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ON METAHARMONIC FUNCTIONS

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Abstract. In this work the author studies some of the principal questions connected with the equation of the form

$$\Delta^n U + a_1 \Delta^{n-1} U + ... + a_n U = 0, \tag{A}$$

where $a_1, ..., a_n$ are constants, Δ is Laplace's operator in the Euclidean p-space ($p \geq 2$). The regular solution of this equation is called an *n-metaharmonic function.* In particular, the regular solution of the equation

$$\Delta U + \lambda^2 U = 0 \qquad (\lambda^2 = const) \tag{M}$$

is called a *metaharmonic function.*

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Introduction of the Editor

Nowadays, it is well known that the problem of scattering by a bounded obstacle (or obstacles) in free space is well posed provided the scattering surface satisfies certain smoothness conditions. That is, there exists a unique solution that is stable with respect to changes in boundary data. Mathematically the problem is to find a solution to the Helmholtz equation

$$(\nabla^2 + k^2)u(x) = 0 \quad \text{for} \quad x \in \mathbb{R}^n \setminus \overline{D}_i, \tag{1}$$

where $x = (x_1, x_2, \cdots, x_n)$ denotes a position vector in $\mathbb{R}^n$, n is the dimension of the space, D_i denotes a bounded, simply connected domain (or a finite number of such domains) with boundary S and $\overline{D}_i = D_i \cap S$ denotes the closure of D_i. The surface S is assumed to be smooth enough to permit the use of the Gauss-Green theorems.

When k is real, the scattered field $u(x)$ is required to satisfy a boundary condition on S and in order to establish uniqueness additional restrictions are needed which have been formulated first time by A.Sommerfeld in 1912 [Som],

$$\left| r^{\frac{n-1}{2}} u(x) \right| = \mathcal{O}(1) \quad \text{as} \quad r = |x| \to \infty, \tag{2}$$

$$\frac{\partial u(x)}{\partial r} - i\,k\,u(x) = o\left(r^{-\frac{n-1}{2}} \right) \quad \text{as} \quad r = |x| \to \infty. \tag{3}$$

The first condition is the so called *finiteness or boundedness condition*, while the second one is the well known *Sommerfeld radiation condition.*

The literature on the Helmholtz equation and radiation conditions is vast (see e.g., [CK], [Nat], [KA] and the references therein).

We would like to treat here only some historical notes concerning the two very important and fundamental papers by F.Rellich [Rel] and I.Vekua [Vek] which appeared simultaneously in 1943.

In both papers it is shown that the Sommerfeld radiation condition (3) implies the boundedness condition (2) and, what is very important and fundamental in the theory of Helmholtz equation, it is proved first time that any solution of homogeneous equation (1) satisfying the condition

$$\lim_{R \to \infty} \int\limits_{\Sigma(R)} |u(x)|^2 d\Sigma(R) = 0, \tag{4}$$

where $\Sigma(R)$ is the surface of a ball of radius R, identically vanishes in $\mathbb{R}^n \setminus \overline{D}_i$.

This lemma plays a crucial role in both the direct and inverse problems. The lemma had been known in the western literature as *Rellich's lemma* since many authors were unacquainted with the paper by I.Vekua which have been published in Russian with an extended summary. Only recently appeared the term *Rellich-Vekua lemma* (and sometimes even *Vekua-Rellich lemma*, see [KA]).

Beside the uniqueness theorems for the basic exterior Dirichlet and Neumann boundary value problems (BVP) which are proved with the help of the fundamental lemma, in the above mentioned paper, I.Vekua studied the existence of solutions by reduction of the BVPs to Fredholm type integral equations. Unfortunately, these equations have countable spectrum with respect to the oscillation parameters. Therefore, the boundary integral equations obtained are not equivalent to the original BVPs for all values of the oscillation parameter. Such types of situations always appear when the direct method is employed, i.e. when the solutions are sought in the form of either a single- or a double-layer potential. To investigate the solvability of the above integral equations one needs to find all eigenvalues and eigenfunctions of the corresponding homogeneous integral equations and their adjoint ones, which is inefficient from the practical point of view. In 1965 this disadvantage has been overcome by several authors simultaneously (see [BW], [Leis], [Pan]) and the BVPs have been reduced to equivalent uniquely solvable integral equations with simple kernel functions explicitly written in terms of fundamental solution of the Helmholtz equation.

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Introduction

The present paper deals with a number of problems connected with the equation

$$\Delta^n U + a_1 \Delta^{n-1} U + \cdots + a_n U = 0 \qquad (A)$$

where $a_1, \cdots, a_n$ are in general complex constants, Δ is the Laplace operator in Euclidean space of p dimensions E_p, i.e. in Cartesian coordinates,

$$\Delta \equiv \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial_2^2} + \cdots + \frac{\partial^2}{\partial x_p^2}, \qquad p \geq 2,$$

$$\Delta^m \equiv \Delta(\Delta^{m-1}), \quad \Delta^0 \equiv 1.$$

Let T be a domain in E_p. The function $U(X)$ of a point X of the domain T, taking, in general, complex values, will be called a n-metaharmonic function or regular solution of equation (A), if it is single-valued and continuous together with its partial derivatives up to and including the $2n$-th order and satisfies equation (A) in the domain T. The coefficients $a_1, \cdots, a_n$ of (A) will be called the parameters of the n-metaharmonic function.

When $a_k = 0$ $(k = 1, \cdots, n)$ the n-metaharmonic function will be said to be n-harmonic. In the case $n = 1$ we describe the n-metaharmonic function as simply metaharmonic, and the n-harmonic function as harmonic.

The paper contains 6 sections, and is divided into two parts. The first part (§§$1-4$) is devoted to the study of basic properties of the solutions of the metaharmonic equation

$$\Delta U + \lambda^2 U = 0 \quad (\lambda = \text{const.}), \qquad (M)$$

and the second part (§§$5-6$) is devoted to the problems concerning the equation (A) with $n > 1$.

Some elementary solutions of equation (M) and their properties are discussed in § 1.

In §2, Green's formulae are derived, giving the integral representations of metaharmonic functions both in finite and infinite domains. In this connection the so-called "Sommerfeld conditions" are stated in a generalized form.

In §3, the expansions of metaharmonic functions into series in Hankel and Hyperspherical functions are given. To make things easier for the reader, the section starts with a detailed treatment of the basic properties of hyperspherical functions. The section ends with proofs of a number of results on the nature of the behaviour of metaharmonic functions at infinity.

In §4, the Dirichlet and Neumann problems are solved for equation (M) in the case of an infinite domain. The existence and uniqueness of solutions of these problems are proved for any boundary data, provided the so-called Sommerfeld conditions are satisfied at infinity.

In §5, the general representation of all solutions of equation (A) is obtained in terms of metaharmonic functions.

In §6, the results in §§4, 5 are used to solve the so-called Riquier problem for equation (A) in the cases of both a finite and an infinite domain.

1 Some Fundamental Properties of Metaharmonic Functions

This part of our paper will be devoted to a study of some fundamental properties of solutions of the equation

$$\Delta U + \lambda^2 U = 0, \tag{M}$$

where λ is in general a complex-valued constant. In accordance with our introductory remarks, every regular solution of this equation will be called a metaharmonic function with the parameter λ.

§ 1. Elementary solutions of equation (M)

1. In our paper very important role will be played in what follows by the so-called elementary solutions of equation (M). We will therefore discuss some of them in this section.

We introduce the polar coordinates

$$\begin{aligned}
&x_1 = r\sin\theta_{p-1}\sin\theta_{p-2}\cdots\sin\theta_2\cos\theta_1\\
&x_2 = r\sin\theta_{p-1}\sin\theta_{p-2}\cdots\sin\theta_2\sin\theta_1\\
&x_3 = r\sin\theta_{p-1}\sin\theta_{p-2}\cdots\cos\theta_2\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\
&x_{p-1} = r\sin\theta_{p-1}\cos\theta_{p-2}\\
&x_p = r\cos\theta_{p-1}
\end{aligned} \tag{1}$$

where

$$r \geq 0,\ \ 0 \leq \theta_1 < 2\pi,\ \ 0 \leq \theta_k \leq \pi\ \ (k = 2, 3, \cdots, p-1).$$

For brevity, we will denote the point with polar coordinates $r, \theta_1, \cdots, \theta_{p-1}$ by (r, Θ), and the function $f(r, \theta_{p-1}, \cdots, \theta_1)$ by $f(r, \Theta)$. In particular, we will denote a point on the unit hypersphere $r = 1$, $(1, \Theta)$ simply by Θ, and a function of this point by $f(\Theta)$.

As is well known, equation (M) in polar coordinates takes the form

$$\frac{\partial^2 U}{\partial r^2} + \frac{p-1}{r}\frac{\partial U}{\partial r} + \frac{1}{r^2}\Lambda U + \lambda^2 U = 0, \tag{M'}$$

where

$$\Lambda U = \frac{1}{h} \sum_{i=1}^{p-1} \frac{\partial}{\partial \theta_i} \left(\frac{h}{h_i} \frac{\partial U}{\partial \theta_i} \right), \tag{2}$$

and

$$\begin{aligned} &h_{p-1} = 1, \;\; h_{p-2} = \sin^2 \theta_{p-1}, \cdots, \;\; h_1 = \sin^2 \theta_{p-1} \sin^2 \theta_{p-2} \cdots \sin^2 \theta_2, \\ &h = \sin^{p-2} \theta_{p-1} \sin^{p-3} \theta_{p-2} \cdots \sin \theta_2. \end{aligned} \tag{3}$$

Notice that the operator Λ is self-adjoint on the unit hypersphere Σ_1 with the centre at the origin, i.e. for any two functions $U(\Theta)$ and $V(\Theta)$ of the point Θ on the unit hypersphere, continuous along with their partial derivatives of the first two orders, we have

$$\int_{\Sigma_1} U \Lambda V d\Sigma_1 = \int_{\Sigma_1} V \Lambda U d\Sigma_1. \tag{4}$$

It can be proved very simply by integrating by parts and using the formula

$$d\Sigma_1 = h d\theta_{p-1} d\theta_{p-2} \cdots d\theta_1, \tag{5}$$

for an element of the area of the unit hypersphere [1].

Let τ_p and σ_p be respectively the volume and surface area of the unit hypersphere in the space E_p. Using (3) and (5), we can easily find that

$$\tau_p = \frac{1}{p} \sigma_p, \quad \sigma_p = \frac{2\pi^{\frac{p}{2}}}{\Gamma(\frac{p}{2})}. \tag{6}$$

2. Let us now find the solutions of equation (M) that depend only on r. All such solutions obviously satisfy the equation

$$\frac{d^2 U}{dr^2} + \frac{p-1}{r} \frac{dU}{dr} + \lambda^2 U = 0.$$

Consequently, they will have the general form

$$\alpha r^{-q} H_q^{(1)}(\lambda r) + \beta r^{-q} H_q^{(2)}(\lambda r) \qquad \left(q = \frac{p-2}{2} \right), \tag{7}$$

where α, β are arbitrary constants, and $H_q^{(1)}, H_q^{(1)}$ are Hankel functions, which are connected with the Bessel and Neumann functions by the relations,

$$H_\nu^{(1)}(x) = J_\nu(x) + i N_\nu(x), \quad H_\nu^{(2)}(x) = J_\nu(x) - i N_\nu(x). \tag{8}$$

It is well known that

$$J_\nu(x) = \sum_{m=0}^{\infty} (-1)^m \frac{(\frac{x}{2})^{2m+\nu}}{m! \Gamma(m+\nu+1)}, \tag{9}$$

[1] It may easily be seen that (4) also holds for any hypersphere concentric with $\sum_1$.

and

$$N_\nu(x) = \frac{J_\nu(x)\cos\nu\pi - J_{-\nu}(x)}{\sin\nu\pi}, \quad \text{for} \quad \nu \neq n, \tag{10}$$

where n is a non-negative integer. If $\nu = n$, then

$$\pi N_n(x) = 2J_n(x)\log\frac{x}{2} - \sum_{s=0}^{n-1}\frac{(n-s-1)!}{s!}\left(\frac{2}{x}\right)^{n-2s}$$

$$-\sum_{s=0}^{\infty}(-1)^s\frac{(\frac{x}{2})^{2s+n}}{s!(s+n)!}\left[\frac{\Gamma'(s+1)}{\Gamma(s+1)} + \frac{\Gamma'(s+n+1)}{\Gamma(s+n+1)}\right], \tag{11}$$

where it should be remembered that, when $n = 0$, the first (finite) sum will be absent.

It is easily seen from equations (8)-(11) that (7) represents a metaharmonic function throughout all space, with the exception of the origin when $\alpha \neq \beta$. In this latter case (7) has a singularity at the origin, of the form

$$\frac{1}{r^{p-2}} \quad \text{or} \quad \log\frac{1}{r},$$

depending on whether $p > 0$ or $p = 2$. Thus functions of type (7) with $\alpha \neq \beta$ belong to the class of so-called elementary solutions (Hadamard's terminology) of equation (M). The point where an elementary solution has a singularity of the above kind is usually called a pole of the solution. Any elementary solution of (M) that has a single pole at the origin must be of the form (7), except for possibly an added term which is a regular solution of (M) throughout all space.

We now consider the elementary solution of the form

$$\Omega(X, X_0) = KR^{-q}Z_q(\lambda R), \tag{12}$$

where R is the distance between the points X and X_0,

$$K = \frac{i}{4(\alpha-\beta)}\frac{\lambda^q}{(2\pi)^q} \quad \left(q = \frac{p-2}{2}\right), \tag{13}$$

$$Z_q(x) = \alpha H_q^{(1)}(x) + \beta H_q^{(2)}(x); \tag{14}$$

moreover, it is obviously assumed here that $\alpha \neq \beta$, Z_q is the so-called cylindrical function of order q.

By passing to the limit in (12) with $p > 2$ as $\lambda \to 0$ we get

$$\frac{\Gamma(q)}{4\pi^{1+q}}\frac{1}{R^{2q}} \equiv \frac{1}{(p-2)\sigma_p}\frac{1}{R^{p-2}}.$$

This is the elementary solution of the Laplace equation when $p > 2$. For $p = 2$ we first subtract from $\Omega(X, X_0)$ the function $(2/\pi)J_0(\lambda R)\log\frac{\lambda}{2}$, which is a regular solution of equation (M). The function obtained will obviously again be an elementary solution of this equation. If we now pass to the limit in this function as $\lambda \to 0$, we get

$$\frac{1}{2\pi} \log \frac{1}{R},$$

i.e. we have the elementary solution of the Laplace equation in two dimensions.

In what follows, we will denote these elementary solutions of the Laplace equation by $\Omega_0(X, X_0)$, i.e.

$$\Omega_0(X, X_0) = \begin{cases} \dfrac{1}{(p-2)\sigma_p} \dfrac{1}{R^{p-2}} & \text{for} \quad p > 2, \\ \dfrac{1}{2\pi} \log \dfrac{1}{R} & \text{for} \quad p = 2. \end{cases} \tag{15}$$

If we now put $\alpha = 1$, $\beta = 0$ in (12), or $\alpha = 0$, $\beta = 1$ we obtain the elementary solutions

$$\Omega_1(X, X_0) = \frac{i\lambda^q}{4(2\pi)^q} R^{-q} H_q^{(1)}(\lambda R), \tag{16}$$

$$\Omega_2(X, X_0) = \frac{q}{4i(2\pi)^q} R^{-q} H_q^{(2)}(\lambda R), \tag{17}$$

respectively.

Close to the pole, the elementary solutions Ω, Ω_1 and Ω_2 behave exactly like the elementary solution Ω_0 of the Laplace equation, but far from the pole their behaviour is essentially different.

In fact, using the asymptotic formulae

$$H_\nu^{(1)}(z) = \sqrt{\frac{2}{\pi z}} e^{i(z - \frac{\nu\pi}{2} - \frac{\pi}{4})} [1 + O(z^{-1})],^{2} \tag{18}$$

when $-\pi < \arg z < 2\pi$ and

$$H_\nu^{(2)}(z) = \sqrt{\frac{2}{\pi z}} e^{-i(z - \frac{\nu\pi}{2} - \frac{\pi}{4})} [1 + O(z^{-1})], \tag{19}$$

for $-2\pi < \arg z < \pi$ and

$$\Omega_1(X, X_0) = K_1 e^{i\lambda R} R^{-q-\frac{1}{2}} [1 + O(R^{-1})], \tag{20}$$

$$\Omega_2(X, X_0) = K_2 e^{-i\lambda R} R^{-q-\frac{1}{2}} [1 + O(R^{-1})], \tag{21}$$

[2]Let us recall the definitions of Landau's symbols $O, o, O(x^n)$(or $o(x^n)$) denotes a magnitude whose ratio to x^n remains bounded (or tends to zero) as $x \to \infty$. In particular, $O(1)$ respectively, denotes a bounded magnitude, and $o(1)$ denotes an infinitesimal.

where

$$K_1 = \frac{i}{4\pi}\left(\frac{\lambda}{2\pi}\right)^{q-\frac{1}{2}} e^{-\frac{1}{2}i\pi(q+\frac{1}{2})},$$
$$K_2 = \frac{1}{4\pi i}\left(\frac{\lambda}{2\pi}\right)^{q-\frac{1}{2}} e^{\frac{1}{2}i\pi(q+\frac{1}{2})}. \tag{22}$$

Comparison of (20) and (21) with (15) shows that, at infinity, the elementary solutions Ω_1 and Ω_2 of equation (M) behave quite differently to the elementary solution of Laplace's equation.

We will show below that, in general, the behaviour at infinity of the entire class of metaharmonic functions is quite different to that of harmonic functions.

The following formulae also immediately follow

$$\frac{d\Omega_1}{dR} - i\lambda\Omega_1 = e^{i\lambda R}O(R^{-q-\frac{3}{2}}), \tag{20a}$$

$$\frac{d\Omega_2}{dR} + i\lambda\Omega_2 = e^{-i\lambda R}O(R^{-q-\frac{3}{2}}). \tag{21a}$$

§ 2. Green's formulae. The Sommerfeld conditions

1. The present section is concerned with deriving Green's formula, giving the integral representations of the metaharmonic functions. The derivation of these formulae presents no difficulty at all if the domain of the metaharmonic functions is finite; if the domain is infinite, however, a difficulty arises because of the fact that the behaviour of the metaharmonic functions at infinity is essentially different from that of the harmonic functions. The point is that, as we know, the condition

$$U = O(r^{-p+2}) \tag{23}$$

is sufficient for the representation of harmonic functions by means of Green's formula in the case of an infinite domain. However, as will be shown below, this condition does not in general hold in the case of metaharmonic functions. Instead, in the case of metaharmonic functions we will deal with conditions of the type

$$L_1(U) = \frac{dU}{dr} - i\lambda U = e^{i\lambda r}o(r^{-q-\frac{1}{2}}) \quad \text{for} \quad \mathrm{Im}(\lambda) \geq 0, \tag{I}$$

$$L_2(U) = \frac{dU}{dr} + i\lambda U = e^{-i\lambda r}o(r^{-q-\frac{1}{2}}) \quad \text{for} \quad \mathrm{Im}(\lambda) \leq 0. \tag{II}$$

When λ is real, these conditions obviously become

$$L_1(U) = \frac{dU}{dr} - i\lambda U = o(r^{-q-\frac{1}{2}}), \tag{I_0}$$

or

$$L_2(U) = \frac{dU}{dr} + i\lambda U = o(r^{-q-\frac{1}{2}}). \qquad (II_0)$$

These latter conditions, in the cases $p = 2, 3$, were first established from physical consideration by Sommerfeld [6, 7]. In the sequel, therefore, we will refer to the conditions (I_0) or (II_0), and also to the more general (I) and (II), as Sommerfeld conditions.

2. Before turning to the derivation of Green's formulae, it is reasonable to introduce some preliminary definitions and concepts concerning the domains and functions with which we will be predominantly concerned later in this section, and in particular, in § 4.

Let S be a closed hypersurface in E_p [3]. Let there exist a finite number of connected pieces of $S : S_1, \cdots, S_m$, satisfying the following conditions: 1) every point of S is an interior point of at least one of the $S_j (j = 1, \cdots, m)$, 2) every $S_j (j = 1, \cdots, m)$ is mapped one-to-one and continuously into a definite domain of space E_{p-1} by equations of the type

$$x_k = x_k(u_1, u_2, \cdots, u_{p-1}) \quad (k = 1, 2, \cdots, p),$$

where the functions

$$f_k \text{ and } \frac{\partial f_k}{\partial u_i} \; (i = 1, \cdots, p-1; \; k = 1, \cdots, p)$$

are continuous, 3) the functional determinants

$$\frac{D(f_1, \cdots, f_{i-1}, f_{i+1}, \cdots, f_p)}{D(u_1, u_2, \cdots, u_{p-1})} \quad (i = 1, \cdots, p)$$

do not vanish simultaneously.

Following Lichtenstein [23], if a hypersurface satisfies the above conditions, we say that it belongs to class A. A hypersurface for which the derivatives $\partial f_k / \partial u_i$ are also continuous in Hölder's sense, [4] will be said to belong to class Ah.

A hypersurface of class A for which the second derivatives $\partial^2 f_k / \partial u_i \partial u_j$ are continuous is said to belong to class B. If, in addition, these derivatives are continuous in Hölder's sense, the hypersurface is said to belong to class Bh.

Let T be a domain in E_p. If its boundary consists of a finite number of hypersurfaces of class A having no common points, we say that the domain T is of class A. Domains of classes Ah, B, Bh are similarly defined.

[3] When $p = 2, 3,$ S will be respectively a plane closed curve and an ordinary closed surface.

[4] A function $f(x)$ of a point of set $\mathfrak{M}$ is said to be continuous in Hölder's sense on this set, given any two points X_1, X_2 of $\mathfrak{M}$, $|f(x_1) - f(x_2)| < M r_{12}^{\lambda}$, where M and λ are positive numbers independent of the choice of the points X_1, X_2, $\lambda \leq 1$ and r_{12} is the Euclidean distance between X_1andX_2.

In the sequel, we will always denote the boundary of a domain by S. We will denote by CT the complement of $T+S$ with respect to the set E_p [5].

Obviously, CT is an open set, which is in general not connected, but consists of a finite number of simply-connected domains. The boundary of CT is obviously S.

Let T be a domain of one of the above-mentioned classes, and let S be its boundary. We will later denote by $X, Y, \cdots$ points lying outside S, and by $x, y, \cdots$ points of the boundary S. Let $F(X)$ be a given function of the point X, defined throughout all space and continuous outside S.

If $F(X)$ has a limit as the point X approaches a boundary point x along any path lying in T, we will denote this limit by the symbol $F^+(x)$, or simply by F^+. If $F^+(x) = F(x)$ everywhere on S and the convergence to the limit is uniform, it may easily be shown that $F(X)$ is continuous in $T+S$.

Now let the point X approach the boundary point x along any path lying in CT. The limit of the function $F(X)$, if it exists, will now be denoted by $F^-(x)$. If $F^-(x) = F(x)$ everywhere on S and the convergence to the limit is uniform, it is easily shown that $F(X)$ is continuous in $CT+S$.

Assume now that $F(X)$ has continuous partial derivatives of the first order outside S. Let n denote the normal to S, directed inwards into the domain T; when necessary, we will denote the normal to S at the point x by n_x. We now consider the derivative

$$\frac{d}{dn_x}F(X).$$

If this derivative has a limit when X approaches x along n_x, we will denote it by one of the symbols

$$\frac{dF^+(x)}{dn_x}, \quad \frac{d}{dn}F^+(x), \quad \frac{dF^+}{dn}.$$

We define $\dfrac{dF^-}{dn_x}$ similarly, when the point X approaches x along n_x from CT.

We now introduce some terminology for the classes of functions with which we will be mainly concerned in what follows.

If the function $F(X)$ is continuous on some point set $\mathfrak{M}$, we say that $F(X)$ belongs to class C on this set and write this symbolically as: $F \in C$ on $\mathfrak{M}$. If the function is continuous in Hölder's sense, we say that it belongs to class Ch. Naturally, we use the similar notation: $F \in Ch$ on $\mathfrak{M}$, in this case.

Now let D_jU denote the partial derivatives of the function U of order j $(j = 0, 1, 2, \cdots)$. We say that U belongs to class G in the domain T if the following conditions hold: 1) U is continuous in $T+S$, while D_1U and D_2U are continuous in T, 2) $dU(X)/dn$ is bounded in T, 3) dU^+/dn exists almost everywhere on S and is a bounded Lebesgue integrable function.

[5]Notice that we denote by E_p the set of all points of space whose Cartesian coordinates are finite numbers.

In particular, we say that U belongs to class G_0(or Gh) if it belongs to class G and in addition dU^+/dn belongs to class C(or Ch) on S.

Functions of class G will play an extremely important role for us in what follows, since the following Green's formula holds for them:

$$\int_T (U\Delta V - V\Delta U)dT = -\int_S \left(U\frac{dV}{dn} - V\frac{dU}{dn}\right) dS, \tag{24}$$

where T is a finite domain of class B and $U,\ V$ are arbitrary functions of class G in T (see [23], p. 211, in this connection where detailed references can be found).

3. We now turn to deriving Green's formulae for metaharmonic functions. We first consider the case of a finite domain.

Let T be a finite domain of class B [6]. We assume that $U(X)$ is a metaharmonic function in the domain T of class G. In this case, we can easily show in the usual way from (24) that

$$U(X) = \int_S \left(U^+\frac{d\Omega(X,y)}{dn_y} - \Omega(X,y)\frac{dU^+}{dn_y}\right) dS_y \quad (X \in T), \tag{25}$$

where Ω is the elementary solution of equation (M) defined by (12). Obviously, the constants α, β in this formula can take completely arbitrary values, provided only that $\alpha \neq \beta$. In particular we can take Ω_1 or Ω_2 as Ω. Notice that the right-hand side of (25) vanishes identically when the point X belongs to CT, i.e. when X lies outside $S+T$.

We will call (25) Green's formula for metaharmonic functions in the case of a finite domain.

It follows at once from (25) that a *metaharmonic function is analytic in the domain where it is metaharmonic.*

4. Now let us turn to the case of an infinite domain. Here, we have the following theorem.

Theorem 1. *Let T be an infinite domain of class B, and let U be a metaharmonic function of class G in this domain, satisfying at infinity one of the following conditions*

$$L_1(U) \equiv \frac{dU}{dr} - i\lambda U = e^{i\lambda r}o(r^{-q-\frac{1}{2}}) \quad for \quad \mathrm{Im}(\lambda) \geq 0, \tag{I}$$

or

$$L_2(U) \equiv \frac{dU}{dr} + i\lambda U = e^{-i\lambda r}o(r^{-q-\frac{1}{2}}) \quad for \quad \mathrm{Im}(\lambda) \leq 0, \tag{II}$$

[6] We restrict our discussion to domains of class B because such domains will be our exclusive concern in § 4. Obviously, many of the results of the present section still remain in force for domains of a wider class, say of class A.

Then

$$U(X)=\int\limits_S \left(U^+\frac{d\Omega_1(X,y)}{dn_y}-\Omega_1(X,y)\frac{dU^+}{dn_y}\right)dS_y,\quad X\in T, \tag{26}$$

or

$$U(X)=\int\limits_S \left(U^+\frac{d\Omega_2(X,y)}{dn_y}-\Omega_2(X,y)\frac{dU^+}{dn_y}\right)dS_y,\quad X\in T, \tag{27}$$

depending on whether condition (I) or (II) holds.

Proof. We will assume for definiteness that condition (I) holds. The theorem will obviously be proved if we can show that

$$\int\limits_{\Sigma_R}\left(U\frac{d\Omega_1(R)}{dR}-\Omega_1(R)\frac{dU}{dR}\right)d\Sigma=0, \tag{28}$$

where Σ_R is the hypersphere with centre at the point X and sufficiently large radius, and

$$\Omega_1(R)=\frac{i}{4}\frac{\lambda^q}{(2\pi)^q}R^{-q}H_q^{(1)}(\lambda R).$$

Condition (28) is obviously equivalent to the condition

$$\Omega_1'(R)\int\limits_{\Sigma_1}U(R,\Theta)d\Sigma-\Omega_1(R)\frac{d}{dR}\int\limits_{\Sigma_1}U(R,\Theta)d\Sigma=0, \tag{29}$$

where Σ_1 is the unit hypersphere with centre at the point X.

We will see below that

$$\int\limits_{\Sigma_1}U(R,\Theta)d\Sigma=A_1R^{-q}H_q^{(1)}(\lambda R)+A_2R^{-q}H_q^{(2)}(\lambda R), \tag{30}$$

where A_1, A_2 are constants, independent of R. We now show that $A_2=0$. In fact, if we apply the operator L_1 to both sides of (30) and take into account condition (I) and formulae (18) and (19), we have

$$A_2e^{-2i\lambda R}[1+O(R^{-1})]=o(1).$$

Hence, since $\mathrm{Im}(\lambda)\geq 0$ it follows at once that $A_2=0$. Thus (30) becomes

$$\int\limits_{\Sigma_1}U(R,\Theta)d\Sigma=A_1R^{-q}H_q^{(1)}(\lambda R). \tag{30a}$$

On substituting this in the left-hand side of (29), we will see that this latter equation holds. The second part of the theorem may be proved in exactly the same way.

5. We will call a function U as being of category I (or II) if it is metaharmonic in a domain T, containing the exterior of some hypersphere, and representable by formula (26) (or (27)).

As we showed above, every function, metaharmonic outside some hypersphere and satisfying condition (I) or (II), belongs respectively to category I or II; but the converse does not in general follow. However, the following facts are obvious: if $\mathrm{Im}(\lambda) \geq 0$ and the function U is of category I, then U satisfies condition (I) at infinity. This is an immediate consequence of (26) taken in conjunction with (20a). It may similarly be shown that, if $\mathrm{Im}(\lambda) \leq 0$, and U is of category II, then U satisfies condition (II) at infinity.

We obtain respectively from (26) and (27), on the basis of (20) and (21):

$$U = e^{i\lambda r} O(r^{-q-\frac{1}{2}}), \tag{31}$$

$$U = e^{-i\lambda r} O(r^{-q-\frac{1}{2}}). \tag{32}$$

These formulae give the asymptotic behaviour of metaharmonic functions of categories I and II, respectively. Let

$$\tau = \mathrm{Re}(\lambda), \quad \sigma = \mathrm{Im}(\lambda), \quad i.e.\ \lambda = \tau + i\sigma.$$

It will be assumed that $\sigma > 0$. Formulae (31) and (32) show that, now, functions of category I are decreasing at infinity at the same rate as $e^{-\sigma r} r^{-q-\frac{1}{2}}$, and of category II are increasing at the same rate as $e^{\sigma r} r^{-q-\frac{1}{2}}$.

Now suppose that λ is a real number ($\sigma = 0$). Conditions (31) and (32) now become

$$U = O(r^{-q-\frac{1}{2}}). \tag{33}$$

This condition was designated by Sommerfeld the "finiteness condition". It figures in Sommerfeld's work (see, e.g., [6] as an independent condition, along with condition (I_0) or (II_0), in the derivation of Green's formula in the case of an infinite domain. We have obtained this condition as a corollary of Green's formula, without making any use of it in deriving the formula [7].

§ 3. Hyperspherical functions. Expansion of metaharmonic functions

In this section we will obtain expansions of the metaharmonic functions into Bessel (Hankel) and hyperspherical functions. To facilitate reading of the present article we will start with a fairly detailed treatment of some fundamental properties of hyperspherical functions. More detailed information can be found in [5].

[7] A similar result was also obtained later (1948) by W. Magnus [1949].

1. Let $U_m(x_1, x_2, \cdots, x_p)$ be a homogeneous harmonic polynomial of degree m in the variables $x_1, x_2, \cdots, x_p$. Passing to polar coordinates (1), we get

$$U_m(x_1, x_2, \cdots, x_p) = r^m Y_m(\theta_1, \theta_2, \cdots, \theta_{p-1}). \tag{34}$$

The functions $Y_m(\theta_1, \theta_2, \cdots, \theta_{p-1})$, or more briefly, $Y_m(\Theta|p)$ are termed hyperspherical functions of order m. By substituting (34) in equation (M'), we see that the hyperspherical functions $Y_m(\Theta|p)$ satisfy the differential equation

$$\Lambda Y + m(m+p-2)Y = 0 \quad (m = 0, 1, 2, \cdots). \tag{35}$$

Hence using (4), we get

$$\int_\Sigma Y_m(\Theta|p) Y_n(\Theta|p) d\Sigma = 0 \quad for \;\; (m \neq n), \tag{36}$$

where Σ is a hypersphere with centre at the origin.

Let

$$Y_m^{(1)}(\Theta|p), \cdots, Y_m^{(k_m)}(\Theta|p) \quad (m = 0, 1, 2, \cdots). \tag{37}$$

be a complete system of linearly independent hyperspherical functions of order m. It is well known (see, e.g., [24], p. 462) that the number of these functions is

$$k_m = \frac{(m+p-2)!}{(p-2)!m!}\left(1 + \frac{m}{m+p-2}\right) \quad (m = 0, 1, 2, \cdots). \tag{38}$$

In the case $p = 2$ this system is the same as the ordinary system of trigonometric functions:

$$1, \cos\theta, \quad \sin\theta, \ldots, \quad \cos m\theta, \quad \sin m\theta, \ldots, \tag{39}$$

while with $p = 3$ we have the system of Laplace spherical functions:

$$P_m(\cos\vartheta), \quad P_{m,k}(\cos\vartheta)\cos k\varphi, \quad P_{m,k}(\cos\vartheta)\sin k\varphi, \tag{40}$$
$$(m = 0, 1, 2, \ldots; \;\; k = 1, 2, \ldots, m).$$

Let

$$r, \theta_{p-1}, \cdots, \theta_1 \;\; and \;\; \rho, \; \vartheta_{p-1}, \cdots, \vartheta_1$$

be the coordinates of the points X, X_0 respectively. We denote by Γ the angle between the vectors $\overrightarrow{OX}$ and $\overrightarrow{OX_0}$, where O is the origin. We will assume that $p > 2$. In this case, we have with $r < \rho$:

$$R^{-p+2} = (r^2 - 2r\rho\cos\gamma + \rho^2)^{-q} = \sum_{n=0}^{\infty} \frac{r^n}{\rho^{n+2q}} P_n(\cos\gamma|P), \tag{41}$$

where

$$P_n(\cos\gamma|p) = \sum_{k=0}^{\leq \frac{1}{2}n} (-1)^k \frac{\Gamma(q+n-k)}{(k!(n-2k)!\Gamma(q)} (2\cos\gamma)^{n-2k}, \tag{42}$$

$$(n = 0, 1, 2, \cdots ; \ \ q = \frac{(p-2)}{2}).$$

The polynomials $P_n(x|p)$ are a generalization of the ordinary Legendre polynomials $P_n(x)$, with which they coincide for $p = 3$ (see [21], p. 451; see also [20], p. 127).

It is easy to show that $r^n P_n(\cos\gamma|p)$ is a homogeneous harmonic polynomial of degree n in the variables $x_1, x_2, \ldots, x_p$. Hence, $P_n(\cos\gamma|p)$ is a hyperspherical function of order n in the arguments $\theta_1, \theta_2, \ldots, \theta_{p-1}$.

Let $U(r, \theta_1, \ldots, \theta_{p-1})$ or more briefly, let $U(r, \Theta)$ be a harmonic function inside the unit hypersphere $\sum_1$, continuous right up to $\sum_1$. Then, as is will known, the value of this function at a point (r, Θ) inside $\sum_1$, is given by the Poisson integral

$$U(r, \Theta) = \frac{\Gamma(q)}{4\pi^{1+q}} \int\limits_{\Sigma_1} U(r, \vartheta) \left(\frac{d}{d\rho} G(X, X_0) \right) \Big|_{\rho=1} d\Sigma \tag{43}$$

where G is the Green function for the unit hypersphere, which has the form

$$G(X, X_0) = (1 - 2r\rho\cos\gamma + r^2\rho^2)^{-q} - (r^2 - 2r\rho\cos\gamma + \rho^2)^{-q}. \tag{44}$$

On substituting this in (43) and expanding the integrand into a Maclaurin series in powers of r, we get

$$U(r, \Theta) = \sum_{n=0} \frac{(n+q)\Gamma(q)}{2\pi^{q+1}} r^n \int\limits_{\Sigma_1} U P_n(\cos\gamma)|p) d\Sigma. \tag{45}$$

If we now replace U by harmonic functions

$$r^m Y_m(\Theta|p) \quad (m = 0, 1, 2, \ldots),$$

we obtain the formulae:

$$\int\limits_{\Sigma_1} P_n(\cos\gamma|p) Y_m(\vartheta|p) d\Sigma_\vartheta = 0, \quad for \quad n \neq m, \tag{46}$$

$$Y_n(\Theta|p) = \frac{\Gamma(q)(q+n)}{2\pi^{q+1}} \int\limits_{\Sigma_1} P_n(\cos\gamma|p) Y_n(\vartheta|p) d\Sigma_\vartheta \quad (n = 0, 1, 2, \ldots,). \tag{47}$$

It may thus be seen that the hyperspherical functions are solutions of the integral equation (47). We will show below that the complete system of solutions of this equation is the same as the complete system of linearly independent hyperspherical functions of order n.

We now prove that the system of hyperspherical functions

$$\{Y_m^{(k)}(\Theta|p)\}$$

is complete, i.e. there exists no continuous function of a point of the hypersphere which is orthogonal to all the functions of system (37) and does not vanish identically.

Assume now the contrary: suppose $\Psi(\Theta)$ is a function, continuous on the hypersphere of unit radius, which does not vanish identically and satisfies the conditions

$$\int_{\Sigma_1} \Psi(\vartheta) Y_m^{(k)}(\vartheta|p) d\Sigma = 0, \quad (m = 0, 1, 2, \ldots; \ \ k = 1, 2, \ldots, k_m). \tag{48}$$

Let $U(r, \Theta)$ be a harmonic function inside the hypersphere Σ_1, satisfying the boundary condition

$$\lim_{r \to 1} U(r, \Theta) = \Psi(\Theta). \tag{49}$$

We now have, by (45)

$$U(r, \Theta) = \sum_{n=0}^{\infty} r^n Y_n(\Theta|p), \tag{50}$$

where

$$Y_n(\Theta|p) = \frac{(n+q)\Gamma(q)}{2\pi^{1+q}} \int_{\Sigma_1} \Psi(\vartheta) P_n(\cos\gamma|p) d\Sigma.$$

We find from (50), using (48) that

$$\int_{\Sigma_1} U(r, \Theta)\Psi(\Theta) d\Sigma = 0 \tag{51}$$

for all $r, 0 \leq r < 1$. But on passing to the limit as $r \to 1$ in (51) and using (49), we obtain

$$\int_{\Sigma_1} \Psi^2(\Theta) d\Sigma = 0,$$

i.e.

$$\Psi(\Theta) \equiv 0,$$

which contradicts our hypothesis. Our assertion is thus proved.

We have carried out the above arguments for the cases $p > 2$. But it may easily be seen that the results still hold when $p = 2$. In this case we are concerned with the ordinary trigonometric system of functions (39), and instead of (43) we have the Poisson integral on the plane.

It is now easy to prove that the complete system of solutions of equation (47) is in fact the same as the system of functions (37). For, if there existed a solution of (47), linearly independent of functions (37), it could be chosen to be orthogonal to the system of functions$\{Y_m^{(k)}\}$, which is impossible, as we have seen.

2. We now take any analytic function

$$U(r,\Theta) \equiv U(r,\theta_1,\cdots,\theta_{p-1})$$

in the domain bounded by two hyperspheres Σ_a and Σ_b with common centre at the origin and radii a,b respectively $(a<b)$.

We prove the following lemma:

Lemma 1. *A function $U(r,\Theta)$, analytic in the domain $a<r<b$, can be expanded into the series*

$$U(r,\Theta)=\sum_{n=0}^{\infty} Y_n(r,\Theta|p), \tag{52}$$

where [8]

$$Y_n(r,\Theta|p)=\frac{(q+n)\Gamma(q)}{2\pi^{1+q}}\int\limits_{\Sigma_1} U(r,\vartheta)P_n(\cos\gamma|p)d\Sigma,\quad n=\overline{0,\infty}, \tag{53}$$

and Σ_1 is the unit hypersphere with centre at the origin.

This series is absolutely and uniformly convergent in any domain of the type $a'\le r\le b'$, where a',b' are arbitrary numbers satisfying the condition $a<a'<b'<b$. In addition, series (52) can be termwise differentiated any number of times, both with respect to r and the variables $\theta_1,\cdots,\theta_{p-1}$.

Proof. We have, by (35)

$$P_n(\cos\gamma|p)=-\frac{1}{n(n+p-2)}\Lambda P_n(\cos\gamma|p)\quad (n=1,2,\cdots). \tag{54a}$$

In view of this, we obtain from (53), using (4)

$$Y_n(r,\Theta|p)=-\frac{(q+n)\Gamma(q)}{2\pi^{1+q}(n+p-2)}\int\limits_{\Sigma_1}\Lambda U\cdot P_n(\cos\gamma|p)d\Sigma \\ (n=0,1,2,\ldots). \tag{54}$$

Since U is analytic, we can repeat this procedure any number of times. We thus obtain

$$Y_n(r,\Theta|p)=(-1)^k\frac{(q+n)\Gamma(q)}{2\pi^{1+q}n^k(n+p-2)^k}\int\limits_{\Sigma_1}\Lambda^k U\cdot P_n(\cos\gamma|p)d\Sigma \\ (n=0,1,2,\ldots), \tag{55}$$

[8]We have in mind the case $p\ge 3$. All our future results will obviously be completely general, however, and remain in force for the case $p=2$, when (52) becomes an ordinary Fourier series.

where k is any positive integer. Hence, applying Schwartz's inequality, we get

$$|Y_n(r,\Theta|p)| \le \frac{(q+n)\Gamma(q)}{2\pi^{q+1}n^k(n+p-2)^k}\Big[\int\limits_{\Sigma_1}(\Lambda^k U)^2 d\Sigma_1\Big]^{\frac{1}{2}}\Big[\int\limits_{\Sigma_1}(P_n^2(\cos\gamma|p)d\Sigma_1\Big]^{1/2}$$
$$(n=0,1,2,\cdots). \tag{56}$$

We know (see e.g. [24], pp. 459-460) that

$$\int\limits_{\Sigma_1} P_n^2(\cos\gamma|p)d\Sigma = \frac{2\pi^{q+1}}{\Gamma(q)\Gamma(2q)}\frac{\Gamma(n+2p)}{n!}\frac{1}{n+q} \quad (n=0,1,2,\cdots). \tag{57}$$

Hence we have from (56)

$$|Y_n(r,\Theta|p)| \le Q_k\frac{\sqrt{(n+q)(n+p-3)(n+p-4)\cdots(n+1)}}{n^k(n+p-2)^k}, \tag{58}$$

where

$$Q_k = \sqrt{\frac{\Gamma(q)}{2\pi^{q+1}\Gamma(2q)}}\ \underset{a'\le r\le b'}{max}\ \Big[\int\limits_{\Sigma_1}(\Lambda^k U)^2 d\Sigma\Big]^{\frac{1}{2}}. \tag{59}$$

If we now take $k > \frac{1}{4}p$, it can easily be shown from (58) that series (52) is in fact absolutely and uniformly convergent in the domain $a' \le r \le b'$. Besides, since the system of hyperspherical functions is complete, we can easily show that the sum of series (52) is $U(r,\Theta)$, whatever the r, $a < r < b$.

We now show that series (52) can be differentiated term by term with respect to r. In fact, on expanding the function $d^sU/dr^s (s \ge 1)$ into a series of the form (52), we have

$$\frac{d^sU(r,\Theta)}{dr^s} = \sum_{n=0}^{\infty}\widetilde{Y}_n(r,\Theta|p), \tag{60}$$

$$\widetilde{Y}_n(r,\Theta|q) = \frac{(q+n)\Gamma(q)}{2\pi^{1+q}}\int\limits_{\Sigma_1}\frac{d^sU(r,\Theta)}{dr^s}P_n(\cos\gamma|p)d\Sigma \quad (n=0,1,2,\cdots).$$

Hence we obviously get

$$\widetilde{Y}_n(r,\Theta|p) = \frac{d^s}{dr^s}Y_n(r,\Theta|p) \quad (n=0,1,2,\cdots), \tag{60a}$$

which proves our assertion.

There is no great difficulty in showing that series (52) can be differentiated term by term any number of times with respect to $\theta_1,\ldots,\theta_{p-1}$. Let us show, for instance, that the operation Λ can be carried out term by term on series (52).

On expanding the function ΛU into a series of the form (52), we get

$$\Lambda U(r,\Theta)=\sum_{n=0}^{\infty}Y_n^*(r,\Theta|p), \tag{60b}$$

where, from (53)

$$Y_n^*(r,\Theta|p)=\frac{(q+n)\Gamma(q)}{2\pi^{q+1}}\int\limits_{\Sigma_1}\Lambda U\cdot P_n(\cos\gamma|p)d\Sigma.$$

Hence we have, by (4), (54a), (53) and (35):

$$Y_n^*(r,\Theta|p)=\Lambda Y_n(r,\Theta|p)\quad (n=0,1,2,\cdots).$$

i.e

$$\Lambda U(r,\Theta)=\sum_{n=0}^{\infty}\Lambda Y_n(r,\Theta|p), \tag{60c}$$

which proves our assertion.

3. We now suppose that U satisfies equation (M), i.e.

$$\frac{\partial^2 U}{\partial r^2}+\frac{p-1}{r}\frac{\partial U}{\partial r}+\frac{1}{r^2}\Lambda U+\lambda^2 U=0,\quad \text{for } a<r<b. \tag{M'}$$

If we define ΛU from this and substitute in (54), we can easily show, using (53), that $Y_n(r,\Theta|p)$, as a function of r, satisfies the ordinary differential equation

$$\frac{d^2Y_n}{dr^2}+\frac{p-1}{r}\frac{dY_n}{dr}+\left(\lambda^2-\frac{n(n+p-2)}{r^2}\right)Y_n=0\quad (n=0,1,2,\ldots). \tag{61}$$

These functions consequently have the form

$$\begin{gathered}Y_n(r,\Theta|p)=r^{-q}H^{(1)}_{q+1}(\lambda r)Y_n'(\Theta|p)+r^{-q}H^{(2)}_{q+n}(\lambda r)Y_n''(\Theta|p)\\ (n=0,1,2,\ldots),\end{gathered} \tag{62}$$

where Y_n' and Y_n'' are hyperspherical functions which are independent of r.

If we use the formula

$$H^{(1)}_\nu(x)\frac{d}{dx}H^{(2)}_\nu(x)-H^{(2)}_\nu(x)\frac{d}{dx}H^{(1)}_\nu(x)=\frac{4}{i\pi x}, \tag{63}$$

together with (62) and (53), we easily find that

$$Y_n'(\Theta|p)=\int\limits_{\Sigma_1}\mathfrak{M}_n'(U)P_n(\cos\gamma|p)d\Sigma, \tag{64}$$

$$Y_n''(\Theta|p) = \int_{\Sigma_1} \mathfrak{M}_n''(U) P_n(\cos\gamma|p) d\Sigma, \tag{65}$$

where

$$\mathfrak{M}_n'(U) \equiv \frac{(q+n)\Gamma(q)}{2\pi^{1+q}} r^{p-1} \Big[r^{-q} H_{q+n}^{(2)}(\lambda r) \frac{dU}{dr} - U \frac{d}{dr}(r^{-q} H_{q+n}^{(2)}(\lambda r)) \Big], \tag{66}$$

$$\mathfrak{M}_n''(U) \equiv \frac{(q+n)\Gamma(q)}{2\pi^{q+1}} r^{p-1} \Big[r^{-q} H_{q+n}^{(1)}(\lambda r) \frac{dU}{dr} - U \frac{d}{dr}(r^{-q} H_{q+n}^{(1)}(\lambda r)) \Big] \tag{67}$$

$$(n = 0, 1, 2, \cdots).$$

Notice that, when $n = 0$, (53) and (62) give us the formula (30) which we used in the previous section, since in this case $P_0(\cos\gamma|p) = 1$, while Y_0', Y_0'' are constants.

On substituting (62) in (52), we obtain

$$U(r, \Theta) = \sum_{n=0}^{\infty} r^{-q} H_{q+n}^{(1)}(\lambda r) Y_n'(\Theta|p) + r^{-q} H_{q+n}^{(2)}(\lambda r) Y_n''(\Theta|p), \tag{68}$$

where Y_n' and Y_n' are hyperspherical functions defined by (64) and (65).

We have thus proved the following:

Theorem 2. *If U is a metaharmonic function in the domain $a < r < b$, it can be expanded into a series of the form (68), which is uniformly and absolutely convergent in any domain of the form $a < a' \le r \le b' < b$. This series can be differentiated term by term any number of times.*

If $a = 0$ and the function U is regular at the origin, series (68) is easily seen to become

$$U(r, \Theta) = \sum_{n=0}^{\infty} r^{-q} J_{q+n}(\lambda r) Y_n(\Theta|p). \tag{69}$$

4. We will now consider the form taken by series (68) for $b = \infty$, if U is subjected to one of the Sommerfeld conditions (I) or (II) at infinity.

Let $b = \infty$. We will suppose first that U satisfies condition (I) at infinity, i.e.

$$L_1(U) \equiv \frac{dU}{dr} - i\lambda U = e^{i\lambda r} o(r^{-q-\frac{1}{2}}), \quad \mathrm{Im}(\lambda) \ge 0. \tag{I}$$

From (62) and (53), we have

$$\begin{aligned} & r^{-q} H_{q+n}^{(1)}(\lambda r) Y_n'(\Theta|p) + r^{-q} H_{q+n}^{(2)}(\lambda r) Y_n''(\Theta|p) \\ & = \frac{(q+n)\Gamma(q)}{2\pi^{q+1}} \int_{\Sigma_1} U(r,\theta) P_n(\cos\gamma|p) d\Sigma \quad (n = 0, 1, 2, \cdots). \end{aligned} \tag{70}$$

On applying the operator L_1 to both sides of (70) and taking into account condition (I) and formulae (18), (19), we easily obtain

$$Y_n'(\Theta|p) e^{-2i\lambda r}[1 + O(r^{-1})] = o(1) \quad (n = 0, 1, 2, \cdots). \tag{71}$$

Hence, since $\mathrm{Im}(\lambda) \geq 0$, it must follow that $Y_n''(\Theta|p) = 0$ $(n = 0, 1, 2, \cdots)$. Consequently, in this case series (68) recasts as

$$U(r, \Theta) = \sum_{n=0}^{\infty} r^{-q} H_{q+n}^{(1)}(\lambda r) Y_n(\Theta|p). \tag{72}$$

We can show in exactly the same way that series (68) becomes

$$U(r, \Theta) = \sum_{n=0}^{\infty} r^{-q} H_{q+n}^{(2)}(\lambda r) Y_n(\Theta|p), \tag{73}$$

if U is a metaharmonic function outside some hypersphere and obeys the condition (II) at infinity.

Theorem 3. *Let $U(r, \Theta)$ be a metaharmonic function outside some hypersphere Σ_a with centre at the origin and radius a. We can now expand $U(r, \Theta)$ for all $r > a$ into a series of the form (72) or (73), according to whether the function obeys condition (I) or (II) at infinity. The series in question are absolutely and uniformly convergent in any domain of the form $a' \leq r \leq b'$, where a', b' are arbitrary numbers satisfying $a < a' < b' < \infty$. In addition, series (72) and (73) can be differentiated term by term any number of times.*

5. We now consider the series

$$\sum_{n=0}^{\infty} r^{-q} H_{q+n}^{(1)}(\lambda r) Y_n'(\Theta|p), \tag{74}$$

$$\sum_{n=0}^{\infty} r^{-q} H_{q+n}^{(2)}(\lambda r) Y_n''(\Theta|p), \tag{75}$$

and suppose that they are absolute and uniformly convergent in any domain of the form $a < a' \leq r \leq b' < \infty$. We will assume in addition that they can be differentiated term by term with respect to r. We denote the sums of the series by U_1 and U_2 respectively. We now show that these functions can be represented by the integrals

$$U_1(X) = \int_{\Sigma_a} \left(U_1 \frac{d\Omega_1(X, y)}{dn_y} - \Omega_1(X, y) \frac{dU_1}{dn_y} \right) d\Sigma_y, \tag{76}$$

$$U_2(X) = \int_{\Sigma_a} \left(U_2 \frac{d\Omega_2(X, y)}{dn_y} - \Omega_2(X, y) \frac{dU_2}{dn_y} \right) d\Sigma_y. \tag{77}$$

We do this by using the following addition formula for cylindrical functions, due to Gegenbauer (see, e.g., [1], p.365):

$$(\lambda R)^{-q} Z_q(\lambda R) = \sum_{n=0}^{\infty} (\lambda r)^{-q} J_{q+n}(\lambda r)(\lambda \rho)^{-q} Z_{q+n}(\lambda \rho) P_m^*(\cos \gamma|p), \tag{78}$$

where

$$P_n^*(\cos\gamma|p) = 2^q(q+n)\Gamma(q)P_n(\cos\gamma|p) \quad \text{for } p > 3 \ \ (n = 0, 1, 2, \ldots), \tag{78a}$$

$$P_n^*(\cos\gamma|2) = \epsilon_n \cos n\gamma \quad (\epsilon_0 = 1,\ \epsilon_n = 2,\ n \geq 1), \tag{78b}$$

and we obviously assume that $r < \rho$, and $R = \sqrt{r^2 - 2\rho r\cos\gamma + \rho^2}$.

Series (78) is absolutely and uniformly convergent in the domain $0 \leq r \leq r' < \rho$, where r' is any number less than ρ. In addition, the series can be differentiated term by term any number of times.

Let up prove (76). The proof of the other formula (77) is exactly similar. Introduce the notation

$$g_n(x) = x^{-q}J_{q+n}(x), \quad f_n(x) = x^{-q}H_{q+n}^{(1)}(x) \quad (n = 0, 1, 2, \cdots). \tag{79}$$

By (78), we now have on the hypersphere Σ_a

$$\Omega_1(X, y) = \frac{i}{4}\frac{\lambda^{2q}}{(2\pi)^q}\sum_{n=0}^{\infty} \mathrm{g}_n(\lambda a)f_n(\lambda r)P_n^*(\cos\gamma|p), \tag{80}$$

$$\frac{d}{da}\Omega_1(X, y) = \frac{i}{4}\frac{\lambda^{2q}}{(2\pi)^q}\sum_{n=0}^{\infty} \lambda \mathrm{g}_n'(\lambda a)f_n(\lambda r)P_n^*(\cos\gamma|p), \tag{80a}$$

where r is the radius vector of the point X, and γ is the angle between the vectors OX and OY, y being a point on Σ_a.

By hypothesis, we also have on Σ_a

$$U_1 = \sum_{n=0}^{\infty} f_n(\lambda a)Y_n'(\Theta|p), \quad \frac{dU_1}{da} = \sum_{n=0}^{\infty} \lambda f_n'(\lambda a)Y_n'(\Theta|p). \tag{81}$$

In view of (46), (47), (78a) and (78b) we also have

$$\int\limits_{\Sigma_a} P_m^*(\cos\gamma|p)Y_n(\Theta|p)d\Sigma = 0 \quad \text{for } n \neq m \tag{82}$$

and

$$Y_n(\Theta|p) = \frac{1}{(2\pi)^{1+q}a^{p-1}}\int\limits_{\Sigma_a} P_n^*(\cos\gamma|p)Y_n(\Theta|p)d\Sigma \quad (n = 0, 1, 2, \cdots). \tag{82a}$$

From (80a), (80), (81), (81a), (82) and (82a), we obtain

$$\int\limits_{\Sigma_a}\left(U_1\frac{d}{da}\Omega_1(X, y) - \Omega_1(X, y)\frac{dU_1}{da}\right)d\Sigma_y$$

$$= \frac{i\pi}{2}(\lambda a)^{p-1}\sum_{n=0}^{\infty}[f_n(\lambda a)\mathrm{g}_n'(\lambda a) - f_n'(\lambda a)\mathrm{g}_n(\lambda a)]f_n(\lambda r)Y_n'(\Theta|p). \tag{83}$$

But, if we recall (8), we have, by (79) and (63)

$$f_n'(\lambda a)\mathrm{g}_n(\lambda a) - f_n(\lambda a)\mathrm{g}_n'(\lambda a) = \frac{2i}{\pi}(\lambda a)^{-p+1} \qquad (n = 0, 1, 2, \cdots).$$

We thus obtain from (83)

$$\int\limits_{\Sigma_a} \left(U_1 \frac{d\Omega_1}{da} - \Omega_1 \frac{d\Omega_1}{da} \right) d\Sigma = \sum_{n=0}^{\infty} f_n(\lambda r) Y_n'(\Theta|p).$$

But the right-hand side of this equation is exactly the expansion (74) of the function U_1, which proves (76).

We have thus proved that the sums U_1 and U_2 of the series (74) and (75) are functions of categories I and II respectively.

It is of course easy to prove the converse, that every function of category I or II can be expanded outside some hypersphere into a series (74) or (75) respectively.

We can in fact express a function of, say, category I by means of (76). But our assertion follows at once from this formula, on using (80) and (80a). The assertion is proved similarly for functions of category II. By (20) and(21), the following asymptotic formulae for U_1 and U_2, may be obtained from (76) and (77) respectively:

$$U_1 = e^{i\lambda r} O(r^{-q-\frac{1}{2}}), \tag{84}$$

$$U_2 = e^{-i\lambda r} O(r^{-q-\frac{1}{2}}). \tag{85}$$

6. We will now prove the following

Theorem 4. *Let U be a metaharmonic function outside some hypersphere. If we have at infinity;*

$$U = e^{-|\sigma| r} o(r^{-q-\frac{1}{2}}) \qquad (\sigma = \mathrm{Im}(\lambda)), \tag{86}$$

then U vanishes identically everywhere.

Proof. We obtain from (70), by (18), (19) and (86)

$$[\alpha_n + O(r^{-1})] e^{i\lambda r} Y_n' + [\beta_n + O(r^{-1})] e^{-i\lambda r} Y_n'' = e^{-|\sigma| r} o(1) \tag{87}$$

$$(n = 0, 1, 2, \ldots),$$

where α_n and β_n are definite non-zero constants. We will assume first that $\sigma > 0$. We now obtain from (87):

$$[\alpha_n + O(r^{-1})] Y_n' + [\beta_n + O(r^{-1})] e^{2\sigma r} \cdot e^{-2i\tau r} Y_n'' = o(1) \tag{88}$$

$$(n = 0, 1, 2, \ldots),$$

where $\tau = Re(\lambda)$. Hence it follows at once that $Y_n' = 0$, $Y_n'' = 0$ for all $= 0, 1, 2, \ldots$. But now, in view of (68), we have $U \equiv 0$, which was to be proved. The proof is exactly similar when $\sigma < 0$. When λ is real, condition (86) becomes

$$U = o(r^{-q-\frac{1}{2}}). \tag{89}$$

Hence it follows from the last condition that, with λ real, $U \equiv 0$.

Theorem 4 was proved even earlier in [4], under the assumption that U is metaharmonic throughout all space.

We now have the following important result: there is no function U, metaharmonic outside some V in E_p, and not identically zero, which can decrease at infinity faster than the function [9]

$$e^{-|\sigma|r} r^{-q-\frac{1}{2}}. \tag{90}$$

In particular, with λ real ($\lambda \neq 0$), we find that metaharmonic functions cannot decrease faster at infinity than

$$r^{-q-\frac{1}{2}}. \tag{91}$$

The behaviour of the metaharmonic functions at infinity is thus very different from that of the harmonic functions, since the latter are well known to include functions which can decrease, say, the rate of

$$r^{-n}, \tag{92}$$

where n is an arbitrary positive integer.

7. We will now prove that the Sommerfeld conditions (I) and (II) are equivalent to conditions (31) and (32) with $\text{Im}(\lambda) \lessgtr 0$ respectively, i.e. it follows from

$$\frac{dU}{dr} - i\lambda U = e^{i\lambda r} o(r^{-q-\frac{1}{2}}), \tag{I}$$

with $\text{Im}(\lambda) > 0$ that

$$U = e^{i\lambda r} O(r^{-q-\frac{1}{2}}), \tag{31}$$

and conversely, while from the condition

$$\frac{dU}{dr} + i\lambda U = e^{-i\lambda r} o(r^{-q-\frac{1}{2}}), \tag{II}$$

with $\text{Im}(\lambda) < 0$ it follows that

$$U = e^{-i\lambda r} O(r^{-q-\frac{1}{2}}), \tag{32}$$

[9] It must be mentioned that these properties of the solutions of equation (M) are completely analogous to the properties of the solutions of the one-dimensional differential equations $u'' + \lambda^2 u = 0$, since solutions of the latter have the form

$$u(x) = Ae^{\sigma x - i\tau x} + Be^{\sigma x + i\tau x}.$$

Obviously, for this case $p = 1$, *i.e.* $q = -\frac{1}{2}$.

and vice versa.

We have already seen (§2, sec. 4) that (31) and (32) are consequences of conditions (I) and (II) respectively. It therefore remains to show that conditions (31), (32) with $\mathrm{Im}(\lambda) > 0$ and $\mathrm{Im}(\lambda) < 0$ imply respectively conditions (I) and (II).

Let U be metaharmonic outside some hypersphere. We will consider first the case $\mathrm{Im}(\lambda) > 0$ and assume that condition (31) holds at infinity. We expand U into the series

$$U(r,\Theta) = \sum_{n=0}^{\infty} r^{-q} H^{(1)}_{q+n}(\lambda r) Y'_n(\Theta|p) + r^{-q} H^{(2)}_{q+n}(\lambda r) Y''_n(\Theta|p), \tag{68}$$

where

$$\begin{aligned} & r^{-q} H^{(1)}_{q+n}(\lambda r) Y'_n(\Theta|p) + r^{-q} H^{(2)}_{q+n}(\lambda r) Y''_n(\Theta|p) \\ & = \frac{(q+n)\Gamma(q)}{2\pi^{q+1}} \int\limits_{\Sigma_1} U(r,\vartheta) P_n(\cos\gamma|p) d\Sigma, \quad (n = 0,1,2,\ldots). \end{aligned} \tag{70}$$

On multiplying both sides of (70) by $e^{-i\lambda r} r^{q+\frac{1}{2}}$, we obtain, by (31), (18) and (19)

$$Y'_n(\Theta|p) O(1) + e^{-2i\lambda r} Y''_n(\Theta|p) \cdot [1 + O(r^{-1})] = O(1) \quad (n = 0,1,2,\ldots).$$

But this latter formula can obviously only hold when $Y''_n(\Theta|p) = 0 \quad (n = 0,1,\cdots)$, since $\mathrm{Im}(\lambda) > 0$. Hence we obtain from (68)

$$U(r,\Theta) = \sum_{n=0}^{\infty} r^{-q} H^{(1)}_{q+n}(\lambda r) Y'_n(\Theta|p). \tag{93}$$

But we have already seen (sec. 5) that metaharmonic functions of this type can be represented by the integral (76). We thus obtain at once from (76), in view of (20a)

$$\frac{dU}{dr} - i\lambda U = e^{i\lambda r} O(r^{-q-\frac{3}{2}}),$$

and this shows that U satisfies condition (I), since $\mathrm{Im}(\lambda) > 0$. The proof of the equivalence of conditions (32) and (II) with $\mathrm{Im}(\lambda) < 0$ is exactly similar.

8. It is now not difficult to see the physical significance of conditions (I) and (II), and also of functions of categories I and II.

Let U be metaharmonic in the domain T containing the exterior of some hypersphere. Let $\lambda = ka,\ a > 0$. Let us consider the "complex monochromatic wave"

$$Ue^{-ikt} \tag{94}$$

where t is time. Let us assume that $U = U_1 + U_2$, where U_1 and U_2 are metaharmonic functions of categories I and II respectively. We now have

$$Ue^{-ikt} = U_1 e^{-ikt} + U_2 e^{-ikt}. \tag{94a}$$

Consequently, by (84) and (85), the wave (94), near infinity, is the result of superposition of waves of the form

$$e^{-ik(t+ar)}O(r^{-q-\frac{1}{2}}), \quad e^{-ik(t-ar)}O(r^{-q-\frac{1}{2}})$$

propagating with velocity $1/a$, the first being waves departing from infinity, and the second being waves travelling to infinity. In physical problems (e.g. in the theory of electromagnetic waves diffraction) cases often occur when waves of one of these types are absent. For instance, if (94a) represents a wave travelling towards infinity, this means that $U_2 \equiv 0$, i.e. in this case U must be a function of category I.

Suppose now that U satisfies the Sommerfeld condition (I). As we have just seen, U will in this case be a function of category I, so that (94) now represents a wave travelling towards infinity with the velocity $1/a$, moreover, if $\mathrm{Im}(k) > 0$, the wave is obviously divergent as $t \to \infty$, and if $\mathrm{Im}(k) = 0$ it is a wave of sinusoidal type with respect to time t.

Suppose that U satisfies condition (II) at infinity. In this case (94a) will represent a wave departing from infinity; if $\mathrm{Im}(k) < 0$, it is a wave that is damped out in the course of time, while if $\mathrm{Im}(k) = 0$, it is a wave of sinusoidal type.

§ 4. Dirichlet and Neumann problems for equation (M)

In the present section we will deal with solving the Dirichlet and Neumann problems for equation (M) in the case of an infinite domain, the discussion being confined to domains of class B (§2, sec. 2). We will not consider the Dirichlet and Neumann problems for equation (M) in the case of finite domains, since these have been fully investigated in the literature (e.g. by the methods of integral equations and of the calculus of variations; see [25]); it should be mentioned, however, that the method described below is also suitable for these problems in the case of a finite domain.

1. Problem D_e. Let T be an infinite domain in E_p of class B, and let S be its boundary. It is required to find a function U, metaharmonic in T, continuous in $T+S$, and satisfying one of the Sommerfeld conditions (I) and (II) at infinity, together with the boundary condition

$$U^+ = f(x) \quad (\text{on } S) \tag{D_e}$$

where $f(x)$ is a given continuous function of the point x of the boundary S.

We will call this the exterior Dirichlet problem for equation (M) and denote it for brevity by D_e; and similarly, we will denote the corresponding homogeneous problem ($f \equiv 0$) by D_e^0.

We next state the exterior Neumann problem.

Problem N_e. Let T be an infinite domain of class B. It is required to find the function U, metaharmonic in T, belonging to the class G_0 (§2, sec. 2) and satisfying one of the Sommerfeld conditions (I) and (II), together with the boundary condition

$$\frac{dU^+}{dn} = f(x), \tag{N_e}$$

where $f(x)$ is a given continuous function on S.

We will denote this problem by N_e, and the corresponding homogeneous problem ($f \equiv 0$) by N_e^0

As it will be seen below, the following problems are closely connected with problems D_e and N_e respectively:

Problem N_i^0. We need to find the function U, metaharmonic on the set $CT(= E_p - T - S)$, belonging to the class G_0, together with the boundary condition

$$\frac{dU^-}{dn} = 0. \tag{N_i^0}$$

Problem D_i^0. We are required to find the function U, metaharmonic on the set CT, belonging to class G_0, together with the boundary condition

$$U^- = 0. \tag{D_i^0}$$

The following results are familiar concerning problems D_i^0 and N_i^0 (see, e.g., [25]):

1) *The problem D^0 only has non-trivial solutions for a discrete set of positive values of the parameter λ^2:*

$$0 < \lambda_1^2 < \lambda_2^2 < \cdots < \lambda_n^2 < \cdots ; \quad \lambda_n^2 \to \infty, \quad as \quad n \to \infty.$$

These values of λ^2 are called the eigenvalues of the problem D_i^0, and the corresponding solutions are called the eigenfunctions.

A similar result holds for problem N_i^0.

2) *If λ^2 and λ'^2 are eigenvalues of problem D_i^0, N_i^0 respectively, then $\lambda^2 \neq \lambda'^2$.*

2. We will first prove uniqueness of solutions of problems D_e and N_e. To do this, we need the following preliminary lemma.

Lemma 2. *Let U be a metaharmonic function in the domain T, belonging to class G (§2, sec. 2) and satisfying at infinity one of the Sommerfeld conditions (I) and (II). Let S' and S'' be parts of S with no common points such that $S = S' + S''$, one of these parts being possibly an empty set. If U satisfies the boundary condition*

$$U^+ = 0 \ \ on \quad S', \tag{D_e^0}$$

$$\frac{dU^+}{dn} = 0 \ \ on \quad S'', \tag{N_e^0}$$

where the latter condition may only hold almost everywhere on S'', then $U = 0$ everywhere in T.

Proof. Let Σ_R be a hypersphere of radius R, the interior of which contains the entire boundary S of the domain T. We denote by T_R the part of T inside Σ_R. The boundary of T_R is obviously $S+\Sigma_R$. We denote by $\overline{U}$ the function which is complex conjugate to U. Since U belongs to class G by hypothesis, $\overline{U}$ will obviously also belong to class G in T.

Suppose for definiteness that $\mathrm{Im}(\lambda)\geq 0$ and that condition (I) is satisfied at infinity, i.e.

$$L_1(U)\equiv \frac{dU}{dr}-i\lambda U=e^{i\lambda r}o(f^{-q-\frac{1}{2}}). \tag{$\bar{I}$}$$

We show as a preliminary that U cannot be real everywhere, unless $U=0$ everywhere in T. In fact, $\overline{U}\equiv U$, we should have, by (I):

$$\frac{dU}{dr}+i\bar{\lambda}U=e^{i\bar{\lambda}r}o(r^{-q-\frac{1}{2}}). \tag{I}$$

Since $\mathrm{Re}(\lambda)\neq 0$ and $\mathrm{Im}(\lambda)\geq 0$, from (I) and ($\bar{I}$) we obtain:

$$U=e^{i\lambda r}o(r^{-q-\frac{1}{2}}).$$

But, by Theorem 4, it follows from the latter condition that $U\equiv 0$. Our assertion is now proved.

Since U and $\overline{U}$ belong to class G, It follows from (24) that we are justified in writing

$$\int\limits_{T_R}(\overline{U}\Delta U-U\Delta\overline{U})dT=-\int\limits_{S+\Sigma_R}\left(\overline{U}\frac{dU}{dn}-U\frac{d\overline{U}}{dn}\right)dS. \tag{95}$$

But, in view of the fact $\Delta U=-\lambda^2 U,\ \Delta\overline{U}=-\lambda^2\overline{U}$, we obtain from (95), by virtue of conditions (D_e^0) and (N_e^0)

$$(\bar{\lambda}^2-\lambda^2)\int\limits_{T_R}U\overline{U}dT=\int\limits_{\Sigma_R}\left(\overline{U}\frac{dU}{dR}-U\frac{d\overline{U}}{dR}\right)s\Sigma_R. \tag{96}$$

Now let us first suppose that $\mathrm{Im}(\lambda)=\sigma>0$. Then we have, by condition (I) and formula (31):

$$\overline{U}\frac{dU}{dR}=R^{-p+1}e^{-2\sigma R}O(1).$$

Hence

$$\int\limits_{\Sigma_R}\left(\overline{U}\frac{dU}{dR}-U\frac{d\overline{U}}{dR}\right)d\Sigma_R=e^{-2\sigma R}O(1),$$

i.e. the integral on the right-hand side of (6) tends to zero as $R\to\infty$. Consequently,

$$\int\limits_{T}U\overline{U}dT=0,$$

i.e. $U \equiv 0$ in T.

We now consider the case $\mathrm{Im}(\lambda) = \sigma = 0$. We now have from (96):

$$\int\limits_{\Sigma_R} \left(\overline{U} \frac{dU}{dR} - U \frac{d\overline{U}}{dR} \right) d\Sigma_R = 0. \tag{97}$$

But, in view of the fact that U satisfies condition (I), from Theorem 3 we have:

$$U(R, \Theta) = \sum_{n=0}^{\infty} f_n(\lambda R) Y_n(\Theta|p), \tag{72}$$

where $f_n(x)$ denotes the function $x^{-q} H^{(1)}_{q+n}(x)$. On substituting this into (97), in view of the orthogonality of the functions $Y_n(\Theta|p)$ we obtain,

$$\sum_{n=0}^{\infty} [\overline{f_n(\lambda R)} f_n'(\lambda R) - \overline{f_n'(\lambda R)} f_n(\lambda R)] \int\limits_{\Sigma_R} Y_n \overline{Y}_n d\Sigma = 0. \tag{98}$$

But, by virtue of (63),

$$\overline{f_n(\lambda R)} f_n'(\lambda R) - \overline{f_n'(\lambda R)} f_n(\lambda R) = \frac{4i}{\pi} (\lambda R)^{-p+1} \quad (n = 1, 2, \ldots).$$

Hence (98) becomes

$$\sum_{n=0}^{\infty} \int\limits_{\Sigma_R} Y_n \overline{Y}_n d\Sigma = 0.$$

It is obvious from this that $Y_n(\Theta|p) = 0 \quad (n = 0, 1, 2, \cdots)$, i.e. in view of (72) and the fact that U is analytic, we have $U \equiv 0$ in the domain T. Our lemma is thus proved. The proof follows exactly similar lines in the case when U satisfies condition (II) at infinity. [10]

We now offer another proof of this lemma, which does not depend on U being able to be expanded in a series of the form (72). We must first prove another lemma.

Lemma 3. *Let $\lambda = \tau + i\sigma$. If a function U, metaharmonic outside some hypersphere, satisfies the condition*

$$\int\limits_{\Sigma_1} U(R\Theta) \overline{U(R\Theta)} d\Sigma = e^{-2|\sigma|R} o(R^{-p+1}), \tag{99}$$

[10] This lemma was first mentioned by A. Sommerfeld [6,7] for the case $p = 2, 3$ and real λ; however, he presented no strict proof. The first strict proof, somewhat different to the above one, was given by G. Freudenthal [10], who considered for simplicity the case $p = 2$; but, as remarked by the author himself (see [10], p. 223, 227), his method is quite general and can be easily extended to the space of any dimension, as was done for the case $p = 3$ by D.Z. Avazashvili (see [9]). Another method of proof of the lemma was proposed even earlier by V. D. Kupradze (see [8]), that appeared almost without change in his book [14]. But in this work, as was mentioned by the author himself, there are a number of ungrounded and incorrect statements.

where Σ_1 is the unit hypersphere with centre at the origin, then $U \equiv 0$.

Proof. Let $Y_n^{(1)}(\Theta|p), \cdots, Y_n^{(k_n)}(\Theta|p)$ be a complete system of orthogonal hyperspherical functions of order n ($n = 0, 1, 2, \cdots$). Let

$$a_{nk}(R) = \int\limits_{\Sigma_1} U(R,\Theta) Y_n^{(k)}(\Theta|p) d\Sigma \tag{100}$$

$$(n = 0, 1, 2, \ldots; \qquad (k = 1, 2, \ldots, k_n).$$

By Parseval's theorem, we obtain from (99):

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} a_{nk}(R)\overline{a_{nk}(R)} = e^{-2|\sigma|R} o(R^{-p+1}).$$

But we obviously have from here:

$$R^{q+\frac{1}{2}} e^{|\sigma|R} |a_{nk}(R)| = o(1) \quad (n = 0, 1, 2, \ldots; \quad k = 1, 2, \ldots, k_n). \tag{101}$$

On the other hand, we know that (§3, sec. 3)

$$a_{nk}(R) = A_{nk} R^{-q} H_{q+n}^{(1)}(\lambda R) + B_{nk} R^{-q} H_{q+n}^{(2)}(\lambda R), \tag{102}$$

where A_{nk} and B_{nk} are constants, independent of R.

On substituting (102) in (101) and taking (18) and (19) into account, we obtain with $\sigma \geq 0$:

$$|A_{nk}[\alpha + O(R^{-1})]e^{i\pi R} + B_{nk}[\beta + O(R^{-1})]e^{-i\pi R} e^{2\sigma R}| = o(1)$$
$$(n = 0,\ 1,\ 2, \ldots; \qquad k = 1,\ 2, \ldots, k_n; \quad \alpha \neq 0, \ \ \beta \neq 0).$$

Hence it follows at once that $A_{nk} = 0$, $B_{nk} = 0, i.e.$ $a_{nk}(R) = 0$ ($n = 0, 1, 2, \cdots;$ $k = 1, 2, \cdots, k_n$). But the latter means, in view of the completeness of the system $\{Y_n^{(k)}(\Theta|p)\}$ that $U \equiv 0$, which was to be proved. The proof follows exactly the same lines when $\sigma < 0$. [11]

[11] In the works [8, 14], Kupradze has the following lemma: A solution of the equation $\Delta u + \lambda^2 u = 0$, satisfying the conditions

$$\lim_{r\to\infty} \sqrt{r}\left(\frac{\partial u}{\partial u} + i\lambda u\right) = 0 \tag{α}$$

$$\lim_{r\to\infty} \int\limits_0^{2\pi} (\sqrt{r}u)^2 d\psi = 0 \tag{β}$$

is identically zero. The case of positive λ and p is obviously under consideration here. Our lemma 3 is obviously more precise than Kupradze's, since it is clear that condition (α) is superfluous. In addition, condition (β) is written incorrectly: the integrand should be $r|u|^2$, since otherwise the lemma contradicts the example $U = H_n^{(2)}(\lambda r)e^{in}\vartheta, \quad n > 0$.

Lemma 2 is an immediate consequence of lemma 3. For, in view of condition (I) and condition (31) that follows from it, (96) becomes (assuming that $\sigma \geq 0$)

$$-4i\sigma\tau \int\limits_{T_R} U\overline{U}dT = 2i(\tau + i\sigma)\int\limits_{\Sigma_R} U\overline{U}d\Sigma + e^{-2i\sigma R}o(1).$$

Hence, when $\lambda = \tau + i\sigma \neq 0, \quad \sigma \geq 0$, we have

$$\int\limits_{\Sigma_1} U\overline{U}d\Sigma = e^{-2\sigma}o(R^{-p+1}). \tag{99}$$

Consequently, by lemma 3, $U \equiv 0$, which was to be proved.

The uniqueness of the solution of problem N_e follows at once from lemma 2, i.e. we have

Theorem 5. *The problem N_e cannot have more than one solution, i.e. the homogeneous problem N_e^o has only the trivial solution $U \equiv 0$.*[12]

We now turn to the proof of the uniqueness of solution of the problem D_e. First, we prove the following preliminary lemma.

Lemma 4. *Let T be a finite or infinite domain of class B. Let U be a metaharmonic function in the domain T, continuous in $T+S$ and satisfying the boundary condition*

$$U^+ = f(x) \quad (\text{on } S), \tag{103}$$

where f is a given continuous function of the point x of the boundary S. We then have:

1. If f belongs to class Ch (§2, sec. 2) on S, then U belongs to class Ch in $T+S$.

2. If f and $D_1 f$ belongs to class Ch on S, then U and $D_1 U$ belong to class Ch in $T+S$.

3. If f, $D_1 f$ and $D_2 f$ belong to class Ch on S, then U, $D_1 U$ and $D_2 U$ belong to class Ch in $T+S$.

Proof. We observe first of all that these propositions are already familiar for harmonic functions (see e.g. [23] p. 243).

Let T be a finite domain. This assumption obviously does not restrict generality, since the propositions of the lemma only require proof in a vicinity of the boundary: inside T the function U is analytic, so that the lemma is obviously true in any closed domain lying inside T.

We represent U as

$$U(X) = \lambda^2 P(X) + U_0(X), \tag{104}$$

[12] It is worth observing that we have proved the uniqueness of a solution of problem N_e not only in the class G_0, but also in the wider class G; besides, it is sufficient that the boundary condition be satified almost everywhere on S with the given function f being bounded and Lebesgue integrable on S.

$$P(X) = \int_T \Omega_0(X,Y)U(Y)dT, \tag{105}$$

and Ω_0 is an elementary solution of Laplace's equation given by (15), and U_0 is a harmonic function [13] in T, which satisfies, by (103), (104) and the boundary condition

$$U_0^+ = \Psi(X) \quad (on\ S), \tag{106}$$

where $\Psi(x) = f(x) - \lambda^2 P(x)$. Since, by hypothesis, U is continuous in $T+S, P$ and $D_1 P$ will belong (see,e.g., [21], p. 82) to the class Ch throughout space. Hence, it $f \in Ch$ or f and $D_1 f \in Ch$ on S, we have respectively $\Psi \in Ch$ or Ψ and $D_1 f \in Ch$ on S. But in this case, in view of the well-known properties of harmonic functions (see [23], p. 243), we must have $U_0 \in Ch$ or U_0 and $D_1 U_0 \in Ch$ in $T+S$, respectively. The proof of the first two propositions of the lemma now follows at once, by (104).

Now let $f, D_1 f$ and $D_2 f \in Ch$ on S. We now always have U and $D_1 U \in Ch$ in $T+S$. But we know that, in this case, the functions, $P, D_1 P$ and $D_2 P \in Ch$ in $T+S$.[14] Hence $\Psi, D_1\Psi$ and $D_2\Psi \in Ch$ on S. Therefore $U_0, D_1 U_0$ and $D_2 I_0 \in Ch$ in $T+S$ (see [23], p. 243). It now follows at once, by (104), that $U, D_1 U$ and $D_2 U \in Ch$ in $T+S$, which was to be proved. The proof of the lemma is complete.

The following is an obvious corollary of Lemma 4.

Corollary. *If T is a domain of class B and U is a metaharmonic function in T, which is continuous in $T+S$ and vanishes on S, then $U, D_1 U$ and $D_2 U \in Ch$ in $T+S$.*

We now prove the uniqueness of solution of problem D_e. Let U_1 and U_2 be two solutions of the problem. The function $U = U_1 - U_2$ will be a solution of the homogeneous problem D_e^0. Hence, by the corollary of Lemma 4, $U \in G_0$, and hence, by Lemma 2, $U \equiv 0$ in T, i.e. we have the theorem.

Theorem 6. *The problem D_e has at most one solution, i.e. the homogeneous problem D_e^0 has only the trivial solution.*

3. Before passing to the direct proof of the existence of solution of problems D_e and N_e, we first give without proof some necessary properties of the so-

[13] In fact, on applying the operator Δ to both sides of (104), and using the fact that $\Delta P = -U$ and $\Delta U = -\lambda^2 U$, we find that $\Delta U_0 = 0$ in T.

[14] Indeed, let $x_1 \cdots, x_p$ and $\xi_1 \cdots, \xi_p$ be the Cartesian coordinates of the points X and Y respectively. It can easily be shown by means of Gauss's formula that

$$\frac{\partial P}{\partial x_k} = \int_T \Omega_0(X,Y)\frac{\partial U}{\partial \xi_k}dT - \int_S \Omega_0(X,y)U^+\cos(n,\xi_k)dS_y = I_1(X) + I_2(x)$$

where I_1 and I_2 denote the integrals over T and S respectively. Obviously, I_1 and $D_1 I_1 \in Ch$ throughout space. As regards I_2, it is the harmonic potential of a simple layer with density $U^+\cos(n,\xi_k)$, which obviously satisfies a Hölder condition. Hence (see [23] p. 201) I_1 and $D_1 I_1 \in T+S$. Consequently $D_2 P \in Ch$ in $T+S$ which is what we wanted to show.

called metaharmonic potentials:

$$V(X;\mu) = \int\limits_S \mu(y)\Omega(X,y)dS_y, \tag{107}$$

$$W(X;\nu) = \int\limits_S \nu(y)\frac{d}{dn_y}\Omega(X,y)dS_y, \tag{108}$$

where Ω is the elementary solution of equation (M) given by (12); μ and ν are continuous functions on the boundary S; $\frac{d}{dn_y}$ denotes the derivative with respect to the normal to S at the point y, directed inwards into the domain T.

The functions $V(X;\nu)$ and $W(X;\nu)$ satisfy equation (M) throughout space except for points of S. We will call these functions the metaharmonic potentials of a simple and double layer respectively, and μ and ν will be called the densities of the respective potentials. We call Ω the kernel of the metaharmonic potentials; in particular, we can take as the kernel the functions Ω_1 and Ω_2 given by (16) and (17) respectively. Notice that, as may easily be seen from formulae (20a) and (21a), the potentials with kernels Ω_1 and Ω_2 satisfy at infinity the Sommerfeld conditions (I) and (II) respectively.

The following fundamental properties of the metaharmonic potentials may be proved in exactly the same way as the corresponding properties of the harmonic potentials (see, e.g., [21]; see also [23], where detailed references to the literature may be found).

A. Let $\mu(x)$ be a continuous function of the point x on the boundary S. In this case the simple layer potential $V(X;\mu)$, given by (107), is a function of class Ch throughout all space. Furthermore, $\frac{dV^+}{dn}$, $\frac{dV^-}{dn}$ exist and belong to class C on S, where

$$\frac{dV^+}{dn_x} = -\frac{1}{2}\mu(x) + \int\limits_S \mu(y)\frac{d}{dn_x}\Omega(x,y)dS_y, \tag{109}$$

$$\frac{dV^-}{dn_x} = \frac{1}{2}\mu(x) + \int\limits_S \mu(y)\frac{d}{dn_x}\Omega(x,y)dS_y, \tag{110}$$

or

$$\frac{dV^+}{dn_x} - \frac{dV^-}{dn_x} = -\mu(x), \tag{111}$$

$$\frac{dV^+}{dn_x} + \frac{dV^-}{dn_x} = 2\int\limits_S \mu(y)\frac{d}{dn_x}\Omega(x,y)dS_y. \tag{112}$$

In addition, $\frac{d}{dn}V(X;\mu)$ is bounded both in T and in CT. Thus $(V(X;\mu)$ belongs to class G_0 both in T and in CT.

B. Let $\nu(x)$ be a continuous function of the point x of the boundary S. Then the double layer potential $W(X;\nu)$, given by (108), has the following properties:

1)W^+ and W^- exist and belong to class C on S;

2) the functions given by

$$W_e^* = \begin{cases} W(X;\nu), & \text{when} \quad X \in T, \\ W^+(X;\nu), & \text{when} \quad X = x \in S; \end{cases} \tag{113}$$

$$W_i^* = \begin{cases} W(X;\nu), & \text{when} \quad X \in CT, \\ W^-(X;\nu), & \text{when} \quad X = x \in S, \end{cases} \tag{114}$$

belong to class C in $T+S$ and in $CT+S$ respectively, which later will be denoted by $W(X;\nu) \in C$ both in $T+S$ and in $Ct+S$.

3)

$$W^+ = \frac{1}{2}\nu(x) + \int\limits_S \nu(y) \frac{d}{dn_y} \Omega(x,y) dS_y, \tag{115}$$

$$W^- = -\frac{1}{2}\nu(x) + \int\limits_S \nu(y) \frac{d}{dn_y} \Omega(x,y) dS_y, \tag{116}$$

or

$$W^+ - W^- = \nu(x), \tag{117}$$

$$W^+ + W^- = 2 \int\limits_S \nu(y) \frac{d}{dn_y} \Omega(x,y) dS_y. \tag{118}$$

4) LetX', X'' be points on the normal n_x at equal distances from x and belonging respectively to T and CT. We denote by σ the distance between the points X' and x. Now, for any small positive ϵ, there exists a positive $\eta(\epsilon)$ independent of x such that

$$\left| \frac{d}{dn_x} W(X') - \frac{d}{dn_x} W(X'') \right| < \epsilon, \quad \text{for } \sigma < \eta(\epsilon), \tag{119}$$

for all x on S.

This is the Tauber-Lyapunov theorem. Hence it follows that, if one of the functions $\frac{dW^+}{dn}$ or $\frac{dW^-}{dn}$ exists, the other exists too and

$$\frac{dW^+}{dn} = \frac{dW^-}{dn} \quad (on\ S). \tag{120}$$

5) Let ν and $D_1\nu \in Ch$ on S. Then $W(x;\nu)$ and $D_1 W(X;\nu) \in Ch$ both in $T+S$ and in $CT+S$. This is a sufficient condition for the existence of the derivatives of the double layer potential on S.

4. We now turn to finding solutions of problems D_e and N_e. Consider first problem N_e.

Let us assume that condition (I) holds at infinity. Now, if problem N_e has a solution, it can be represented in accordance with Theorem 2 by Green's formula

$$U(X) = \int\limits_S U^+ \frac{d}{dn_y}\Omega_1(X,y)dS_y - \int\limits_S \Omega_1(X,y)\frac{dU^+}{dn_y}dS_y. \tag{26}$$

The first term on the right-hand side of this formula is the metaharmonic double layer potential with density U^+, the second term on the right-hand side is the simple layer potential with density $\frac{dU^+}{dn}$. Let us always denote in what follows the simple and double layer potentials with kernel Ω_1 and densities μ and ν by $V(X;\nu)$ and $W(X;\nu)$ respectively.

Since, by hypothesis,

$$\frac{dU^+}{dn} = f(x) \quad (on\ S), \tag{N_e}$$

we can now write (26) in the alternative form

$$U(X) = W(X;U^+) - V(X;f) \quad (X \in T). \tag{121}$$

On passing to the limit as the point X approaches the boundary point x from the domain T, we obtain by A and B_3:

$$U^+ = \frac{1}{2}U^+ + W^+(x;U^+) - V(x;f)$$

or

$$U^+ = 2W^+(x;U^+) - 2V(x;f). \tag{122}$$

If we introduce the notation

$$U^+ = \varphi(x), \tag{123}$$

$$K(x;y) = 2\frac{d}{dn_y}\Omega_1(x,y), \tag{124}$$

$$-2V(x;f) = \mathrm{g}(x), \tag{125}$$

equation (122) becomes

$$\varphi(x) - \int\limits_S K(x,y)\varphi(y)dS_y = \mathrm{g}(x). \tag{126}$$

We have now arrived at a Fredholm integral equation. It will be shown below that this equation always has a solution - by means of which the solution of our problem can be found.

Consider now the homogeneous equation corresponding to (126):

$$\nu(x) - \int\limits_S K(x,y)\nu(y)dS = 0 \tag{126_0}$$

and the adjoint equation

$$\mu(x) - \int\limits_S K(y,x)\mu(x)dS = 0. \tag{$126'_0$}$$

Let $\nu_1, \cdots, \nu_k$ and $\mu_1, \cdots, \mu_k$ be complete systems of linearly independent solutions [15] of these equations. It is easy to show that these solutions are continuous on S.

Let us consider the potentials

$$V(X;\nu_i), \quad W(X;\mu_i) \quad (i = 1, 2, \cdots, k).$$

Using (A) and (B), we obtain from equations (126_0) and ($126'_0$):

$$W^-(x;\nu_i) = 0 \quad (i = 1, \cdots, k), \tag{127}$$

$$\frac{d}{dn}V^+(x;\nu_i) = 0 \quad (i = 1, \cdots, k). \tag{128}$$

Hence we have, using (111) and (117):

$$\frac{d}{dn}V^-(x;\mu_i) = \mu_i \quad (i = 1, \cdots, k), \tag{129}$$

$$W^+(x;\nu_i) = \nu_i(x) \quad (i = 1, \cdots, k). \tag{130}$$

But, since the potentials $V(X,\nu_i) \in G_0$ both in T and in CT, and, in view of the boundary condition (128), satisfy all the conditions of Theorem 5, we have

$$V(X;\mu_i) \equiv 0 \quad (in\ \ T) \quad (i = 1, \cdots, k).$$

We obtain from here, since the functions $V(X,\mu_i)$ are continuous,

$$V^-(x;\mu_i) = 0 \quad (i = 1, 2, \cdots, k). \tag{131}$$

It follows from (127) and (131) that $V(X;\mu_i)$ and $W(X;\nu_i)(i = 1, \cdots, k)$ are solutions of the problem D_i^0.

Let us show that none of these functions can vanish identically in CT if the corresponding densities (μ_i, ν_i) are non-zero. Indeed, if we had $V(X,\mu_i) \equiv 0$ in CT, we should have $\frac{d}{dn}V^-(x;\mu_i) \equiv 0$. Hence, by (129), we get $\mu_i \equiv 0$. Suppose now that $W(X;\nu_i) \equiv 0$ in CT. Then $\frac{d}{dn}W^-(x;\nu_i) \equiv 0$ and by B_4, $\frac{d}{dn}W^+(x;\nu_i) \equiv 0$. Hence, we have, by Theorem 5, $W(X;\nu_i) \equiv 0$ in T. Consequently, $W^+(x;\nu_i) \equiv 0$, and by (130), $\nu_i \equiv 0$. Our assertion is proved.

[15] In what follows, under solutions of homogeneous equations or homogeneous boundary value problems we will mean only those solutions which are not identically zero.

We can show by exactly the same arguments that each of the systems $V(X;\mu_i)$ $(i=1,\cdots,k)$ *and* $W(X;\nu_i)$ $(i=1,\cdots,k)$ *is a linearly independent system of functions in* CT.

We now observe that $V(X;\mu_i)$ and $W(X;\nu_i)$ satisfy the conditions or the corollary to Lemma 4 in CT. Hence

$$V(X;\mu_i),\ \ W(X;\nu_i),\ \ D_jV(X;\mu_i)\ \text{ and }\ D_jW(X;\nu_i)\in\ Ch \text{ in } CT+S$$

$$(i=1,\cdots,k;\quad j=1,2).$$

If we now make use of (129), we obtain the following lemma, which will be useful later.

Lemma 5. *Let* μ *be a solution of equation* $(126'_0)$*. Then* μ *and* $D_1\mu\in Ch$ *on* S.

We now show that each of the systems $V(X;\mu_i)$ $(i=1,\ldots,k)$, $W(X;\nu_i)$ $(i=1,\ldots,k)$ *is a complete system of solutions of problem* D_i^0.

Let U be a solution of problem D_i^0. Since $U^-\equiv 0$, by the corollary to Lemma 4, U, D_1U and $D_2U\in Ch$ in CT. We thus have, by Green's formula (25)

$$U(X)=\int\limits_S\Omega_1(X,y)\frac{dU^-}{dn_y}dS. \tag{132}$$

If we now differentiate both sides of this formula with respect to the normal direction at the point x and then take the limit, we obtain, by (110)

$$\frac{dU^-}{dn_x}-\int\limits_S K(y,x)\frac{dU^-}{dn_y}dS=0,$$

i.e. $\dfrac{du^-}{dn}$ satisfies the integral equation $(126'_0)$. Thus

$$\frac{dU^-}{dn}=\alpha_1\mu_1+\cdots+\alpha_k\mu_k, \tag{133}$$

where $\alpha_1,\cdots,\alpha_k$ are constants. On substituting (133), into (132) we get

$$U(X)=\alpha_1V(X;\mu_1)+\cdots+\alpha_kV(X;\mu_k),$$

which proves our assertion regarding the functions $V(X;\mu_i)$ $(i=1,\cdots,k)$. On recalling the linear independence of the functions $W(X;\nu_i)$ $(i=1,\cdots,k)$, we can now easily show that these latter also form a complete system of solutions of the problem D_i^0.

Consequently, *the problem* D_i^0 *obviously has solution if the homogeneous integral equation* (126_0) *or* $(126'_0)$ *has solution, and vice versa.*[16]

[16]By another method this statement was proved by Kupradze ([13], p. 565; [14], p. 88).

We now show that *the integral equation* (126) *always has a solution if its right-hand side has the form* (125).

In fact, the conditions for (126) to have a solution are

$$\int_S \mathrm{g}\mu_i dS = 0 \quad (i = 1, \cdots, k). \tag{134}$$

But, by (125), these conditions take the form

$$\int_S f(x) V^-(x, \mu_i) dS = 0 \quad (i = 1, \cdots, k),$$

which actually hold, in view of (131). The existence of a solution of (126) is thus proved.

Let $\varphi(x)$ be any given solution of (126). Obviously, $\varphi(x)$ will be continuous if $\mathrm{g}(x)$ is continuous. But $\mathrm{g}(x)$ is shown to be continuous by (125), since $f(x)$ is continuous by hypothesis. Hence, by (116), we can rewrite (126) as

$$W^-(x; \varphi) - V^-(x; f) = 0, \tag{135}$$

i.e. the function $W(X; \varphi) - V(X; f)$ is a solution of the problem D_i^0. Thus,

$$W(X; \varphi) - V(X; f) = \sum_{i=1}^{k} \alpha_i W(X; \nu_i) \quad in \quad CT, \tag{136}$$

where $\alpha_1, \cdots, \alpha_k$ are definite constants.

We now introduce the function

$$U(X) \equiv W(X; \varphi) - V(X; f) - \sum_{i=1}^{\kappa} \alpha_i W(X; \nu_i). \tag{137}$$

It will be shown that U is a solution of problem N_e. It is obvious that U is a metaharmonic function in T, satisfying condition (I) at infinity and continuous in $T + S$. It therefore remains to prove that U is a function of class G_0 and that it satisfies the boundary condition (N_e).

Because of A, $V(X; f)$ belongs to class G_0 in T and in CT. As we have seen above, $W(X; \nu_i)$ also belongs to the class G_0, since they are solutions of problem D_i^0. Consequently, by (136), $W(X; \varphi)$ belongs to the class G_0 in CT. We thus have because of (136)

$$\frac{d}{dn} W^-(x; \varphi) = \frac{d}{dn} V^-(x; f) + \sum_{i=1}^{\kappa} \alpha_i \frac{d}{dn} W^-(x; \nu_i). \tag{138}$$

But, in view of B_4 and (111), this equation becomes

$$\frac{d}{dn} W^+(x; \varphi) - \frac{d}{dn} V^+(x; f) - \sum_{i=1}^{\kappa} \alpha_i \frac{d}{dn} W^+(x; \nu_i) = f(x),$$

and, by (137), is equivalent to

$$\frac{dU^+}{dn} = f(x).$$

We have thus shown that U is a function of class G_0, satisfying the condition (N_e), i.e. the function U given by (137) is a solution of problem N_e.

We observe finally that our discussion refers to the case when the homogeneous equation (126_0) has non-trivial solutions. But it may easily be seen that, when this equation has only the trivial solution, our conclusion still holds, provided we put all α_i in (137) equal to zero, i.e. the solution of problem N_e in this case is

$$U(X) = W(X; \varphi) - V(X; f). \tag{137a}$$

We have assumed up to now that condition (I) holds at infinity. If condition (II) holds at infinity, our discussion naturally still holds provided we now use the potentials with kernel Ω_2.

Note. It may easily be seen that the above method enables us to show that problem N_e also has a solution which is unique in class G, if the given function f is bounded and integrable on S. We only need to observe that the simple layer potential $V(X; f)$ is a function of class G both in T and in CT, provided f is a bounded integrable function on S.

5. We now turn to the solution of problem D_e; here we will assume first that condition (I) holds at infinity.

To solve problem D_e we use a method somewhat different to that used above for solving problem N_e. If the latter method is used, we meet a serious difficulty, which can only be overcome by imposing further restrictions (as well as continuity) on the given function f. In fact, if we assume that problem D_e has a solution in the class G, then, by Green's formula (26), we have

$$U(X) = W(X; f) - V\Big(X; \frac{dU^+}{dn}\Big). \tag{139}$$

Thus, this formula will give the solution of problem D_e provided we can find $\frac{dU^+}{dn}$. If now differentiate both sides of (139) with respect to the normal and then take the limit, we obtain the following integral equation for $\frac{dU^+}{dn}$:

$$\frac{dU^+}{dn_x} + \int\limits_S K(y, x) \frac{dU^+}{dn_y} dS_y = 2\frac{d}{dn} W^+(x; f), \tag{140}$$

where $K(x, y)$ is the kernel given by (124). It is clear from here that our purpose is achieved only when $\dfrac{d}{dn} W^+(x; f)$ is at least an integrable function. However, this is not in general the case if f is merely continuous. We must therefore use a different method to show that problem D_e always has a solution for any continuous function f.

We seek the solution in the form [17]:

$$U(X) = W(X;\nu) + V(X;\mu), \tag{141}$$

where ν and μ are as yet undetermined continuous functions of a point of the boundary S. On passing to the limit as the point X approaches the boundary point x from the domain T, we have, by (115):

$$\nu(x) + \int\limits_S K(x,y)\nu(y)dS_y = h(x), \tag{142}$$

where $K(x,y)$ is the kernel given by (124), and

$$h(x) = 2f(x) - 2V(x;\mu). \tag{143}$$

We have thus obtained the Fredholm equation (142) for defining the function ν. As regards μ, we will leave it undefined for the moment. Obviously, if ν is a solution of equation (142), then (141) will give the solution of problem D_e whatever the function μ. Our immediate problem is therefore to select μ in such a way that equation (142) has a solution.

Suppose first that the homogeneous equation

$$\nu(x) + \int\limits_S K(x,y)\nu(y)dS = 0 \tag{142_0}$$

has only the trivial solution. Equation (142) then obviously always has a solution. In particular, therefore, we can put $\mu = 0$. The solution of problem D_e may now be written as the double layer potential,

$$U(X) = W(X;\nu), \tag{141a}$$

where ν is the solution of equation (142) with $\mu = 0$. This solution obviously belongs to the class C in $T + S$, since ν is a continuous function.

We now turn to the case when equation (142_0) has non-trivial solutions. We consider the equation adjoint to (142_0):

$$\mu(x) + \int\limits_S K(y,x)\mu(y)dS = 0. \tag{$142'_0$}$$

Let $\mu_1, ..., \mu_k$ be a complete system of linearly independent solutions of this equation. Obviously, $\mu_1, ..., \mu_k$ are continuous functions. We consider the simple layer potentials $V(X;\mu_1), ..., V(X;\mu_k)$ which, in view of ($142'_0$) and (110), satisfy the conditions

$$\frac{d}{dn}V^-(x;\mu_i) = 0 \qquad (i = 1, 2, ..., k). \tag{144}$$

[17]The problem D_e was solved by this method in the reference [16].

It follows from here, by (111), that

$$\frac{d}{dn}V^+(x;\mu_i) = -\mu_i \qquad (i = 1, 2, ..., k). \tag{145}$$

We now show that $V(X;\mu_i)$ $(i = 1, ..., k)$ do not vanish identically in CT and are linearly independent.

Let $V(X;\mu_i) \equiv 0$ in CT. Then $V^-(x;\mu_i) = V^+(x;\mu_i) \equiv 0$. Consequently, by Theorem 6, $V(X;\mu_i) \equiv 0$ in T. Hence $\dfrac{d}{dn}V^+(x;\mu_i) \equiv 0$, i.e. $\mu_i \equiv 0$, which contradicts our assumption.

It can similarly be proved that the functions $V(X;\mu_i)$ $(i = 1, ..., k)$ are linearly independent.

Thus, by (144), the functions $V(X;\mu_i)$ $(i = 1, ..., k)$ are linearly independent solutions of problem N_i^0.

We show that these functions represent a complete system of solutions of the latter problem (see, [13, 14]).

In fact, let U be any solution of problem N_i^0. In view of condition (N_i^0), we obtain with the aid of Green's formula (25):

$$U(X) = -W(X;U^-) \qquad in \quad CT. \tag{146}$$

Hence we have, by (116)

$$U^-(x) + \int\limits_S K(x,y)U^-(y)dS = 0, \tag{142$_0$}$$

i.e.

$$U^- = \alpha_1\nu_1 + ... + \alpha_k\nu_k, \tag{147}$$

where $\nu_1, ..., \nu_k$ is a complete system of solutions of equation (142_0), and $\alpha_1, ..., \alpha_k$ are constants. By (147), we obtain from (146):

$$U(X) = -\alpha_1 W(X;\nu_1) - ... - \alpha_k W(X;\nu_k) \qquad (X \in CT). \tag{148}$$

We have thus shown that any solution of problem N_i^0 is a linear combination of the potentials $W(X;\nu_i)$ $(i = 1, ..., k)$.

We now show that these potentials do not vanish identically in CT.

In fact, it follows from (142_0) that

$$W^+(x;\nu_i) = 0 \qquad (i = 1, ..., k). \tag{149}$$

In view of this, we have by (117):

$$W^-(x;\nu_i) = -\nu_i \qquad (i = 1, ..., k). \tag{150}$$

It is hence obvious that $W(X;\nu_i) \not\equiv 0)$ in CT, and that these functions are linearly independent in CT.

In addition, by (148), these functions represent a complete system of solutions of problem N_i^0. It may easily be seen from here that the functions $V(X;\mu_i)$ $(i=1,...,k)$ also represent a complete system of solutions of problem N_i^0.

We now show that a functions μ can always be chosen in (143) such that equation (142) has a solution. Indeed, the conditions for this equation to have a solution are

$$\int\limits_S h\mu_i dS = 0 \qquad (i=1,...,k). \tag{151}$$

But, by (143), these conditions become

$$\int\limits_S \mu(x)V(x;\mu_i)dS = f_i \qquad (i=1,...,k), \tag{152}$$

where

$$f_i = \int\limits_S \mu_i f dS \qquad (i=1,...,k).$$

We showed above that $V(X;\mu_i)$ are solutions of problem N_i^0. Consequently, by (148),

$$V(X;\mu_i) = \sum_{j=1}^{k} \beta_{ij} W(X;\nu_i) \qquad (i=1,...,k), \tag{153}$$

in CT, where β_{ij} are definite constants, and the determinant $|\beta_{ij}|$ is obviously non-zero; otherwise, $V(X;\mu_i)$ would be linearly dependent. We obtain from (153), by (150),

$$V^-(x;\mu_i) = \sum_{j=1}^{k} \beta_{ij} W^-(x;\nu_j) = -\sum_{j=1}^{k} \beta_{ij}\nu_j. \tag{154}$$

We introduce the notation

$$\nu_i^* = -\sum_{j=1}^{k} \beta_{ij}\nu_j \qquad (i=1,...,k). \tag{155}$$

Obviously, the functions ν_i^* $(i=1,...,\kappa)$ are linearly independent. By (155) and (154), conditions (152) become

$$\int\limits_S \mu_i \nu_i^* dS = f_i \qquad (i=1,...,k). \tag{156}$$

We now put

$$\mu = \sum_{i=1}^{k} A_i \bar{\nu}_i^*, \tag{157}$$

where the constants $A_1, ..., A_k$ are given by the equations

$$a_{i1}A_1 + a_{i2}A_2 + ... + a_{ik}A_k = f_i \qquad (i = 1, ..., k), \tag{158}$$

and

$$a_{ij} = \int\limits_S \nu_i^* \bar{\nu_j^*} dS \qquad (i, j = 1, ..., k).$$

The system (158) obviously has a solution, since the determinant $|a_{ij}|$ is non-zero because of the linear independence of ν_i^* $(i = 1, ..., k)$. If we now take as μ the function (157), the conditions (156) will evidently be satisfied. Consequently, equation (142) will have a solution, provided its right-hand side is taken equal to

$$h(x) = 2f(x) - 2\sum_{i=1}^{\kappa} A_i V(x; \bar{\nu_i^*}). \tag{159}$$

If we now replace ν in (141) by any solution of equation (142), and μ by the function (157), we obtain the solution of problem D_l in the form

$$U(X) = W(X; \nu) + \sum_{i=1}^{\kappa} A_i V(X; \bar{\nu_i^*}). \tag{160}$$

We prove that the functions $V(X; \bar{\nu_i^*})$ $(i = 1, ..., k)$ do not in general vanish simultaneously in the domain T. If this were to happen, equation (142) would have a solution whatever its right-hand side, which is of course impossible since there are cases when the corresponding homogeneous equation (142_0) has a nontrivial solution. In fact, if λ is an eigenvalue of problem N_i^0, then, as we saw above, (142_0) has a nontrivial solution, and vice versa.

In this way we finally obtain the following theorem.

Theorem 7. *The problem D_e always has a solution. The solution is represented as the double layer potential if λ is not an eigenvalue of the corresponding homogeneous problem N_i^0, but if λ is an eigenvalue of the corresponding homogeneous problem N_i^0, then the solution of problem D_e can not be represented, in general, in the form of the double layer potential. In this case, the solution has the form (160).*

The problem D_e may be solved in a similar way in the case when condition (II) holds at infinity, provided we make use of the potentials with kernel Ω_2.

6. We can now propose a further method for solving problem N_e.

We seek the solution of the problem as

$$U(X) = V(X; \mu) + U_0(X), \tag{161}$$

where μ is a continuous function of a point of the boundary S, and U_0 is the solution of a specific problem D_e to be defined bleow; we will only assume for the moment that U_0 belongs to class G_0 in T.

On differentiating both sides of (161) with respect to the normal direction and taking the limit, we obtain, by (109) and the condition (N_e), the Fredholm integral equation

$$\mu(x) - \int_S K(y,x)\mu(y)dS = g^*(x), \tag{162}$$

where

$$g^*(x) = -2f(x) + 2\frac{dU_0^+}{dn_x}. \tag{163}$$

Having found μ from this equation and substituting it in (161), we obtain the solution of problem N_e. Consequently, it remains for us to select the metaharmonic function U_0 of class G_0 in such a way that the integral equation (162) has a solution.

We take the homogeneous equation corresponding to (162):

$$\mu(x) - \int_S K(y,x)\mu(y)dS = 0 \tag{162_0}$$

and its adjoint

$$\nu(x) - \int_S K(x,y)\nu(y)dS = 0. \tag{$162'_0$}$$

Equations $(162'_0)$ and (162_0) are the same as (126_0), $(126'_0)$ respectively, which we discussed in details in sec. 4 of the present part.

We suppose first that (162_0) has only the trivial solution. Obviously, (162) will now have a solution whatever its right-hand side, and, in particular, we can put $U_0 \equiv 0$. Formula (161) becomes in this case

$$U(X) = V(X;\mu), \tag{161a}$$

where μ is the solution of equation (162) with $g^* = -2f$. Consequently, when equation (162_0) has only the trivial solution (and this is only the case when problem D_i^0 has only the trivial solution), we can write the solution of problem N_e as the simple layer potential (161a).

Now let the homogeneous equation (162_0) have a nontrivial solution. Let $\mu_1, ..., \mu_\kappa$ and $\nu_1, ..., \nu_\kappa$ be complete systems of solutions of equations (162_0) and $(162'_0)$ respectively. As we showed in sec. 4 of this part, the potentials $V(X;\mu_i)$ $(i = 1, ..., \kappa)$, and also $W(V;\nu_i)$ $(i = 1, ..., \kappa)$, represent complete systems of linearly independent solutions of problem D_i^0. Consequently, the case we are discussing can only hold when λ is an eigenvalue of problem D_i^0, i.e. in our case λ is positive.

We now write down the conditions for (162) to have a solution

$$\int_S g^*\nu_j dS = 0 \qquad (j = 1, ..., k), \tag{164}$$

which become, by (163) and (130)

$$\int_S \frac{dU_0^+}{dn} W^+(x;\nu_j) dS = f_j \qquad (j = 1, ..., k). \tag{165}$$

As we showed above (sec. 4), the functions $W(X;\nu_j)$ $(j = 1, ..., k)$ belong to class G_0 both in the domain T and in CT. In addition, U_0 belongs to class G_0 in T by hypothesis. Hence we have, by (24):

$$\begin{aligned}&\int_S \Big(\frac{dU_0^+}{dn} W^+(x,\nu_j) - U_0^+ \frac{d}{dn} W^+(x,\nu_j)\Big) dS \\ &= \int_{\Sigma_R} \Big(\frac{dU_0}{dR} W(X;\nu_j) - U_0 \frac{d}{dR} W(X,\nu_j)\Big) d\Sigma,\end{aligned} \tag{166}$$

where Σ_R is a hypersphere of sufficiently large radius R, containing S.

But the functions U_0 and $W(X;\nu_j)$ $(j = 1, ..., k)$ satisfy one of the Sommerfeld conditions (I_0) and (II_0) at infinity. Suppose that condition (I_0) holds, i.e.

$$\frac{dU}{dr} = i\lambda U + o(r^{-q-\frac{1}{2}}). \tag{I_0}$$

From this condition, and the condition that follows from (I_0):

$$U = O(r^{-q-\frac{1}{2}}), \tag{33}$$

we obtain

$$\frac{dU_0}{dR} W(X;\nu_j) - U_0 \frac{d}{dR} W(X;\nu_j) = R^{-p+1} o(1).$$

On substituting this in the right-hand side of (166), we easily find that

$$\int_S \Big[\frac{dU_0^+}{dn} W^+(x;\nu_j) - U_0^+ \frac{d}{dn} W^+(x;\nu_j)\Big] dS = 0 \qquad (j = 1, ..., k),$$

i.e. conditions (165) become

$$\int_S U_0^+ \frac{d}{dn} W^+(x;\nu_j) dS = f_j \qquad (j = 1, ..., k). \tag{167}$$

But in CT, as we already know (sec. 4),

$$W(X;\nu_j) = \sum_{i=1}^{k} \alpha_{ji} V(X;\mu_i) \quad (j = 1, ..., k), \tag{168}$$

where α_{ji} are constants, and the determinant $|\alpha_{ji}| \neq 0$. We obtain from this, in view of B_4 and (129):

$$\frac{d}{dn}W^+(x;\nu_j) = \frac{d}{dn}W^-(x;\nu_j) = \sum_{i=1}^{k}\alpha_{ji}\frac{d}{dn}V^-(x;\mu_i) = \sum_{i=1}^{k}\alpha_{ji}\mu_i \equiv \mu_i^* \quad (169)$$
$$(j = 1, ..., k).$$

The functions μ_j^* $(j = 1, ..., k)$ obviously form a complete system of solutions of equation $(162_0')$, i.e. (126_0).

Hence, by Lemma 5, μ_i^* and $D_1\mu_i^* \in Ch$ on S $(i = 1, ..., k)$.

If now we substitute (169) in (167), we get

$$\int_S U_0^+\mu_i^* dS = f_i \qquad (i = 1, ..., k). \tag{170}$$

These conditions will obviously hold if

$$U_0^+ = \sum_{j=1}^{k} B_j\bar{\mu}_j^*, \tag{171}$$

where B_j are constants, defined by the equations

$$b_{i1}B_1 + b_{i2}B_2 + ... + b_{ik}B_k = f_i \qquad (i = 1, ..., k),$$

where

$$b_{ij} = \int_S \bar{\mu}_i^*\mu_j^* dS \qquad (i, j = 1, ..., k);$$

the determinant $|b_{ij}|$ obviously does not vanish.

Thus, to find the function U_0, we have to solve problem D_e with the boundary condition (171). But we showed in the previous section that this problem always has a solution. It now remains to show that U_0 is a function of class G_0 in the domain T. But this is obvious from Lemmas 4 and 5, since, by (171), U_0^+ and $D_1U_0^+ \in Ch$ on S.

If we now substitute our function U_0 in the right-hand side of (162), we arrive at an integral equation which has a solution for any function f. Replacing μ in (161) by the solution of this equation, where U_0 is a solution of problem D_e with boundary condition (171), we obtain the required solution of problem N_e.

It is now easily shown that the functions U_0 cannot in general be represented by the simple layer potential. For, if this were possible, we would find that (162) has a solution whatever its right-hand side, no matter what the parameter λ. But this cannot be true, since the homogeneous equation (162_0) has a nontrivial solution whenever λ is an eigenvalue of the problem D_i^0.

We thus have the following

Theorem 8. *The problem N_e always has a solution. Its solution can be represented by the simple layer potential if λ is not an eigenvalue of the problem D_i^0. If λ is such an eigenvalue, then the solution of problem N_e cannot in general be represented by the simple layer potential, but instead it has the form (161).*

7. We now introduce Green's functions corresponding to problems D_e and N_e.

Let X and X_0 be respectively a variable and a fixed point of the domain T. We will denote by (I, D_e) and (I, N_e) the problems D_e, N_e respectively when condition (I) is satisfied at infinity. The symbols $(II,\ D_e)$ and $(II,\ N_e)$ are similarly defined.

Green's function $G_I(X, X_0)$ corresponding to problem (I, D_e) is defined as follows: the function

$$g_I(X, X_0) = G_I(X, X_0) - \Omega_1(X, X_0) \tag{172}$$

is metaharmonic in T, satisfies condition (I) at infinity, and satisfies the boundary condition

$$g_I^+(x, X_0) = -\Omega_1(x, X_0) \qquad (x \in S). \tag{173}$$

Green's function $\Gamma_I(X, X_0)$ corresponding to problem (I, N_e) is defined thus: the function

$$\gamma_I(X, X_0) = \Gamma_I(X, X_0) + \Omega_1(X, X_0) \tag{174}$$

is metaharmonic in T satisfies condition (I) at infinity and satisfies the boundary condition

$$\frac{d}{dn}\gamma_I^+(x; X_0) = \frac{d}{dn}\Omega_1(x; X_0) \qquad (x \in S). \tag{175}$$

Green's functions $G_{II}(X, X_0)$ and $\Gamma_{II}(X, X_0)$ corresponding to problems (II, D_e) and (II, N_e) are similarly defined; we only have to take Ω_1 instead of Ω_2 in these cases.

Obviously, all these functions exits and, with X_0 fixed, belong to the class G_0 in the domain T.

The solutions of problems (I, D_e) and (I, N_e) can obviously be expressed respectively by the formulae:

$$U(X_0) = \int\limits_S U^+ \frac{d}{dn_x} G_I(x; X_0) dS, \tag{176}$$

$$U(X_0) = \int\limits_S \Gamma_I(x; X_0) \frac{dU^+}{dn_x} dS. \tag{177}$$

Similar expressions can be obtained for the solutions of problems (II, D_e) and (II, N_e) in terms of the functions $G_{II}(X, X_0)$ and $\Gamma_{II}(X, X_0)$ respectively.

8. We now take as an example Green's function $G_I(X,X_0)$ for the domain T bounded by the hypersphere Σ_a of radius a with centre at the origin.

Since $g_I(X,X_0)$ satisfies condition (I) at infinity, it can be expanded, by Theorem 3 (§ 3, sec. 3), into a series

$$g_I(X,X_0)=\sum_{n=0}^{\infty} r^{-q}H^{(1)}_{q+n}(\lambda r)Y_n(\Theta|p), \tag{178}$$

where $r,\ \Theta(\theta_1,...,\theta_{p-1})$ are the polar coordinates of the variable point X.

In view of (80), we have

$$\Omega_1(x,X_0)=\frac{i}{4}\frac{1}{(2\pi)^q}\sum_{n=0}^{\infty} a^{-q}J_{q+n}(\lambda a)\rho^{-q}H^{(1)}_{q+n}(\lambda\rho)P^*_n(cos\gamma|p), \tag{179}$$

where x is a point of the hypersphere Σ_a $(S\equiv\Sigma_a)$ of radius a, ρ is the radius vector of the fixed point X_0, γ is the angle between the vectors OX_0 and Ox, P^*_n are the functions defined by (78a) or (78b).

By (178) and (179), the boundary condition (173) gives:

$$\sum_{n=0}^{\infty} a^{-q}H^{(1)}_{q+n}(\lambda a)Y_n=\frac{1}{4i(2\pi)^q}\sum_{n=0}^{\infty} a^{-q}J_{q+n}(\lambda a)\rho^{-q}H^{(1)}_{q+n}(\lambda\rho)P^*_n. \tag{180}$$

We obtain at once from this, in view of (82) and (82a):

$$Y_n(\Theta|p)=\frac{1}{4i(2\pi)^q}\frac{J_{q+n}(\lambda a)\rho^{-q}H^{(1)}_{q+n}(\lambda\rho)}{H^{(1)}_{q+n}(\lambda a)}P^*_n(cos\gamma|p). \tag{181}$$

On substituting (181) in (178), we obtain the required function

$$g_I(X,X_0)=\frac{1}{4i(2\pi)^q}\sum_{n=0}^{\infty}\frac{J_{q+n}(\lambda a)r^{-q}H^{(1)}_{q+n}(\lambda r)\rho^{-q}H^{(1)}_{q+n}(\lambda\rho)}{h^{(1)_{q+n}}(\lambda a)}P^*_n(cos\gamma|p). \tag{182}$$

On substituting this in (172), we find from the latter Green's function $G_I(X,X_0)$ for the domain bounded by the hypersphere Σ_a.

All the other Green's functions mentioned above can be constructed for this domain in a similar manner. We will not discuss this in detail here.

Note 1. The methods described above for solving the exterior Dirichlet and Neumann problems can also be used for solving the analogous problems in the case of general differential equations of the elliptic type. The methods obviously can also be used in the case of Laplace's equation.

Generally speaking, these methods differ from the usual methods of potentials, widely employed in solving boundary value problems in connection with elliptic differential equations [23, 26, 27, 28] only in central modifications of a methodological kind. The modifications amount to the fact that, according to our method, it is always possible to form a combination of simple and double layer potentials which will lead to a Fredholm integral equation equivalent to the Dirichlet or Neumann boundary value problem in question.

Note 2. The solution by means of potentials of problems D_e and N_l in the cases $p = 2, 3$, and real λ, was also considered by Kupradze [11-15].

2 Some Fundamental Properties of Solutions of Equation (A)

The second half of the paper is devoted to the general representation of all solutions of equation (A) by means of metaharmonic functions (§5). In addition, the Riquier boundary value problem connected with equation (A) will be solved in §6 by means of this representation.

§ 5. General representation of solutions of equation (A)

1. We can write equation (A) as [18]

$$\Delta^k(\Delta + \chi_1)^{k_1}...(\Delta + \chi_m)^{k_m}U = 0, \tag{A}$$

where $\chi_1, ..., \chi_m$ are distinct roots of the equation

$$\chi^n - a_1\chi^{n-1} + ... + (-1)^n a_n = 0, \tag{a}$$

and $k_1, ..., k_m$ are the respective multiplicities of these roots; it is assumed here that

$$\chi_i \neq 0 \qquad (i = 1, ..., m);$$

k is the multiplicity of the root $\chi = 0$. Obviously, $k \geq 0,\ k_i \geq 1\ (i = 1, ..., m)$ and

$$k + k_1 + ... + k_m = n.$$

It will be assumed that χ_i, like the coefficients of equation (A), are in general complex.

We will assume that, in general, the solutions of (A) are complex-valued functions. We will denote the solutions symbolically by $U(X; A)$, where X is a point of space, and A indicates the dependence of the solution on the parameters $a_1, ..., a_n$.

We will prove the following

Theorem 9. *Every solution of equation (A), regular in some domain T, has the form*

$$U(X; A) = V(X) + \sum_{i=1}^{m}\sum_{j=0}^{k_1-1} r^j \frac{\partial^j U_{ij}(X; \chi_i)}{\partial r^j}, \tag{1}$$

[18]My attention to this form of writing equation (A) was drawn by Acad. S.L. Sobolev.

where $V(X)$ is a k-harmonic function, i.e. a solution of the equation $\Delta^k V = 0$, and $U_{ij}(X;\chi_i)$ $(j = 0, 1, ..., k_{i-1})$ are metaharmonic functions with the parameters χ_i $(i = 1, ..., m)$, $\Delta U_{ij} + \chi_i U_{ij} = 0$ $(i = 1, ..., m;\ \ j = 0, 1, ..., k_i - 1)$.

These functions are uniquely defined by means of the corresponding solution of equation (A).

Conversely, formula (1) gives the solution of equation (A), where V is any k-harmonic function, and U_{ij} $(j = 0, 1, ..., k_i - 1)$ are any metaharmonic functions with the parameters χ_i $(i = 1, ..., m)$.

Proof. 1^0. We first take the case $k = 0,\ k_i = 1$ $(i = 1, ..., n)$. In this case equation (A) and formula (1) become

$$(\Delta + \chi_1)(\Delta + \chi_2)...(\Delta + \chi_n)U = 0, \tag{2}$$

$$U(X;A) = U_1(X;\chi_1) + U(X;\chi_2) + ... + U_n(X;\chi_n), \tag{3}$$

where $U_1(X;\chi_1), ..., U_n(X;\chi_n)$ are metaharmonic functions with the parameters $\chi_1, ..., \chi_n$ respectively. It may easily be seen that expression (3) is always a solution of equation (2). It remains to show that any solution of this equation can be expressed by (3).

Formula (3) obviously holds with $n = 1$. Suppose now that this formula holds for $n = k$, where k is any positive integer, and consider the equation

$$(\Delta + \chi_1)...(\Delta + \chi_k)(\Delta + \chi_{k+1})U = 0. \tag{4}$$

This equation is obviously equivalent to

$$(\Delta + \chi_1)...(\Delta + \chi_k)U = U'_{k+1}(X;\chi_{k+1}), \tag{5}$$

where

$$U'_{k+1}(X;\chi_{k+1})$$

is a metaharmonic function with parameter χ_{k+1}. It may easily be verified that equation (5) is satisfied by the metaharmonic function

$$U_{k+1}(X;\chi_{k+1}) = \frac{U'_{k+1}(X;\chi_{k+1})}{(\chi_1 - \chi_{k+1})...(\chi_k - \chi_{k+1})}. \tag{6}$$

Hence, in view of our assumption, the general solution of equation (5), or what amounts to the same thing, of equation (4), is

$$U_1(X;\chi_1) + ... + U_k(X;\chi_k) + U_{k+1}(X;\chi_{k+1}), \tag{7}$$

where

$$U_1(X;\chi_1), ..., U_k(X;\chi_k)$$

are metaharmonic functions with parameters $\chi_1, ..., \chi_k$ respectively. But, since the validity of (3) is obvious with $n = 1$, we find at once from (7) that it also holds for any positive integer n, which was to be proved.

If we now apply the operator

$$(\Delta + \chi_1)...(\Delta + \chi_{i-1})(\Delta + \chi_{i+1})...(\Delta + \chi_n),$$

to both sides of (3), we easily obtain the formula

$$U_i(X;\chi_i) = \frac{(\Delta + \chi_1)...(\Delta + \chi_{i-1})(\Delta + \chi\_i + 1)...(\Delta + \chi_n)U(x;A)}{(\chi_1 - \chi_i)...(\chi_{i-1} - \chi_i)(\chi_{i+1} - \chi_i)...(\chi_n - \chi_i)}. \qquad (8)$$

This shows that the metaharmonic functions appearing in (3) are uniquely determined by the n-metaharmonic function $U(X;A)$.

2^0. We now take the case when $k = 0$, while $k_1, ..., k_m$ are arbitrary positive integers. In this case equation (A) and formula (1) become respectively

$$(\Delta + \chi_1)^{k_1}(\Delta + \chi_2)^{k_2}...(\Delta + \chi_m)^{k_m}U = 0, \qquad (9)$$

$$U(X;A) = \sum_{i=1}^{m}\sum_{j=0}^{k_i-1} r^j \frac{\partial^j U_{ij}(X;\chi_i)}{\partial r^j}. \qquad (10)$$

Let the latter formula hold for

$$k_1 = p_1, ..., k_m = p_m,$$

where $p_1, ..., p_m$ are arbitrary positive integers; we consider the equation

$$(\Delta + \chi_1)^{p_1+1}(\Delta + \chi_2)^{p_2}...(\Delta + \chi_m)^{p_m}U = 0, \qquad (11)$$

which is obviously equivalent to

$$(\Delta + \chi_1)^{p_1}(\Delta + \chi_2)^{p_2}...(\Delta + \chi_m)^{p_m}U = U_1'(X;\chi_1), \qquad (12)$$

where $U_1'(X;\chi_1)$ is a metaharmonic function with parameter χ_1.

We introduce for brevity the notation

$$\Delta_k \equiv r^k \frac{\partial^k}{\partial r^k} \qquad (k = 0, 1, ...;\ \Delta_0 \equiv 1) \qquad (13)$$

and show that equation (12) is satisfied by

$$\Delta_{p_1} U_1(X;\chi_1), \qquad (14)$$

where $U_1(X;\chi_1)$ is the metaharmonic function with parameter χ_1, given by

$$U_1(X;\chi_1) = \frac{(-1)^{p_1} U_1'(X;\chi_1)}{2^{p_1} p_1! \chi_1^{p_1} \prod\limits_{i \neq 1} (\chi_i - \chi_1)^{p_i}}. \qquad (15)$$

To prove this we need to make use of the formula

$$(\Delta + \chi)\Delta_k U(X;\chi) = -2\chi k \Delta_{k-1} U(X;\chi) - \chi k(k-1)\Delta_{k-2} U(X;\chi) \qquad (16)$$
$$(k = 0, 1, 2, ...; \quad \Delta_{-1} = \Delta_{-2} \equiv 0),$$

where $U(X;\chi)$ is any metaharmonic function with the parameter χ. This formula is easily shown to hold by using the familiar expression for the operator Δ in polar coordinates (see §1).

Let U be a regular solution of equation (11) in the domain T. Now, by (12) and (15), the function $U_1(X;\chi_1)$ will be metaharmonic in the domain T, hence it is analytic in the same domain T.

If we now apply the operator $\Delta+\chi$ successively $k-1$ times to both sides of (16) and use this formula each time, we easily find that

$$(\Delta+\chi)^k\Delta_k U(X;\chi) = (-1)^k 2^k k!\chi^k U(X;\chi) \qquad (k=0,1,2,...). \tag{17}$$

On putting

$$k=p_1, \quad \chi=\chi_1, \quad U(X;\chi)=U_1(X;\chi_1),$$

in this formula, where $U_1(X;\chi_1)$ is the function defined by (15), and applying to both sides of the resulting equation the operator $(\Delta+\chi_2)^{p_2}...(\Delta+\chi_m)^{p_m}$, we obtain by the formula

$$(\Delta+\chi_j)^{p_j}U(X;\chi_j) = (\chi_j-\chi_i)^{p_j}U(X;\chi_i), \tag{18}$$

together with (15)

$$(\Delta+\chi_2)^{p_2}...(\Delta+\chi_m)^{p_m}\Delta_{p_1}U_1(X;\chi_1) = U_1'(X;\chi_1),$$

which was to be proved.

According to our assumption, (10) holds for

$$k=p_1...,k_m=p_m.$$

The general solution of (12), or what amounts to the same thing, of equation (11), is therefore

$$\sum_{i=1}^{m}\sum_{j=0}^{p_i-1}\Delta_j U_{ij}(X;\chi_i)+\Delta_{p_1}U_1(X;\chi_1).$$

But, if we introduce the notation

$$U_1(X;\chi_1)=U_{1,p_1}(X;\chi_1),$$

this expression is the same as (10) with

$$k_1=p_1+1,\ k_2=p_2,...,k_m=p_m.$$

By applying this result and remembering that we have already proved (10) above for $k_i=1$ $(i=1,...,n)$, we find immediately that (10) holds for any positive integers $k_1,...,k_m$.

We now proved that the metaharmonic functions $U_{ij}(X;\chi_i)$ appearing in (10) are uniquely defined by the corresponding n-metaharmonic function $U(X;A)$ and that, in addition, (10) is always a solution of equation (A).

We obviously have from (17):

$$(\Delta+\chi)^{k+l}\Delta_k U(X;\chi)=0, \tag{19}$$

where l is any positive integer, and $U(X;\chi)$ is any metaharmonic function with parameter χ.

We introduce into discussion the operators

$$F_{j,s}(\Delta)\equiv\frac{(-1)^s(\Delta+\chi_j)^s\prod\limits_{i\neq j}(\Delta+\chi_i)^{k_i}}{2^s s!\chi_j^s\prod\limits_{i\neq j}(\chi_i-\chi_j)^{k_i}} \tag{20}$$
$$(j=1,...,m;\quad s=0,1,...,k_j).$$

When $s=k_j$ the operator $F_{j,s}(\Delta)$ is the same, up to a constant factor, as the operator on the left-hand side of (9).

On applying the operator $F_{j,s}(\Delta)$ to both sides of (10) and taking (18) and (19) into account, we get

$$U_{js}(X;\chi_j)=F_{j,s}(\Delta)U(X;A)-F_{j,s}(\Delta)\sum_{i=s+1}^{k_i-1}\Delta_i U_{ji}(X;\chi_j) \tag{21}$$
$$(j=1,...,m;\ s=0,1,...,k_j-1).$$

On putting $s=k_j-1$ in this formula, we get

$$U_{j,k_j-1}(X,\chi_j)=F_{j,k_j-1}(\Delta)U(X;A),\qquad (j=1,...,m), \tag{22}$$

i.e. the metaharmonic functions

$$U_{j,k_j-1}(X;\chi_j),$$

appearing in (10) are uniquely defined by the n-metaharmonic function $U(X;A)$.

We now assign to the index s in (21) the value k_j-2 and use the fact that the functions U_{j,k_j-1} are already given by (22), and thus obtain an expression for the functions

$$U_{j,k_j-2}(X;\chi_j)\qquad (j=1,2,...,m).$$

On proceeding further in this way, we can find an expression for all the metaharmonic functions appearing in (10) in terms of the n-metaharmonic function $U(X;A)$, these metaharmonic functions being uniquely defined, as is clear from the method of obtaining them. In addition, as may easily be seen from (19), expression (10) is always a solution of equation (A).

3^0. It now remains to consider the most general case, when $k,k_1,...,k_m$ are arbitrary positive integers.

In this case, by virtue of (10), equation (A) is equivalent to

$$\Delta^k U = \sum_{i=1}^{m} \sum_{j=0}^{k_i-1} \Delta_j U'_{ij}(X;\chi_i), \tag{23}$$

where

$$U'_{ij}(X;\chi_i)$$

are metaharmonic functions with the parameters χ_i respectively. We show that the constants

$$B^{i,s}_{j,0} \qquad (i=1,...,m; \;\; s=0,1,...,k_i-1; \;\; j=0,1,...,s)$$

can always be chosen in such a way that the function

$$W(X) = \sum_{i=1}^{m} \sum_{s=0}^{k_i-1} \sum_{j=0}^{s} B^{i,s}_{j,0} \Delta_j U'_{is}(X;\chi_i) \tag{24}$$

satisfies equation (23).

On applying the operator Δ to both sides of (24) and taking into consideration (16), we get

$$\Delta W = \sum_{i=1}^{m} \sum_{s=0}^{k_i-1} \sum_{j=0}^{s} B^{i,s}_{j,1} \Delta_j U'_{is}(X;\chi_i), \tag{25}$$

where

$$B^{i,s}_{j,1} = -\chi_i [B^{i,s}_{j,0} + 2(j+1)B^{i,s}_{j+1,0} + (j+1)(j+2)B^{i,s}_{j+2,0}] \tag{26}$$
$$(i=1,..,m; \;\; s=0,1,...,k_i-1; \;\; j=0,1,...,s; \;\; B^{i,s}_{k_j,0} = B^{i,s}_{k_i+1,0} = 0).$$

If we now apply the operator Δ successively to both sides of (25) and use formula (26) each time, we obtain the general formula

$$\Delta^l W(X) = \sum_{i=1}^{m} \sum_{s=0}^{k_i-1} \sum_{j=0}^{s} B^{i,s}_{j,l} \Delta_j U'_{is}(X;\chi_i), \tag{27}$$

where

$$B^{i,s}_{j,l} = (-1)^l \chi_i^l [B^{i,s}_{j,l-1} + 2(j+1)B^{i,s}_{j+1,l-1} + (j+1)(j+2)B^{i,s}_{j+2,l-1}] \tag{28}$$

$$(i=1,...,m; \;\; s=0,1,...,k_i-1; \;\; j=0,1,...,s;$$
$$l=0,1...,; \;\; B^{i,s}_{k_i,l} = B^{i,s}_{k_i+1,l} = 0).$$

If we now put $l = k$ in (27) and (28) and

$$B^{i,s}_{s,k} = 1, \quad B^{i,s}_{s-1,k} = 0, ..., \quad B^{i,s}_{0,k} = 0 \qquad (29)$$
$$(i = 1, ..., m; \quad s = 0, 1, ..., k_i - 1),$$

then (27) becomes

$$\Delta^k W(X) = \sum_{i=1}^{m} \sum_{s=0}^{k_i-1} \Delta_s U'_{is}(X; \chi_i).$$

Thus it may be seen that the function W given by (24) is in fact a solution of equation (23), provided conditions (29) hold. We now show that the constants $B^{i,s}_{j,0}$ are in fact uniquely defined by these latter conditions.

It may easily be seen that the constants $B_{j,k-1}$ are uniquely defined from (29). By using them, we can find the constants $B^{i,s}_{j,k-2}$ from (28) with $l = k - 1$. On continuing this process, we arrive finally at equation (26) whence the required constants $B^{i,s}_{j,0}$ may be found, which was to be proved.

If we now introduce the notation

$$U_{ij}(X; \chi_i) = \sum_{s=j}^{k_i-1} B^{i,s}_{j,0} U'_{is}(X; \chi_i), \qquad (30)$$

expression (24) becomes

$$W(X) = \sum_{i=1}^{m} \sum_{j=0}^{k_i-1} \Delta_j U_{ij}(X; \chi_i)$$

and the general solution of equation (23) becomes

$$U(X; A) = V(X) + \sum_{i=1}^{m} \sum_{j=0}^{k_i-1} \Delta_j U_{ij}(X; \chi_i),$$

where $V(X)$ denotes a k-harmonic function.

We have thus proved (1) in the most general case, and the proof of theorem 9 is consequently complete.

2. We now find the general form of solution of equation (A), depending only on r. If $U(X; A)$ is such a solution, the metaharmonic functions appearing in (1) will also be functions of r only, i.e. they will have the form

$$U_{ij}(X; \chi_i) = \alpha_{ij} r^{-q} H^{(1)}_q(\lambda_i r) + \beta_{ij} r^{-q} H^{(2)}_q(\lambda_i r) \qquad (31)$$
$$(i = 1, ..., m; \quad j = 0, 1, ..., k_i - 1; \quad \lambda_i = \sqrt{\chi_i}),$$

where α_{ij}, β_{ij} are arbitrary complex constants. If we now use successively the formula

$$\frac{d}{dx}[x^{-q} H^{(\nu)}_q(x)] = -x^{-q} H^{(\nu)}_{q+1}(x) \qquad (\nu = 1, 2),$$

we obtain from (1), on substituting (31) in it, the expression

$$\sum_{i=1}^{m}\sum_{j=0}^{k_i-1} A_{is}r^{-q+s}H^{(1)}_{q+s}(\lambda_i r) + B_{is}r^{-q+s}H^{(2)}_{q+s}(\lambda_i r), \tag{30a}$$

where $A_{is},\ B_{is}$ are arbitrary constants which can be expressed linearly in terms of the constants α_{ij} and β_{ij}.

Expression (31) represents the most general form of the solution of equation (A) which is a function of only r. Provided not all the A_{is} are respectively equal to B_{is}, this expression will have a singularity of the form $1/r^{p-2}$ or $log 1/r$ at a fixed point, according to whether $p > 2$ or $p = 2$. The function (30a) belongs to the class of so-called elementary solutions of the equation (A)[19].

§ 6. Riquier's boundary value problem for equation (A)

We will deal here with some simple applications of the above results to the solution of boundary value problems connected with equation (A).

1. Let T be a finite domain, with respect to which any Dirichlet problem (with continuous data) for the Laplace equation has a solution. We consider the following boundary value problem:

Problem R. Find a solution of equation (A), regular in the domain T, from the boundary conditions

$$U = f_0(x),\ \ \Delta U = f_1(x), ...,\ \Delta^{n-1}U = f_{n-1}(x), \tag{R}$$

where $f_0(x),\ f_1(x), ..., f_{n-1}(x)$ are given continuous functions of the boundary point x[20].

We will confine ourselves for simplicity to the case when the equation

$$\chi_n - a_1\chi^{n-1} + ... + (-1)^n a_n = 0 \tag{a}$$

has only simple non-zero roots $\chi_1, ..., \chi_n$, i.e. $k = 0,\ k_i = 1\ (i = 1, ..., n)$.

The general case can be treated in exactly the same way.

By Theorem 9, the required solution is given in this case by

$$U(X; A) = U_1(X; \chi_1) + ... + U_n(X; \chi_n), \tag{3}$$

[19]These solutions were obtained by another method by Ghermanescu [18], where, incidentally, a fairly complete list of references to articles concerned with an equation of the type (A) can be found.

[20]This problem was first considered by Riquier [17] for the n-harmonic equation in the case $p = 2$. The major part of Ghermanescu's article (Ghermanescu [18]) is concerned with the solution of the problem R in the case of equation (A), though it gives no criterion for the existence of a solution of the problem; as it will be seen below, the criterion is in fact extremely simple.

where $U_1, ..., U_n$ are metaharmonic functions in the domain T with parameters $\chi_1, ..., \chi_n$ respectively, i.e.

$$\Delta U_i + \chi_i U_i = 0 \qquad (i = 1, ..., n). \tag{31}$$

From (8) and the boundary conditions (R), we find that

$$U_i^+(x; \chi_i) = \varphi_i(x) \qquad x \in S \quad (i = 1, ..., m). \tag{32}$$

where

$$\varphi_i(x) = A_{i1} f_0(x) + A_{i2} f_1(x) + ... + A_{in} f_{n-1}(x), \tag{33}$$

and the A_{ik} are known constants, independent of the choice of functions $f_0, f_1, ..., f_{n-1}$, and such that the determinant $|A_{ik}| \neq 0$.

The problem R thus reduces to the solution of n independent Dirichlet problems for metaharmonic functions. It may easily be seen that these latter problems have a solution if the problem R has a solution, and vice versa.

Consequently, the necessary and sufficient condition for problem R to have a solution for any continuous boundary data is that the roots $\chi_1, ..., \chi_n$ of equation (a) are not eigenvalues of the Dirichlet problem for the metaharmonic equation; the solution of problem R is unique if it exists.

This theorem still holds when equation (a) has multiple roots.

Hence the sufficient condition for problem R to have a solution is that the equation (a) has no positive roots.

2. We now consider problem R in the case of an infinite domain. Let T be an infinite domain of class B. The problem R now obviously again reduces to finding the metaharmonic functions $U_1, ..., U_n$ from the boundary conditions (32). But in order for the problem to be well-pesed, we must add to these boundary conditions further conditions at infinity. The latter conditions have the form

$$\frac{dU_k}{dr} - i\lambda_k U_k = e^{i\lambda_k r} o(r^{-q-\frac{1}{2}}) \quad (\lambda_k^2 = \chi_k), \tag{I_k}$$

when $Im(\lambda_k) \geq 0$, and

$$\frac{dU_k}{dr} + i\lambda_k U_k = e^{-i\lambda_k r} o(r^{-q-\frac{1}{2}}), \tag{II_k}$$

when $Im(\lambda_k) \leq 0$, where, by (8), the U_k are metaharmonic functions connected with the required solution of equation (A) by the formulae

$$U_k(X; \chi_k) = \frac{\prod\limits_{i \neq k} (\Delta + \chi_k) U(X; A)}{\prod\limits_{i \neq k} (\chi_i - \chi_k)} \qquad (k = 1, ..., n). \tag{34}$$

The problem R for an infinite domain thus reduces to n independent problems D_e. But we showed in §4 that these problems always have a solution. We therefore have:

The problem R always has a solution in the case of an infinite domain T of class B (whatever the continuous boundary data).

A completely analogous result can be obtained when the equation (a) has multiple roots.

A similar method of solution may easily be seen to be possible for the problem with boundary condition of the type

$$\frac{dU}{d\nu} = f_0(x), \quad \frac{d\Delta U}{d\nu} = f_1(x), ..., \frac{d\Delta^{n-1} U}{d\nu} = f_{n-1}(x), \qquad (R')$$

where ν is the normal to S.

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