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A Note on Diophantine Equations and Maximization of Products

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A sufficient condition for solvability in natural numbers of linear Diophantine equation is obtained and a problem on maximizing the product of powers natural numbers whose corresponding weighted sum is given is investigated.

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1. Introduction

For natural numbers $n > 1$, $p_i > 0, i = 1, \dots, n$ and $L \geq \sum_{k=1}^{n} p_k$ we find a condition for solvability of the Diophantine equation

$$\sum_{i=1}^{n} p_i x_i = L\,. \qquad (DE)$$

in natural numbers $x_i, i = 1, \dots, n$ and calculate the maximal value of the product

$$\prod_{i=1}^{n} x_i^{p_i}$$

when $x_i, i = 1, \dots, n$ are natural numbers, satisfying (DE).

A solution of a related *extremum problem* when $p_i = 1, i = 1, \dots, n$ and $L \geq n$ is contained in [2, Proposition 3.1].

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2. Main result

Below for a real number x by $[x]$ is denoted the integer part of x.

Theorem 2.1 : *Let $n > 1$, L and $p_i, i = 1, \dots, n$ be natural numbers, such that*

$$L \geq m := \sum_{k=1}^{n} p_k$$

Let, moreover

$$q := \frac{L}{m}, \;\; r := L - [q]m \, .$$

and assume that the following condition is satisfied:
(NC) for some proper subset $J \subset \{1, \dots, n\}$ we have

$$\sum_{k \in J} p_k = r \, .$$

We agree $\sum_{k \in \emptyset} p_k := 0$.
Then
(1) the natural numbers $x_k, k = 1, \dots, n$ defined by equalities

$$x_k = 1 + [q], \textit{for } k \in J \textit{ and } x_k = [q] \;\; \textit{for } k \in \{1, \dots, n\} \setminus J \, . \qquad (SO)$$

satisfy (DE).
(2) Maximum of the product

$$\prod_{i=1}^{n} x_i^{p_i}$$

when $x_i, i = 1, \dots, n$ are natural numbers, satisfying (DE), is

$$(1 + [q])^r [q]^{m-r}$$

and this maximum is attained when $x_i, i = 1, \dots, n$ are defined by (SO).

Proof: (1) If $J = \emptyset$, then $r = 0$ and for the natural numbers $x_i, i = 1, \dots, n$ defined by (SO) we have:

$$\sum_{i=1}^{n} p_i x_i = \sum_{i=1}^{n} p_i q = mq = L \, ,$$

hence, they satisfy (DE) in this case.

If $J \neq \emptyset$, then $0 < r < m$ and for the natural numbers $x_i, i = 1, \dots, n$ defined by

(SO) we have:

$$\sum_{i=1}^{n} p_i x_i = \sum_{i \in J} p_i x_i + \sum_{i \in \{1,\dots,n\} \setminus J} p_i x_i$$

$$= \sum_{i \in J} p_i (1 + [q]) + \sum_{i \in \{1,\dots,n\} \setminus J} p_i [q] = r(1 + [q]) + (m - r)[q] = r + m[q] = L\,,$$

hence, they satisfy (DE) in this case too.

(2) Note first of all that from the mean-arithmetic–mean-geometric inequality for positive numbers L and $x_i, i = 1, \dots, n$ satisfying (DE) we have:

$$\prod_{i=1}^{n} x_i^{p_i} \le \left(\frac{L}{m} \right)^m . \tag{1}$$

When $r = 0$; i.e., when m divides L, the statement follows from inequality (1).

In general, the statement can be derived from [2, Proposition 3.1], which asserts, in particular, that the maximum of the product

$$\prod_{i=1}^{m} y_i$$

when $y_i, i = 1, \dots, m$ are natural numbers such that $\sum_{i=1}^{m} y_i = L$ is

$$(1 + [q])^r [q]^{m-r}$$

and this maximum is attained for natural numbers $y_i, i = 1, \dots, m$ for which the equalities

$$\text{card}\left(\{i \in \{1, \dots, m\} : y_i = 1 + [q]\}\right) = r$$

$$\text{and}\ \ \text{card}\left(\{i \in \{1, \dots, m\} : y_i = [q]\}\right) = m - r$$

hold. □

Remark 1 : Note that

(a) The condition (NC) of Theorem 2.1(1) provides only a sufficient condition for solvability in natural numbers of the equation (DE), *which may not be necessary.* For example, for the equation

$$2x_1 + 3x_2 = 16$$

the condition (NC) of Theorem 2.1(1) is not satisfied, but it has the solution $x_1 = 5$ and $x_2 = 2$.

The same example shows that *the first part of Theorem 2.1(2) is not true without assuming the condition* (NC).

(*b*) Theorem 2.1(1) for the equation

$$2x_1 + 3x_2 = 37$$

gives the solution $(8, 7)$; *however this equation has other solutions too*:

$$(3k - 1, 13 - 2k),\ k = 1, 2, 4, 5, 6\,.$$

It can be checked directly that

$$(3k - 1)^2(13 - 2k)^3 < 8^2 \cdot 7^3 = 21352, \text{ for } k = 1, 2, 4, 5, 6\,;$$

in particular, that Theorem 2.1(2) is true in this case.

(*c*) Using the inequality (1) we get that maximum of the product

$$\prod_{i=1}^{n} x_i^{p_i}$$

when $x_i, i = 1, \ldots, n$ are natural numbers satisfying (DE) is not greater than

$$\left[\left(\frac{L}{m}\right)^m\right].$$

From this and Theorem 2.1(2) we conclude that

$$(1 + [q])^r [q]^{m-r} \le \left[\left(\frac{L}{m}\right)^m\right].$$

It can be shown, that in this inequality the equality we have only in very special cases; cf. [1], where the case $p_i = 1, i = 1, \ldots, n$ is treated.

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