Bulletin of TICMI
Vol. 27, No. 2, 2023, 51–57

Consistent Hypothesis Testing Criteria in the Banach Space of Measures for Haar Statistical Structures

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(*Received April 20, 2023; Revised May 05, 2023; Accepted June 20, 2023*)

In this article, we define Haar statistical structures. Necessary and sufficient conditions for the existence of consistent criteria for testing hypotheses in the Banach space of measures of the Haar statistical structure are proved.

Keywords: Haar measure, Haar statistical structure, Banach space of measures, consistent criterion, hypotheses testing.

AMS Subject Classification: 62H05, 62H12.

1. Introduction

Statistics of random processes is used in various fields of science and technology (for example, in theoretical physics, genetics, economics, radio physics, ...). When using random processes as models of real phenomena, the question of determining the probabilistic characteristics of the process arises. To determine these characteristics statistical methods should be used. This article is devoted to the question of the existence of consistent criteria for hypotheses testing for Haar statistical structure and the method of their finding. Recall that a statistical criterion is any measurable mapping from the set of all possible sample values to the set of hypotheses. It is said that an error of the h-th type of the δ criterion occurs, if the criterion rejected the main hypothesis of H_h. The following probability $\alpha_h(\delta) = \mu_h(\{x : \delta(x) \neq h\})$ is called the probability of an error of the h-th kind for a given criterion δ.

Remark 1 : Let the Haar statistical structure $\{E, S, \mu_h, h \in H\}$ admit a consistent criterion δ for hypothesis testing, then the probability of an error of all types is equal to zero for the criterion δ.

By (ZFC) we denote the formal system of Zermelo-Fraenkel with the addition of axiom of choice (AC), i.e. (ZFC)=(ZF)&(AC). By (ZFC)&(MA) we denote the theory with the addition of Martin's axiom (MA).

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ISSN: 1512-0082 print

2. The consistent criteria for hypotheses testing for Haar Statistical Structures

Let (E, S) be a measurable space. The following definitions are taken from [1] - [5].

Definition 2.1: Let E be an arbitrary locally compact and σ-compact topological group and $B(E)$ is σ-algebra of subsets of E. We say that measure μ defined on $B(E)$ is Haar measure if μ is a regular measure and

$$\mu(sX) = \mu(X), \quad \forall s \in E, \quad \forall X \in B(E).$$

Definition 2.2: An object $\{E, S, \mu_h, h \in H\}$ is called a Haar statistical structure, where $\{\mu_h,\ h \in H\}$ is a family of Haar probability measures on $(E, B(E))$.

Definition 2.3: A Haar statistical structure $\{E, S, \mu_h, h \in H\}$ is called an orthogonal (singular) Haar statistical structure if the family of probability measures $\{\mu_h,\ h \in H\}$ are pairwise singular measures.

Definition 2.4: A Haar statistical structure $\{E, S, \mu_h, h \in H\}$ is called a strongly weakly separable statistical structure if there exists a family S-measurable sets $\{X_h, h \in H\}$ such that the realations are fulfilled:

$$\mu_h(X_{h'}) = \begin{cases} 1, & if\ h = h'; \\ 0, & if\ h \neq h' \end{cases} \qquad (h, h' \in H).$$

Let $\{\mu_h, h \in H\}$ be Haar probability measures defined on the measurable space (E, S). For each $h \in H$ denote by $\overline{\mu}_h$ the completion of the measure μ_h, and denote by $dom(\overline{\mu}_h)$ the σ-algebra of all $\overline{\mu}_h$-measurable subsets of E. Let

$$S_1 = \cap_{h \in H} dom(\overline{\mu}_h).$$

Definition 2.5: A Haar statistical structure $\{E, S, \mu_h, h \in H\}$ is called strongly separable if there exists the family of S_1-measurable sets $\{Z_h, h \in H\}$ such that the following realations are fulfilled:
1) $\overline{\mu}_h(Z_h) = 1,\ \ \forall h \in H;$
2) $Z_{h_1} \cap Z_{h_2} = \emptyset,\ \ \forall h_1 \neq h_2;\ \ h_1, h_2 \in H;$
3) $\cup_{h \in H} Z_h = E.$

Let H be the set of hypotheses and let $B(H)$ be σ-algebra of subsets of H which contains all finite subsets of H.

Definition 2.6: We will say that the statistical structure $\{E, S_1, \overline{\mu}_h, h \in H\}$ admits a consistent criterion for hypothesis testing if there exists at least one measurable mapping $\delta : (E, S_1) \longrightarrow (H, B(H))$, such that

$$\overline{\mu}_h(\{x : \delta(x) = h\}) = 1,\ \ \forall\ h \in H.$$

Let M^σ be a real linear space of all alternating finite measures on S.

Definition 2.7: A linear subset $M_B \subset M^\sigma$ is called a Banach space of measures if:

1) On M_B one can introduce the norm so that M_B is a Banach space by this norm, and also for any orthogonal measures $\mu, \nu \in M_B$ and real number $\lambda \neq 0$ the equality $||\mu + \lambda\nu|| \geq \mu$ is true;

2) If $\nu \in M_B$, f is a real measurable function with $|f(x)| \leq 1$ and $A \in S$, then

$$\nu_f(A) = \int_A f(x)\nu(dx) \in M_B,$$

and $||\nu_f|| \leq ||\nu||$;

3) If $\nu_n \in M_B$, $\nu_n > 0$, $\nu_n(E) < \infty$, $n = 1, 2, ...$ and $\nu_n \downarrow 0$, then for any linear functional $l^* \in M_B^*$:

$$\lim_{n\to\infty} l^*(\nu_n) = 0,$$

where M_B^* is a linear space, congugate to M_B.

Let I be a set of indexes, and let M_{B_i} $(i \in I)$ be a Banach space.

Definition 2.8: The Banach space

$$M_B = \{\{X_I\}_{i\in I};\ X_i \in M_{B_i}, \forall i \in I;\ \sum_{i\in I} ||X_i||_{M_{B_i}} < +\infty\},$$

with the norm

$$||\{X_I\}_{i\in I}|| = \sum_{i\in I} ||X_i||_{M_{B_i}}$$

is called the direct sum of Banach spaces M_{B_i}, and is denoted by

$$M_B = \oplus_{i\in I} M_{B_i}.$$

The following theorem was proved in [3].

Theorem 2.9: *Let M_B be a Banach space of measures. Then there exists a family of pairwise orthogonal probability measures $\{\mu_i, i \in I\}$ from this space such that*

$$M_B = \oplus_{i\in I} M_B(\mu_i),$$

where $M_B(\mu_i)$ is a Banach space of elements ν of the form

$$\nu(B) = \int_B f(x)\mu_i(dx),\ B \in S,\ \int_E |f(x)|\mu_i(dx) < \infty,$$

with the norm

$$||\nu||_{M_B(\mu_i)} = \int_E |f(x)|\mu_i(dx).$$

Remark 1: It is obvious that any Banach space of measures is a Banach space whose elements are alternating measures, but not vice versa.

We define by $F = F(M_B)$ the set of real function f such that $\int_E f(x)\overline{\mu}_h(dx)$ is defined for all $\overline{\mu}_h \in M_B$.

Theorem 2.10 : *Let*

$$M_B = \oplus_{h\in H} M_B(\overline{\mu}_h)$$

be a Banach space of measures, $cardH \leq c$, let E be a complete separable metric space, let $S_1 = \oplus_{h\in H} dom(\overline{\mu}_h)$ be a Borel σ-algebra on E. In order for the Borel Haar orthogonal statistical structure $\{E, S_1, \overline{\mu}_h, h \in H\}$ to admit a consistent criterion for hypotheses testing in the theory of (ZFC) & (MA) it is necessary and sufficient that the correspondence $f \longleftrightarrow \psi_f$ defined by the equality

$$\int_E f(x)\overline{\mu}_h(dx) = l_f(\overline{\mu}_h), \quad \overline{\mu}_h \in M_B$$

was one-to-one (here l_f is a linear continuous functional on M_B, $f \in F(M_B)$).

Proof : Necessity. The existence of a consistent criterion for hypotheses testing $\delta : (E, S_1) \longrightarrow (H, B(H))$ implies that $\overline{\mu}_h(\{x : \delta(x) = h\}) = 1, \ \ \forall h \in H$. Setting $X_h = \{x : \delta(x) = h\}$ for $h \in H$ we get:
1) $\overline{\mu}_h(X_h) = 1, \ \ \forall h \in H$;
2) $X_{h'} \cap X_{h''} = \emptyset$ for all different parameters $h^{'}$ and $h^{''}$ from H;
3) $\cup_{h\in H} X_h = \{x : \delta(x) \in H\} = E$.

Therefore the Haar statistical structure $\{E, S_1, \overline{\mu}_h, h \in H\}$ is strongly separable, hence, there exist S_1-measurable sets X_h $(h \in H)$, such that

$$\overline{\mu}_h(X_{h'}) = \begin{cases} 1, & if\ h = h^{'}; \\ 0, & if\ h \neq h^{'}. \end{cases}$$

We put the linear continuous functional l_{c_h} into correspondence to a function $I_{c_h}(x) \in F(M_B)$ by the formula

$$\int_E I_{c_h}(x)\overline{\mu}_h(dx) = l_{c_h}(\overline{\mu}_h) = ||\overline{\mu}_h||_{M_B(\overline{\mu}_h)}.$$

Next, we put the linear continuous functional $l_{f_{\psi_1}}$ into correspondence to the function $f_{\psi_1}(x) = f_1(x)I_{c_h}(x) \in F(M_B)$. Then for any $\psi_2 \in M_B(\overline{\mu}_h)$:

$$\int_E f_{\psi_1}(x)\psi_2(dx) = \int_E f_1(x)I_{c_h}(x)\psi_2(dx) = \int_E f(x)f_1(x)I_{c_h}(x)\overline{\mu}_h(dx)$$

$$= l_{f_{\psi_1}}(\psi_2) = ||\psi_2||_{M_B(\overline{\mu}_h)}.$$

Let Σ be the collection of extensions of functionals l satisfying the condition $l_f \leq p(x)$ on those subspace where they are defined. We introduce a partial ordering on $\in$, assuming $l_{f_1} < l_{f_2}$ if l_{f_2} is defined on a set larger than l_{f_1} and $l_{f_1}(x) = l_{f_2}(x)$ where they are both defined.

Let $\{l_{f_h}\}_{h\in H}$ be a linear ordered subset of Σ, and let $M_B(\overline{\mu}_h)$ be the subspace on which l_{f_h} is defined. Define l_f on $\cup_{h\in H}M_B(\overline{\mu}_h)$ by assuming $l_f(x) = l_{f_h}(x)$ if $x \in M_B(\overline{\mu}_h)$. It is obvious that $l_{f_h} < l_f$. Since any linearly ordered subset Σ has an upper bound, by Chorn's lemma Σ contains a maximal element λ defined on the same set $X^{'}$ and satisfying the condition $\lambda(x) \leq p(x)$ for $x \in X^{'}$. But $X_{'}$ must coincide with the entire space M_B, otherwise we could extend λ to a wider space by adding one more dimension. This contradicts the maximality of λ. Hence, $X^{'} = M_B$. Therefore, the extension of the functional is defined everywhere.

Let l_f be the linear functional corresponding to the function

$$f(x) = \sum_{h\in H} g_h(x) I_{X_h}(x) \in F(M_B).$$

Then we have

$$\int_E f(x)\overline{\mu}(dx) = ||\overline{\mu}|| = \sum_{h\in H} ||\overline{\mu}_h||_{M_B(\overline{\mu}_h)},$$

where

$$\overline{\mu}(B) = \sum_{h\in H} \int_B g_h(x)\overline{\mu}_h(dx), \quad B \in S.$$

Sufficiency. For $f \in F(M_B)$ we define a linear continuous functional l_f by

$$l_f(\nu) = \int_E f(x)\nu(dx).$$

Denote by H_f a countable subset of H for which

$$\int_E f(x)\overline{\mu}_h(dx) = 0 \ \text{ for } \ h \notin H_f.$$

Consider the functional l_{f_h} on $M_B(\overline{\mu}_h)$ corresponding to f_h. Then for $h_1, h_2 \in M_B(\overline{\mu}_h)$ we have

$$\int_E f_{h_1}(x)\overline{\mu}_{h_2}(dx) = l_{f_{h_1}}(\overline{\mu}_{h_2}) = \int_E f_1(x) f_2(x)\overline{\mu}_h(dx) = \int_E f_{h_1}(x) f_2(x)\overline{\mu}_h(dx).$$

Therefore $f_{h_1} = f_1$ a.e. with respect to the measure $\overline{\mu}_h$. Let $f_h(x) > 0$ a.e. with respect to the measure $\overline{\mu}_h$ and

$$\int_E f_h(x)\overline{\mu}_h(dx) < \infty, \quad \overline{\mu}_h^*(C) = \int_C f_h(x)\overline{\mu}_h(dx),$$

then

$$\int_E f_{\overline{\mu}_h^*}(x)\overline{\mu}_{h'}(dx) = 0, \ \ \forall h^{'} \neq h.$$

Denote $C_h = \{x : f_{\overline{\mu}_h^*}(x) > 0\}$, then

$$\int_E f_{\overline{\mu}_h^*}(x)\overline{\mu}_{h'}(dx) = l_{\overline{\mu}_h^*}(\overline{\mu}_{h'}) = 0, \ \ \forall h^{'} \neq h.$$

Hence it follows that $\overline{\mu}_{h'}(C_h) = 0$, $\forall h^{'} \neq h$.

On the other hand,

$$\overline{\mu}_h^*(E \setminus C_h) = \int_{E \setminus C_h} f_{\overline{\mu}_h}(x)\overline{\mu}_h(dx) = \int_E f_{\overline{\mu}_h}(x) I_{\{E\setminus C_h\}}\overline{\mu}_h(dx)$$

$$= \int_E f_{\overline{\mu}_h^*}(x) I_{\{E\setminus C_h\}}\overline{\mu}_h(dx) = 0,$$

since $f_{\overline{\mu}_h^*}(x) = f_{\overline{\mu}_h}(x)$ a.e. with respect to the measure $\overline{\mu}_h$ and $f_{\overline{\mu}_h^*}(x) I_{\{E\setminus C_h\}} \equiv 0$. Thus, the Haar statistical structure $\{E, S_1, \overline{\mu}_h, h \in H\}$ is strongly separable.

Let us define an ω_α-sequence of parts of the space E so that the following relations hold:

1) $(\forall h)$ $(h < \omega_\alpha \Rightarrow B_h$ is a Borel subset of $E)$;

2) $(\forall h)$ $(h < \omega_\alpha \Rightarrow B_h \subset X_h)$;

3) $(\forall h_1)$ $(\forall h_2)$ $(h_1 < \omega_\alpha)\&(h_2 < \omega_\alpha)\&(h_1 \neq h_2) \Rightarrow B_{h_1} \cap B_{h_2} = \emptyset$;

4) $(\forall h)$ $(h < \omega_\alpha \Rightarrow \overline{\mu}_h(B_h) = 1)$.

Suppose $B_0 = X_0$. Let further the partial sequence $\{B_{h'}\}_{h' < h}$ be already defined for $h < \omega_\alpha$. It is clear that $\overline{\mu}_h^*(\bigcup_{h' < h} B_{h'}) = 0$. Thus, there exists a Borel subset Y_h of the space E such that the following relations hold: $\bigcup_{h' < h} B_{h'} \subset Y_h$ and $\overline{\mu}_h(Y_h) = 0$. Suppose $B_h = X_h \setminus Y_h$. Thus the ω_α-sequence $\{B_{h'}\}_{h' < \omega_\alpha}$ – of disjunctive measurable subsets of the space E constructed. Therefore $(\forall h)$ $(h < \omega_\alpha \Rightarrow \overline{\mu}_h(B_h) = 1)$, where ω_α denotes the first ordinal number of the power of the set H.

For $x \in E$ we put $\delta(x) = h$, where h is the unique hypothesis from the set H for which $x \in B_h$. Let now $Y \subset B(H)$. Then $\{x : \delta(x) \in Y\} = \cup_{h\in Y} B_h$. We must show that $\{x : \delta(x) \in Y\} \in dom(\overline{\mu}_h)$, $\forall h \in H$.

If $h_0 \in Y$, then

$$\{x : \delta(x) \in Y\} = \cup_{h\in Y} B_h = B_{h_0} \cup (\cup_{h\in Y\setminus\{h_0\}} B_h).$$

It follows that

$$B_{h_0} \in S_1 = \cap_{h\in H} dom(\overline{\mu}_h) \subseteq dom(\overline{\mu}_{h_0}).$$

The validity of the condition

$$\cup_{h\in Y\setminus\{h_0\}} B_h \subseteq (E \setminus B_{h_0})$$

implies that

$$\overline{\mu}_{h_0}(\cup_{h\in Y\setminus\{h_0\}} B_h) = 0.$$

The last equality yields that

$$\cup_{h\in Y\setminus\{h_0\}} B_h \in dom(\overline{\mu}_{h_0}).$$

Since $dom(\overline{\mu}_{h_0})$ is a σ-algebra, we conclude that

$$\{x : \delta(x) \in Y\} = B_{h_0} \cup (\cup_{h\in Y\setminus h_0} B_h) \in dom(\overline{\mu}_{h_0}).$$

The last relation implies that

$$\{x : \delta(x) \in Y\} \in dom(\overline{\mu}_{h_0}).$$

If $h_0 \notin Y$, then

$$\{x : \delta(x) \in Y\} = \cup_{h\in Y} B_h \subseteq (E \setminus B_{h_0})$$

and we conclude that $\overline{\mu}_{h_0}\{x : \delta(x) \in Y\} = 0$. The last relation implies that

$$\{x : \delta(x) \in Y\} \in dom(\overline{\mu}_{h_0}).$$

Thus we have shown the validity of the relation

$$\{x : \delta(x) \in Y\} \in dom(\overline{\mu}_{h_0})$$

for an arbitrary $h_0 \in H$. Hence,

$$\{x : \delta(x) \in Y\} \in \cap_{h\in H} dom(\overline{\mu}_h) = S_1.$$

We have shown that the map $\delta : (E, S_1) \longrightarrow (H, B(H))$ is a measurable map. Since $B(H)$ contains all singletons of H we ascertain that

$$\overline{\mu}_h(\{x : \delta(x) = h\}) = \overline{\mu}_h(B_h) = 1, \ \ \forall h \in H.$$

□

Acknowledgment

The work was partially supported by the grant STEM-22-226.

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