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Enhancing Tail Risk Measurement: A Practical Approach to Managing Model Risk of Tail Risk

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The tail market risk measurement has become one of the most crucial assignments of financial institutions' risk management units. At the same time various papers show that tail risk measures are especially sensitive to model misidentification. In this paper we consider this practical problem. We propose model risk robust approach for measuring tail risk based on superposed risk measures. Superposed risk measures consider novel approach to measure market and model risk in a consistent way. This paper has two major goals. First, we investigate several practical superposed market risk measures under extreme value theory. Second, we demonstrate our results via the case study of DAX 30 index.

Keywords: Market risk, superposed risk measure, market risk measure, superposed value at risk.

AMS Subject Classification: 60H07, 60H30, 62P05.

1. Introduction

The modern risk management requires monitoring tail events that are rare in frequency, but they are associated with large losses. Stock crashes, unexpected news in capital markets, political instability, oil shocks can lead to extreme unexpected losses. Extreme value theory is a framework that enables statistical modeling of tail events [9]. The extreme value theory was first introduced by [8] and [10]. In this paper we focus on the parametric approach of extreme value theory based on the generalized Pareto (GP) distribution. [14] understand under model the probability distribution. Following this approach, we consider a complete model set that includes all probability distributions modelling extreme tail risk of financial position. Our case study shows that the tail risk measures are highly sensitive on model misspecification. Further, it is surprisingly robust and easy to construct the market tail risk measures that capture model risk. Finally, we show that model risk of tail market risk measures can be effectively managed. [8] and [10] show that under some technical assumptions the normalized maximum process of any unknown distributions follows the generalized Extreme Value distribution. Further, [1] and

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[19] state that the distribution of tail events follow the GP distribution. The later theorem enables constructing a model set that includes all possible EVT models.

The model risk management literature has become richer. After the pioneering work of [6] and [11], the scholars have started paying attention to model risk. [11], [13], [22] and [3] agree that model risk has hidden nature and if a financial institute ignores it, then it could face huge financial losses. We follow [14] approach that considers probability distribution as a model.

2. Measuring market risk under model risk

Let $(\Omega, \mathcal{F}, m)$ be a probability space. Further, let $(M, \mathcal{M})$ be a measurable space of models. We associate a model with a probability distribution on $(\Omega, \mathcal{F})$. Consider a random variable $X : (\Omega, \mathcal{F}) \to (\mathbb{R}, \mathcal{B})$ that denotes financial position. $\mathcal{B}$ stands for Borel $\sigma-$algebra in $\mathbb{R}$. We denote $\mathcal{X} \subseteq L^0$ the linear space containing constants. Further, $L^\infty \subseteq L^0$ is a linear space of bounded functions.

Traditional monetary risk measures often require assumption on the probability distribution of financial position. These assumptions can significantly change the final outcome of market risk. Thus, every risk measure that depends on the probability distribution of financial position is model sensitive. To overcome this shortcoming [14] propose market risk measures that incorporate model risk. We summarize their methodology briefly. Let $\boldsymbol{\rho}$ denote the family of risk measures. We assume that $\boldsymbol{\rho}$ is bounded. The first intuitive choice of market risk measure that ensures 'safety' against model risk is the worst case market risk measure.

Definition 2.1: The worst case market risk measure ρ_{WC} is defined as

$$\rho_{WC}(X) = \sup_{m \in M} (\rho_m(X)). \tag{1}$$

The worst case market risk measure highly overstates the market risk. Hedging or insuring against the worst case scenario is usually very expensive. Neither the risk management practice, nor the regulatory requirements ask to be immune against the worst case risk and therefore Definition 2.1 is impractical.

In order to provide more useful risk measures, [14] introduce general superposed market risk measures that consider superpositions of risk measures on $\mathcal{X}$ and on $L^\infty(M, \mathcal{M})$. Where $L^\infty(M, \mathcal{M})$ states for a linear space of bounded functions on the measurable model space $(M, \mathcal{M})$. Assume that $\boldsymbol{\rho}$ is $\mathcal{M}$-measurable and bounded. Let $\zeta : L^\infty(M, \mathcal{M}) \to \mathbb{R}$ be a monetary risk measure on $L^\infty(M, \mathcal{M})$.

Definition 2.2: The superposed risk measure $\zeta \circ \boldsymbol{\rho}$ is defined by

$$\zeta \circ \boldsymbol{\rho}(X) = \zeta(-\boldsymbol{\rho}(X)), \; X \in \mathcal{X}. \tag{2}$$

The superposed market risk measures are very flexible and they enable to investigate different aspects of market risk and model risk. [14] prove that, if $\boldsymbol{\rho}$ is a family of coherent (convex) risk measures and ζ is coherent (convex), then model superposed market risk measures are coherent (convex). The most intuitive example of superposed market risk measure is model weighted market risk measure.

Definition 2.3: Let μ be a probability measure on the measurable model space

$(M, \mathcal{M})$. The model weighted market risk measure $\boldsymbol{\rho} * \mu$ is defined as

$$\boldsymbol{\rho} * \mu\,(X) = \int_M \rho_m\,(X)\,d\mu\,(m). \tag{3}$$

The model weighted market risk measure is a 'model robust' risk measure. Other interesting special cases of superposed market risk measures are listed below.

Definition 2.4:

(i) The superposed value at risk of $\boldsymbol{\rho}\,(X)$, with confidence level $\alpha \in [0,1]$ is defined as

$$\begin{aligned} VaR_{\alpha,\boldsymbol{\rho},\mu}(X) &= VaR_{\alpha,\mu}\,(-\boldsymbol{\rho}(X)) \\ &= \inf\{x \in \mathbb{R} : \mu\,(\{m : \rho_m(X) > x\}) \leq 1-\alpha\}. \end{aligned} \tag{4}$$

(ii) The superposed expected shortfall of $\boldsymbol{\rho}(X)$, with the confidence level $\alpha \in [0,1]$ is defined as

$$ES_{\alpha,\boldsymbol{\rho},\mu}(X) = ES_{\alpha,\mu}\,(-\boldsymbol{\rho}(X)) = \frac{1}{1-\alpha}\int_0^1 VaR_{\gamma,\boldsymbol{\rho},\mu}(X)d\gamma. \tag{5}$$

3. Measuring tail risk

Extreme value theory focuses on modeling the tail of some unknown distribution without examining central tendency. How bad could things go if the rare but plausible market event occurs? This is a central question EVT tries to answer. As the majority of financial risk measures concentrate on the tail events, EVT gives a natural settings to calculate market risk. The peaks-over-threshold (POT) method sets specific threshold u and considers all observations that exceed u as extreme outcomes. The aim is to describe the distribution of these extreme outcomes and based on this distribution calculate risk measures. Figure 1 demonstrates this approach.

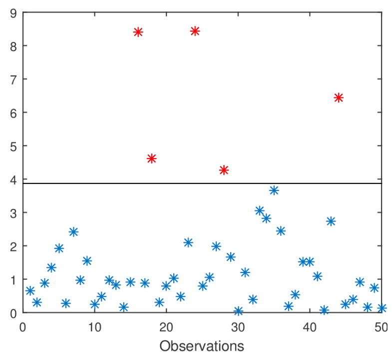

Figure 1. The peaks-over-threshold method.

Let $(\Omega, \mathcal{F}, \mathbf{P}$ be a probability space. $x_1, ..., x_n$ let be n i.i.d. random variables. Denote the cumulative distribution function of model $\mathbf{P}$ with F. Consider the maximum,

$$MAX_n = \max(x_1, ..., x_n). \tag{6}$$

Next, suppose that there exist sequences of real numbers $a_n, b_n > 0$, such that

$$\mathbf{P}\left\{\frac{\mathrm{MAX_n - a_n}}{b_n} \leq x\right\} = F^n(b_n x + a_n) \to H(x), \text{ as } n \to \infty, \tag{7}$$

where H is some non-degenerate distribution function. If Equation (7) holds we say that F is in the maximum domain of attraction of H, i.e., $F \in \mathrm{MDA(H)}$. Denote $z \in \mathbb{R} \cup \infty$ right endpoint of distribution F. Then, the probability distribution function of excesses over u is described as, see e.g., [15], [21]

$$\mathbf{P}\left\{X - u \leq x | X > u\right\} = \frac{F(x+u) - F(u)}{1 - F(u)}, x \in [0, z-u). \tag{8}$$

Theorem 3.1: *([1] and [19]). Let* $F \in \mathrm{MDA(H)}$*, then the limiting distribution for the distribution of the excesses as* $u \to \infty$ *is a generalized Pareto distribution.*

Definition 3.2: The probability distribution function of the generalized Pareto (GP) distribution with a shape parameter ξ and a scale parameter β is given by

$$GP(x|\xi, \beta) = \begin{cases} 1 - \left(1 + \frac{\xi x}{\beta}\right)^{-\frac{1}{\xi}}, & \text{if } \xi \neq 0 \\ 1 - \exp\left\{-\frac{x}{\beta}\right\}, & \text{if } \xi = 0, \end{cases} \tag{9}$$

where, $\beta > 0$ and the support is $x \geq 0$ when $\xi \geq 0$ and $x \in [0, -\beta/\xi]$ when $\xi < 0$. When $\xi > 0$ generalized Pareto distribution becomes Pareto distribution; if $\xi = 0$ we have an exponential distribution and if $\xi < 0$ GP gives type II Pareto distribution.

4. Case Study

In this case study, we analyze the tail market risk utilizing the extreme value theory. We study the tail risk of DAX 30 index and show model sensitivity of risk measures defined in Section 3. Finally, we compute model risk measures via the reference models. We collect daily historical data between Jan. 02, 1970 and Aug. 31, 2018, of DAX 30 index [1]. This timeframe includes several interesting events that affected financial prices including the collapse of the Bretton Woods system in the 1970s, "Black Monday" – Oct. 19, 1987, the Asian crisis in 1997, the collapse of LTCM after the Russian debt crisis in 1998, Sep. 11, 2001 terrorist attacks, the global financial crises during 2007-2008, and the European debt crisis during 2011.

Figure 2 illustrates the historical price evolution and log-returns of DAX 30 index. The stock return volatility is time inhomogeneous. We observe clear volatility clusters and have high excess kurtosis.

[1] Source Bloomberg

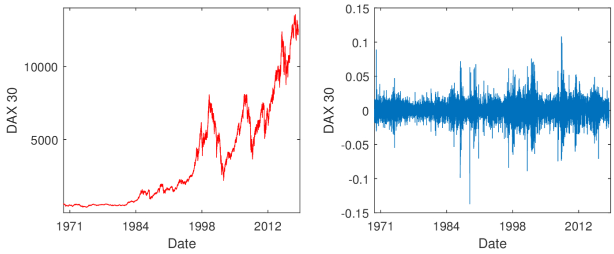

Figure 2. Daily close prices and log-returns of DAX 30 (prices are quoted in Euro).

The statistical facts like volatility clustering show that many financial returns time series are not i.i.d., see e.g., [5]. The extreme value theory is based on the i.i.d. assumption. In order to deal with this limitation, we apply GARCH filter for DAX 30 index returns prior to the EVT, as proposed by [16]. We consider GARCH(1,1) filter with Students $t-$ distribution.

In order to capture the market risk for both long and short positions, we consider the absolute values of standardized returns. We set the threshold at 90% of the total observations. We consider the following model set that includes all Generalized Pareto distributions,

$$M = \{GP_{\xi,\beta} | \xi \in \mathbb{R}, \beta \in \mathbb{R}_+\}. \tag{10}$$

Let Y be the vector of log-returns. Further, let $(M, \mathcal{M})$ be a measurable space. A model $m \in M$ is a probability measure on $(\Omega, \mathcal{F})$. The model set M contains parameterized models from one distribution class. Therefore, under the model m we understand the parameter vector $\theta \in \Theta$. Let $\pi(\theta)$ denote the prior distribution of the unknown parameter vector θ. Applying the Bayes theorem, the posterior distribution μ on the model space is given by

$$\mu(\theta | Y = y) = \frac{m_\theta(Y = y)\pi(\theta)}{\int_\Theta m_\theta(Y = y)\pi(\theta)\, d\theta}. \tag{11}$$

Equation (11) connects the realized observed data and prior subjective believes in the posterior distribution of the model. The Bayesian computation not only gives the best possible point estimate of model parameters but also characterizes the full model distribution.

Solving Equation (11) usually involves the numerical integration on the high-dimensional space, which considers the significant numerical challenge. The popular way to deal with this problem is Markov chain Monte Carlo (MCMC) simulation methods that originate from [17] and [12]. Metropolis-Hastings algorithm can be applied to sample from an arbitrarily complex distribution; the algorithm is very efficient, as it does not require the evaluation of high-dimensional integrals. For further details about the MCMC methods, we refer to [20], [4].

To get a posterior distribution of models, we employ R package 'MCMC4Extremes' [7]. The package employs the Metropolis-Hastings algorithm to get posterior sample of GP distribution parameters. We run 10'000'000 Iterations. We burn in the first 90% of iterations. We consider four financial positions EUR 1,000 investment in DAX 30 index. Based on the MCMC simulation results we calculate 10 days VaR and ES with 95% and 99% confidence levels. In order to adjust the variance, we simulate GARCH(1,1) process with Student's $t-$ distribution.

Figure 3 provides the boxplots of market risk measures. The distribution of expected shortfall has larger variance than the distribution of VaR. Further, risk measures with a 99% confidence level are more prone to model risk than the risk measures with 95% confidence level. Finally, the distribution of spectral risk measure with risk aversions parameter $\gamma = 0.01$ has higher variance than the distribution of spectral risk measures with $\gamma = 0.02$. These results are consistent with literature see e.g., [14], [18] and [2].

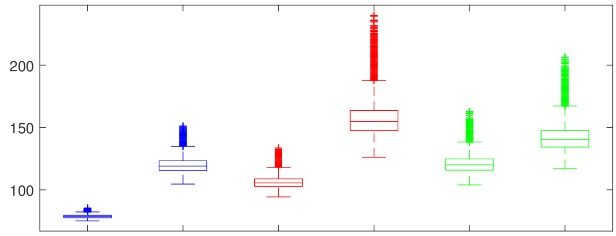

Figure 3. Boxplots of market risk measures. From left to right: $\mathrm{VaR}_{0.95}$, $\mathrm{VaR}_{0.99}$, $\mathrm{ES}_{0.95}$, $\mathrm{ES}_{0.99}$, spectral risk measure with $\gamma = 0.02$, spectral risk measure with $\gamma = 0.01$.

5. Conclusion

Our methodology is based on the pioneering work of [14]. We consider different superposed tail market risk measures. These risk measures can describe different aspects of market risk and simultaneously captures the model risk. We demonstrate our theoretical results via the case study, which highlights that the model risk is an important factor for the tail market risk measures. Even in the case of univariate modeling of equity index, the posterior tail risk measures have significant model risk and employing point estimators of the tail risk could be highly misleading. Expected shortfall and spectral risk measures are more affected by model risk than value at risk. In addition, risk measures with 99% confidence levels are more model sensitive than the risk measures with 95% confidence level. The spectral risk measures with high subjective risk aversion are also more model risk prone than the same spectral risk measure with the low risk aversion. These results are consistent with the previous studies of [14] and [2].

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