + u_{m,\ell})] \\ -2 [2\nu u_{\ell,\ell} u_{i,j} + (1-2\nu) u_{\ell,j}\ (u_{\ell,i} + u_{i,\ell})]\},$$

$$d_{ij} = \frac{E}{4(1+\nu)(1-2\nu)} \left\{ \frac{1}{2}\delta_{ij} \left[\frac{2(1-\nu)}{1-2\nu}\ u_{\ell,\ell} u_{m,m} + \frac{1-2\nu}{2(1-\nu)} u_{\ell,m}(u_{\ell,m} - u_{m,\ell}) \right] - \left[u_{\ell,\ell}(u_{i,j} - u_{j,i}) + \frac{1-2\nu}{2(1-\nu)} (u_{i,\ell} - u_{\ell,i})(u_{j,\ell} - u_{\ell,j}) \right] \right\}, \tag{9.4}$$

it turns out that b_{ij} and d_{ij} have quite a similar appearance.

The Eshelby tensor b_{ij} serves as integrand of Rice's J-integral as

$$J_i = \int_S b_{ji} n_j dA\ . \tag{9.5}$$

Accordingly, we introduce an N-integral in which d_{ij} serves as integrand

$$N_i = \int_S d_{ji} n_j dA\ . \tag{9.6}$$

In plane fracture mechanics, the J-integral is used to calculate stress-intensity factors K_I and K_{II} (cf., e.g., [35]). On evaluating both integrals along a path Γ within the near-crack-tip field around a crack tip under mixed-mode-loading conditions in plane elasticity (see Fig. 15) it turns out that the following relations hold

$$J_1 = \frac{K_I^2 + K_{II}^2}{E^*}, \quad N_1 = \frac{K_I^2 - K_{II}^2}{E^*}, \\ J_2 = -\frac{2K_I K_{II}}{E^*}, \quad N_2 = -\frac{2K_I K_{II}}{E^*}, \tag{9.7}$$

with E^*, as before,

$$E^* = \begin{cases} E & \text{for plane stress,} \\ \dfrac{E}{1-\nu^2} & \text{for plane strain.} \end{cases} \tag{9.8}$$

As discussed in [19], linear combinations of J_1 and N_1 provide favorable tools to calculate K_I and K_{II} separately

$$K_I = \sqrt{\tfrac{E^*}{2}(J_1 + N_1)}, \\ K_{II} = \sqrt{\tfrac{E^*}{2}(J_1 - N_1)}.$$

Also, advantages in the numerical implementation and the obtained accuracy are reported.

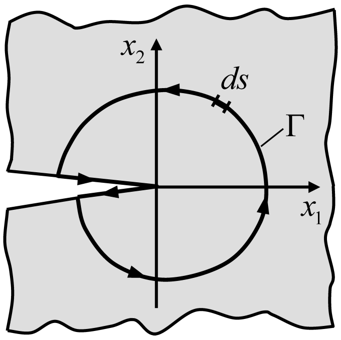

Figure 15: Integration path Γ in the vicinity of a crack tip under mixed-mode-loading conditions in plane elasticity.

10 Conclusions

The presented lecture notes aim at creating interest in the Mechanics of Material Space or Configurational Mechanics. Whereas configurational forces are well established in fracture mechanics, further applications have been dealt with quite recently, e.g., the accuracy of numerical computations can be improved by modifying the grid in order to minimize spurious configurational forces. Also the applications in damage mechanics, plasticity, phase-transformation and phase-transition problems are of ongoing interest. Gérard Maugin, a most active proponent of the subject put it in the following phrase: "..., it contributes one of the latest and most fruitful advances in macroscopic field theories, an area that may have considered a completely closed field of research offering no further progress and therefore no true scientific interest, for quite a long time" [4, page 6].

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