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Canonical Commutation Relation for Orbital Operators Corresponding to Creation and Annihilation Operators

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In this article the orbits of creation, annihilation and numerical operators at the states of quantum Hilbert spaces are created. The Hilbert space of finite orbits and the Frechet-Hilbert space of all orbits for these operators are created. The orbital operators corresponding to these operators in the spaces of orbits are defined and studied. Generalization of well-known canonical commutation relations for orbital operators corresponding to creation and annihilation operators are established.

Keywords: Creation operator, annihilation operator, numerical operator, orbits of operator, canonical commutation relation.

AMS Subject Classification: 81S05, 46N50, 47B47,47B93.

1. Introduction

Introduced by Paul Dirac creation and annihilation operators have widespread applications in quantum mechanics, notably in the study of quantum harmonic oscillators and many-particle systems. Modern quantum physics almost unthinkable without them. We create finite orbits and orbits of creation, annihilation and numerical operators at the states of quantum Hilbert space $L^2(\mathbb{R})$ ("quantum Hilbert space" means simply the Hilbert space associated with a given quantum system ([1], Sect.13.1, p.255)). The Hilbert space of finite orbits and the Frechet-H ilbert space (note that,initially, Frechet-Hilbert spaces were always supposed to be the strict projective limits of a sequence of Hilbert spaces; in modern literature, however, the requirement that the projective limit is strict, is omitted) of all orbits which elements are the orbits of these operators at some elements of the space $L^2(\mathbb{R})$ are definite and studied. Moreover, the notion of orbital operators corresponding to these operators in the spaces of orbits is introduced and studied. We establish well-known canonical commutation relations for orbital operators corresponding to the creation and annihilation operators in the Hilbert space of finite orbits and in the Frechet-Hilbert space of all orbits. The orbital spaces and orbital operators for Hamiltonian of quantum harmonic oscillator are constructed in [2].

The definitions of finite orbits and of the operator at the states was introduced, respectivelly, in [3] and [4]. We present the following reasoning from [4]: let H be a Hilbert space and $F : D(F) \subset H \to H$ be a linear operator with the domain of

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definition $D(F)$. We shall call the sequence $\mathrm{orb}(F,x) = (x, Fx, F^2x, \cdots)$ the orbit of the operator F at the point x, i.e. $\mathrm{orb}(F,x)$ is an element of the Frechet-Hilbert space H^N. If $F^jx \in H$ for $j = 0, 1, \cdots, n$, then we denote the finite sequence $(x, Fx, \cdots, F^nx)$ by $\mathrm{orb_n}(F,x)$ and call n-orbit of the operator F at the point $x \in H$, i.e. $\mathrm{orb_n}(F,x)$ is an element of the space H^{n+1}. The space of such elements we denote by $D(F^n), n \in N_0 = \{0, 1, 2, \cdots\}$, besides F^0 is the identical operator. Algebraically $D(F^n)$ is a subset of H. In what follows we consider the space $D(F^n)$ with the inner product

$$\langle \mathrm{orb}_n(F,\varphi), \mathrm{orb}_n(F,\chi)\rangle_n$$

$$=< \varphi, \chi > + < F\varphi, F\chi > + \cdots + < F^n\varphi, F^n\chi >, \ n \in N_0,$$

and with the corresponding norm

$$||\mathrm{orb}_n(F,x)||_n = (||x||^2 + ||Fx||^2 + \cdots + ||F^nx||^2)^{1/2},$$

where $< \cdot, \cdot >$ and norm $|| \cdot ||$ are inner product and norm in the space of H, i.e. the space $(D(F^n), || \cdot ||_n)$ is isometrically embedded in the space H^{n+1}.

In [5] the following concept was introduced: "Let X be a linear metric space. Let F be a linear continuous operator mapping X into itself. Let $x \in X$ and consider the set $\vartheta(F,x) = \{F^nx; n \in \mathbb{N}_0\}$. We shall call $\vartheta(F,x)$ an orbit of x with respect to the operator F." Note that, in this case the set $\vartheta(F,x)$ is a subset of a linear metric space X. Thus, in [4] the notion of an orbit of the operator F at a point and in [5] the notion of orbit of x with respect to the operator are introduced, i.e. these notions are different as subsets, are different as terms and with notations. In [4] the continuity of the operator F is not assumed. We also consider the concepts of an orbital operator F_n [3] that acts in the Hilbert spaces of finite orbits and orbital operator F^∞ that acts in the Frechet-Hilbert spaces of all orbits [4] (see also [6]).

In this article the study of the corresponding to a creation operator C and a annihilation operator A is carried out within the framework of the orbital quantum mechanics, the concept of which was formulated in [7].

In the second section finite orbits of the creation operator C and of the annihilation operator A at the states, as well n-orbital operators C_n and A_n corresponding to creation and annihilation operators in the Hilbert space of finite orbits are defined. According to the definition of orbital operators C_n and A_n it is naturally to determinate its value on the element $(\varphi_0, \varphi_1 \cdots, \varphi_n) \in (D(C))^{n+1} \cap (D(A))^{n+1}$. We need this while proving of canonical commmutation relations between C_n and A_n because we must also consider the value C_n on the orbits of the operator A and the value A_n on the orbits of the operator C at some states. When proving the canonical commutative relation, one has to consider the value of the operato C_n on the orbits of the operator A and the value of the operato A_n on the orbits of the operator C. Some relations between orbital operators N_n corresponding to numerical operator N and with the operators C_n and A_n are also established. The generalized canonical commutation relations between C_n and A_n are proved that in the case $n = 0$ coincides with the classical one.

In the third section orbits of creation and annihilation operators at states, the Frechet-Hilbert spaces of all orbits $D(C^\infty)$ and $D(A^\infty)$, the orbital operators C^∞

and A^∞ in these spaces are studied and generalized canonical commutation relation is proved. The analogous relationship between orbital operator N^∞, C^∞ and A^∞ is established.

2. Orbital operators corresponding to the creation and annihilation operators in the Hilbert space of finite orbits

A creation operator is a differential operator that has the following form ([8], p.541)

$$C = -d/dx + x/2. \tag{1}$$

An annihilation operator is usually denoted by ([8], p.541)

$$A = d/dx + x/2. \tag{2}$$

Note that under the names of creation and annihilation operators, the lightly modified operators $\frac{1}{\sqrt{2}}(d/dx+x)$ and $\frac{1}{\sqrt{2}}(-d/dx+x)$ are often considered and denoted, respectively, by a^* and a ([1], ch., 11.4). As well, they are denoted by $A^\dagger$ and A ([10], ch. V). They are often also denoted by $\hat{a}^\dagger$ and $\hat{a}$, or by a^+ and a. The annihilation operator thus defined reduces the number of particles in a given state by one, and the creation operator increases this number by one. Neither the creation nor the annihilation operator are defined as mappings on the entire Hilbert space $L^2(\mathbb{R})$ into itself. After all, for $\varphi \in L^2(\mathbb{R})$ the functions $C\varphi$ and $A\varphi$ may fail to be in $L^2(\mathbb{R})$. By definition, the domain of definition $D(C)$ of the operator C consists of all $\psi \in L^2(\mathbb{R})$ such that $C\psi \in L^2(\mathbb{R})$. The operators C and A are unbounded operators in $L^2(\mathbb{R})$.

It is well-known that the creation and the annihilation operators do not commute, but satisfy the relation

$$[A, C] = AC - CA = I \tag{3}$$

on $D([A, C]) = D(AC) \cap D(CA)$, $D(CA) = \{u \in D(A), A(u) \in D(C)\}$ and likewise for $D(AC)$. In (3) $[A, C]$ is the commutator and I is identity operator on the space $L^2(\mathbb{R})$. Really

$$AC = x^2/4 - d^2/dx^2 + 1/2I,\ CA = x^2/4 - d^2/dx^2 - 1/2I,\ AC - CA = I.$$

The relation (3) is known as the canonical commutation relation.

n-orbits of the annihilation and creation operators (1) and (2) in the state φ are defined as

$$\mathrm{orb}_n(A, \varphi) = (\varphi, A\varphi, A^2\varphi, \cdots, A^n\varphi) = (\varphi, (d/dx + x/2)\varphi, \cdots, (d/dx + x/2)^n\varphi),$$

and

$$\mathrm{orb}_n(C, \varphi) = (\varphi, C\varphi, C^2\varphi, \cdots, C^n\varphi)$$

$$= (\varphi, (-d/dx + x/2)\varphi, \cdots, (-d/dx + x/2)^n\varphi) \qquad (4).$$

It is well known ([8], formula (56)) that

$$C\psi_j = \sqrt{j+1}\psi_{j+1}, \tag{5}$$

where

$$\psi_j(x) = (-1)^j(2\pi)^{-1/4}(j!)^{-1/2}exp(x^2/4)d^jexp(-x^2/2)/dx^j,\ j \in N_0, \tag{6}$$

are wave functions of harmonic oscillator.

For an acting in a Hilbert space H operator F we introduce the acting in H^{n+1} operator F_n, which is defined as

$$F_n(\varphi_0, \varphi_1, \cdots, \varphi_n) := (F\varphi_0, F\varphi_1, \cdots, F\varphi_n),$$

$$(\varphi_0, \varphi_1, \cdots, \varphi_n) \in D(F_n) = (D(F))^{n+1}.$$

For the orbit of creation operator (1) in the state ψ_j we have

$$\mathrm{orb}_n(C, \psi_j) = \{\psi_j, C\psi_j, \cdots, C^n\psi_j\}$$

$$= (\psi_j, \sqrt{j+1}\psi_{j+1}, \sqrt{j+1}\sqrt{j+2}\psi_{j+2}, \cdots, \sqrt{j+1}\cdots\sqrt{j+n}\psi_{j+n})$$

and

$$C_n\mathrm{orb}_n(C, \psi_j) = (C\psi_j, C^2\psi_j, \cdots, C^{n+1}\psi_j)$$

$$= (\sqrt{j+1}\psi_{j+1}, \sqrt{j+1}\sqrt{j+2}\psi_{j+2}, \cdots, \sqrt{j+1}\cdots\sqrt{j+n+1}\psi_{j+n+1}).$$

It is well-known([8], formula (53)), that

$$A\psi_j = \sqrt{j}\psi_{j-1}. \tag{7}$$

Therefore

$$\mathrm{orb}_n(A, \psi_j) = (\psi_j, A\psi_j, A^2\psi_j, \cdots, A^n\psi_j)$$

$$= (\psi_j, \sqrt{j}\psi_{j-1}, \sqrt{j}\sqrt{j-1}\psi_{j-2}, \cdots, \sqrt{j}\sqrt{j-1}\cdots\sqrt{j-n+1}\psi_{j-n})$$

and

$$A_n\mathrm{orb}_n(A, \psi_j) = (A\psi_j, A^2\psi_j, \cdots, A^{n+1}\psi_j)$$

$$= (\sqrt{j}\psi_{j-1}, \sqrt{j}\sqrt{j-1}\psi_{j-2}, \cdots, \sqrt{j}\sqrt{j-1}\cdots\sqrt{j+n}\psi_{j-n-1}).$$

We have

$$AC\psi_j = A(\sqrt{j+1}\psi_{j+1}) = \sqrt{j+1}A\psi_{j+1} = (j+1)\psi_j.$$

The operator

$$N = CA = x^2/4 - d^2/dx^2 - 1/2,$$

is called the number operator. We have

$$N\psi_j = CA\psi_j = C(\sqrt{j}\psi_{j-1}) = \sqrt{j}C\psi_{j-1} = j\psi_j$$

and

$$N_n(\varphi_0, \cdots, \varphi_n) = (N\varphi_0, \cdots, N\varphi_n) \text{ for } (\varphi_0, \cdots, \varphi_n) \in D(N_n) = (D(N))^{n+1}.$$

Theorem 2.1 : *The following representations are valid:*
a. If $(\varphi_0, \varphi_1, \cdots, \varphi_n) \in D(N_n)$, *then*

$$N_n(\varphi_0, \varphi_1, \cdots, \varphi_n) = C_n A_n(\varphi_0, \varphi_1, \cdots, \varphi_n).$$

b. For the functions ψ_j, *defined by formula (6), we have*

$$N_n \mathrm{orb}_n(A, \psi_j) = (j\psi_j, (j-1)\sqrt{j}\psi_{j-1},$$

$$(j-2)\sqrt{j}\sqrt{j-1}\psi_{j-2}, \cdots, (j-n)\sqrt{j}\sqrt{j-1}\cdots\sqrt{j-n+1}\psi_{j-n}),\ j \in \mathbb{N}_0,$$

$$\psi_{j-n} = 0, \text{if } j < n.$$

c. For the functions ψ_j, *defined by formula (6), we have*

$$N_n \mathrm{orb}_n(C, \psi_j) = (j\psi_j, \sqrt{j+1}(j+1)\psi_{j+1},$$

$$\sqrt{j+1}\sqrt{j+2}(j+2)\psi_{j+2}, \cdots, \sqrt{j+1}\sqrt{j+2}\cdots\sqrt{j+n}(j+n)\psi_{j+n}).$$

d. $\mathrm{orb}_n(C + A, \psi) \neq \mathrm{orb}_n(C, \psi_j) + \mathrm{orb}_n(A, \psi_j)$, *if* $n \geq 2$.

Proof : a. Let $(\varphi_0, \varphi_1, \cdots, \varphi_n) \in D(N_n)$, then

$$\begin{aligned} N_n(\varphi_0, \varphi_1, \cdots, \varphi_n) &= (N\varphi_0, N\varphi_1, \cdots, N\varphi_n) = (CA\varphi_0, CA\varphi_1, \cdots, CA\varphi_n) \\ &= C_n(A\varphi_0, A\varphi_1, \cdots, A\varphi_n) = C_n A_n(\varphi_0, \varphi_1, \cdots, \varphi_n). \end{aligned}$$

b. Taking into account that $N\psi_j = CA\psi_j = C(\sqrt{j}\psi_{j-1}) = j\psi_j, \quad j \in \mathbb{N}_0$, we have

$$
\begin{aligned}
& N_n \text{ or } b_n(A,\psi_j) \\
&= C_nA_n(\psi_j, \sqrt{j}\psi_{j-1}, \sqrt{j}\sqrt{j-1}\psi_{j-2}, \cdots, \sqrt{j}\sqrt{j-1}\cdots, \\
&\qquad \sqrt{j-n+1}\psi_{j-n}) \\
&= (N\psi_j, N\sqrt{j}\psi_{j-1}, N\sqrt{j}\sqrt{j-1}\psi_{j-2}, \cdots, N\sqrt{j}\sqrt{j-1}\cdots, \\
&\qquad \sqrt{j-n+1}\psi_{j-n}) \\
&= (j\psi_j, (j-1)\sqrt{j}\psi_{j-1}, (j-2)\sqrt{j}\sqrt{j-1}\psi_{j-2}, \cdots, \\
&\qquad (j-n)\sqrt{j}\sqrt{j-1}\cdots\sqrt{j-n+1}\psi_{j-n}), \quad \psi_{j-n}(x) = 0, \text{ if } j < n;
\end{aligned}
$$

c.

$$
\begin{aligned}
& N_n \text{ or } b_n(C,\psi_j) \\
&= C_nA_n(\psi_j, \sqrt{j+1}\psi_{j+1}, \sqrt{j+1}\sqrt{j+2}\psi_{j+2}, \cdots, \sqrt{j+1}\sqrt{j+2}\cdots, \\
&\qquad \sqrt{j+n}\psi_{j+n}) \\
&= (N\psi_j, N\sqrt{j+1}\psi_{j+1}, N\sqrt{j+1}\sqrt{j+2}\psi_{j+2}, \cdots, N\sqrt{j+1}\sqrt{j+2}\cdots, \\
&\qquad \sqrt{j+n}\psi_{j+n}) \\
&= (j\psi_j, \sqrt{j+1}(j+1)\psi_{j+1}, \sqrt{j+1}\sqrt{j+2}(j+2)\psi_{j+2}, \cdots, \\
&\qquad \sqrt{j+1}\sqrt{j+2}\cdots\sqrt{j+n}(j+n)\psi_{j+n}).
\end{aligned}
$$

d. The proof is clear.

We prove now the generalized canonical commutation relations between operators C_n and A_n. These relations, in the case $n = 0$ coincide with the classical one. □

Theorem 2.2 : *For the commutator $[A_n, C_n] = A_nC_n - C_nA_n$ the following relations are hold:*
a. If $(\varphi_0, \varphi_1, \cdots, \varphi_n) \in D([A_n, C_n]) = D(A_nC_n) \cap D(C_nA_n)$, then

$$A_nC_n(\varphi_0, \varphi_1, \cdots, \varphi_n) - C_nA_n(\varphi_0, \varphi_1, \cdots, \varphi_n) = (\varphi_0, \varphi_1, \cdots, \varphi_n).$$

b. If $\mathrm{orb}_n(A,\varphi) \in D(C_nA_n)$ and $\mathrm{orb}_n(C,\varphi) \in D(A_nC_n)$, then

$$A_nC_n\mathrm{orb}_n(C,\varphi) - C_nA_n\mathrm{orb}_n(A,\varphi) = (I\psi, AC^2\varphi - CA^2\varphi, \cdots, AC^{n+1}\varphi - CA^{n+1}\varphi).$$

c. If $\mathrm{orb}_n(A,\varphi) \in D(A_nC_n)$ and $\mathrm{orb}_n(C,\varphi) \in D(C_nA_n)$, then

$$A_nC_n\mathrm{orb}_n(C,\varphi) - C_nA_n\mathrm{orb}_n(A,\varphi) = (I\psi, ACA\varphi - CAC\varphi, \cdots, ACA^n\varphi - CAC^n\varphi).$$

Proof: a. $A_nC_n(\varphi_0, \varphi_1, \cdots, \varphi_n) - C_nA_n(\varphi_0, \varphi_1, \cdots, \varphi_n)$

$$= A_n(C\varphi_0, C\varphi_1, \cdots, C\varphi_n) - C_n(A\varphi_0, A\varphi_1, \cdots, A\varphi_n)$$
$$= (AC\varphi_0, AC\varphi_1, \cdots, AC\varphi_n) - (CA\varphi_0, CA\varphi_1, \cdots, CA\varphi_n)$$
$$= ((AC - CA)\varphi_0, (AC - CA)\varphi_1, \cdots, (AC - CA)\varphi_n) = (\varphi_0, \varphi_1, \cdots, \varphi_n).$$

b. $A_nC_n\text{orb}_n(C, \varphi) - C_nA\text{orb}_n(A, \varphi)$

$$= A_nC_n(\varphi, C\varphi, C^2\varphi, \cdots, C^n\varphi) - C_nA_n(\varphi, A\varphi, A^2\varphi, \cdots, A^n\varphi)$$
$$= A_n(C\varphi, C^2\varphi, \cdots, C^{n+1}\varphi) - C_n(A\varphi, A^2\varphi, \cdots, A^{n+1}\varphi)$$
$$= (AC\varphi - CA\varphi, AC^2\varphi - CA^2\varphi, \cdots, AC^{n+1}\varphi - CA^{n+1}\varphi)$$
$$= (I\varphi, AC^2\varphi - CA^2\varphi, \cdots, AC^{n+1}\varphi - CA^{n+1}\varphi).$$

Analogously will be proved the statement
c. $A_nC_n\text{orb}_n(A, \varphi) - C_nA_n\text{orb}_n(C, \varphi)$

$$= A_nC_n(\varphi, A\varphi, A^2\varphi, \cdots, A^n\varphi) - C_nA_n(\varphi, C\varphi, C^2\varphi, \cdots, C^n\varphi)$$
$$= A_n(C\varphi, CA\varphi, \cdots, CA^n\varphi) - C_n(A\varphi, AC\varphi, \cdots, AC^n\varphi)$$
$$= (AC\varphi - CA\varphi, ACA\varphi - CAC\varphi, \cdots, ACA^n\varphi - CAC^n\varphi)$$
$$= (I\varphi, ACA\varphi - CAC\varphi, \cdots, ACA^n\varphi - CAC^n\varphi).$$

The statements a. and b. give us the direct generalization of canonical commutation relation. The statements c. and d. also are generalization of the canonical commutation relation. □

Corollary 2.3: *From part a. of Theorem 2.2 it follows that:*
a. If $\text{orb}_\text{n}(C, \varphi) \in D([A_n, C_n]) = D(C_nA_n) \cap D(A_nC_n)$*, then*

$$A_nC_n\text{orb}_\text{n}(C, \varphi) - C_nA_n\text{orb}_\text{n}(C, \varphi) = \text{orb}_\text{n}(C, \varphi).$$

b. If $\text{orb}_\text{n}(A, \varphi) \in D(A_nC_n - C_nA_n) = D(C_nA_n) \cap D(A_nC_n)$*, then*

$$A_nC_n\text{orb}_\text{n}(A, \varphi) - C_nA_n\text{orb}_\text{n}(A, \varphi) = \text{orb}_\text{n}(A, \varphi).$$

c. $[N_n, C_n] = C_n$ *and* $[N_n, A_n] = -A_n$*.*
According to the well-known distributional property, we have

$$[N_n, C_n] = [C_nA_n, C_n] = C_n[A_n, C_n] + [C_n, C_n]A_n = C_n.$$

As well

$$[N_n, A_n] = [C_nA_n, A_n] = C_n[A_n, A_n] + [C_n, A_n]A_n = -A_n.$$

If we introduce in $D(C^n)$ *the inner product*

$$\langle \text{orb}_n(C, \varphi), \text{orb}_n(C, \chi) \rangle_n$$

$$=< \varphi, \chi > + < C\varphi, C\chi > + \cdots + < C^n\varphi, C^n\chi >, \; n \in N_0, \tag{8}$$

and the corresponding norm

$$||\mathrm{orb}_n(C,\varphi)||_n = (||\varphi||^2 + ||C^2\varphi||^2 + \cdots + ||C^n\varphi||^2)^{1/2}, \tag{9}$$

where $< \cdot, \cdot >$ and $|| \cdot ||$ are inner product and norm in the space $L^2(\mathbb{R})$, then it will turn into a prehilbert space. The same can be said about $D(A^n)$. The operator C_n is a linear unbounded operator in the space $D(C^n)$ with a dense image. Analogously is defined the Hilbert space $D(A^n)$ in which, the inner product and the norm are defined by formulas (8) and (9) with the replacement of C by A. The spaces $D(C^n)$ and $D(A^n)$ can be turned into Hilbert spaces by changing the domains of the operators A and C. Namely, as the domain of definition of the operators (1) and (2) we consider the set $U \cap V$. The set U consits of all functions $\varphi \in L^2(\mathbb{R})$ which are absolutely continouos on every finite interval on $\mathbb{R}$ and such that $\varphi' \in L^2(\mathbb{R})$. The set V consists of all functions $\psi \in L^2(\mathbb{R})$ such that $x\psi(x) \in L^2(\mathbb{R})$. It is well-known that the operator $i\frac{d}{dx}$ with the domain of definition U is selfadjoint ([12], pp.396). Taking into account that a function $\varphi \in U$ satisfies the equality $\varphi(-\infty) = \varphi(\infty) = 0$ ([12], p.394), we verify that the operators $\frac{d}{dx}$ and $-\frac{d}{dx}$ with the domain of definition U are conjugate with each other. If we take into account yt selfadjointness of the position operator of quantum mechanics $X\psi(x) = x\psi(x), \; \psi \in V$, we obtain that the annihilation and creation operators (1) and (2) with the domain of definition $U \cap V$, are conjugate with each other. Every adjoint operator is closed ([12], p.353). Therefore, the operators A and C with the domain of definition $U \cap V$ are closed and we can turn $D(C^n)$ into a Hilbert space with the inner product (8) and corresponding norm (9). The same can be said about $D(A^n)$.

Theorem 2.4: *If as the domain of definition of the operators (1) and (2) is considered the set $U \cap V$, then the sequence $\{\mathrm{orb_n}(A, \psi_k)\}$ (resp. $\{\mathrm{orb_n}(C, \psi_k)\}$) , $n, k \in \mathbb{N}_0$, is an orthogonal basis on $D(A^n)$, (resp. on $D(C^n)$).*

Proof: We prove Theorem for the operator A (for the operator C proof is carried out in a similar way). The orthogonality of the sequence $\{\mathrm{orb_n}(A, \psi_k)\}$ in the space $D(A^n)$ follows from the orthogonality of $\{\psi_k(x)\}$ in $L^2(\mathbb{R})$ and from the formulae (5) and (7). Because of the sequence $\{\psi_k(x)\}$ is a basis in $L^2(\mathbb{R})$, we have for $\psi(x) \in L^2(\mathbb{R})$ that

$$\psi(x) = \sum_{k=0}^{\infty} a_k \psi_k(x),$$

where

$$a_k = \int_{\mathbb{R}} \psi(x)\overline{\psi_k(x)}dx, \; k \in \mathbb{N}_0.$$

Because of $A^j\psi \in L^2(\mathbb{R}), \; j = 1, 2, \cdots, n,$ we have for $\psi \in L^2(\mathbb{R})$

$$A^j\psi(x) = \sum_{k=0}^{\infty} b_k^j \psi_k(x),$$

where

$$b_k^j = \int_{\mathbb{R}} A^j \psi(x) \overline{\psi_k(x)} dx.$$

Due to the fact that the operators A and C are mutually conjugate to each other, we obtain

$$b_k^j = \int_{\mathbb{R}} \psi(x) C^j \overline{\psi_k(x)} dx.$$

In its turn

$$C^j \psi_k = C^{j-1} C \psi_k = C^{j-1} \sqrt{k+1} \psi_{k+1} = \sqrt{k+1}\sqrt{k+2} \cdots \sqrt{k+j} \psi_{k+j}.$$

Therefore

$$\begin{aligned} b_k^j &= \sqrt{(k+1)(k+2)\cdots(k+j)} \int_{\mathbb{R}} \psi(x) \overline{\psi_{k+j}(x)} dx \\ &= \sqrt{(k+1)(k+2)\cdots(k+j)} a_{k+j} \end{aligned}$$

and

$$\begin{aligned} A^j \psi(x) &= \sum_{k=0}^{\infty} \sqrt{(k+1)(k+2)\cdots(k+j)} a_{k+j} \psi_k(x) = \sum_{k=0}^{\infty} a_{k+j} A^j \psi_{k+j}(x) \\ &= \sum_{k=j}^{\infty} a_k A^j \psi_k(x). \end{aligned}$$

We have

$$\begin{aligned} A^j \psi_k &= A^{j-1}(A\psi_k) = \sqrt{k} A^{j-1} \psi_{k-1} = \sqrt{k(k-1)} A^{j-2} \psi_{k-2} \\ &= \sqrt{k!} A^{j-k} \psi_0, \ j-k \geq 1. \end{aligned}$$

But

$$A\psi_0 = \text{const}\Big(\frac{d}{dx} e^{-x^2/4} + \frac{x}{2} e^{-x^2/4}\Big) = 0.$$

Thus $A^j \psi_k(x) = 0$, if $k = 0, 1, \cdots, j-1$, and it is proved that

$$A^j \psi(x) = \sum_{k=0}^{\infty} a_k A^j \psi_k(x).$$

Therefore, for the $\mathrm{orb_n}(A, \psi)$ the following representation is valid

$$\mathrm{orb_n}(A, \psi) = \sum_{n=0}^{\infty} a_k \mathrm{orb_n}(A, \psi_k).$$

This equality proves the Theorem 2.4. □

3. Orbital operators corresponding to the creation and annihilation operators in the Frechet-Hilbert space of all orbits

In this section the orbital operators corresponding to the creation and annihilation operators in the Frechet-Hilbert space of all orbits are constructed. Note that for the general operator F with the discrete spectrum, the space $D(F^{\infty})$ circumstantially was studied in ([9], Chapt. 8), where $D(F^{\infty})$ was the whole symbol and F^{∞}, if taken separately, meant nothing. $D(F^{\infty})$ is the intersection $\cap_{n=0}^{\infty} D(F^n)$ of the spaces $D(F^n)$. This means that, on the function of $D(F^{\infty})$ one can apply the operator F infinitely many times. In [4] we have defined the operator F^{∞} as follows

$$F^{\infty}(\varphi, F\varphi, \cdots, F^n\varphi, \cdots) = (F\varphi, F^2\varphi, \cdots, F^{n+1}\varphi, \cdots),$$

or

$$F^{\infty}\mathrm{orb}(F,\varphi) = \mathrm{orb}(F, F\varphi). \tag{10}$$

Due tu this notation, the space $D(F^{\infty})$ acquare new meaning that differs from the classical case. Namely, now $D(F^{\infty})$ is also the domain of definition of the operator F^{∞}, defined by equality (10). According to ([10], Sect.X.6), $D(F^{\infty})$ is the set of infinitely differentiated elements of F and is denoted as $C^{\infty}(F)$.

The space $D(F^{\infty})$ is isomorphic to the space of all orbits $\mathrm{orb}(F,\varphi) = \{\varphi, F\varphi, \cdots, F^n, \cdots\}$ of the operator F at the states φ and this isomorphism is obtained by the mapping $D(F^{\infty}) \ni \varphi \to \mathrm{orb}(F,\varphi)$. It is easy to prove that algebraically $D(H^{\infty}) \subset D(C^{\infty}) \subset D(C^n)$, where C is the creation operator and H is the Hamiltonian of quantum harmonic oscillator. $D(H^{\infty})$ is isomorphic to the Schwartz space of rapidly decreasing functions [2] and is a nonempty set of the second category. The topology of the space $D(C^{\infty})$ is generated with the sequence of norms (6). As well $D(C^{\infty})$ is also the domain of definition of the operator C^{∞} defined by the equality

$$C^{\infty}(\varphi(x), C\varphi(x), \cdots, C^{n-1}\varphi(x), \cdots) = \mathrm{orb}(C, C\varphi)$$

$$= (C\varphi(x), C^2\varphi(x), \cdots, C^{n+1}\varphi(x), \cdots), \tag{11}$$

It will be also noted that the space $D(C^{\infty})$ is represented as projective limit of the sequence of the Hilbert spaces $\{D(C^n)\}$ and is a Frechet-Hilbert space.

Problem 1. It is not known whether the Frechet space $D(C^{\infty})$ is nuclear and countable-Hilbert (The example of a nuclear Frechet space that is not countable-Hilbert, was constructed in [11]).

In the case of annihilation operator A analogously is defined the space of all orbits $D(A^{\infty})$. The space $D(A^{\infty})$ is also the domain of definition of the operator A^{∞} defined by the equality

$$A^{\infty}(\varphi, A\varphi, \cdots, A^n\varphi, \cdots) = (A\varphi, A^2\varphi, \cdots, A^{n+1}\varphi, \cdots), \tag{12}$$

This means that $A^\infty(\varphi, A\varphi, \cdots, A^n\varphi, \cdots) = (d/dx + x/2)^\infty \mathrm{orb}(A, \varphi)$, where the operator $A^\infty \mathrm{orb}(A, \varphi)$ is really defined by the equality

$$(d/dx + x/2)^\infty \mathrm{orb}(A, \varphi)$$
$$= ((d/dx + x/2)\varphi, (d/dx + x/2)^2\varphi, \cdots, (d/dx + x/2)^{n+1}\varphi, \cdots).$$

According to the assertion a. of Theorem 2.1, we have

$$N^\infty \mathrm{orb}(C, \psi_j) = C^\infty A^\infty \mathrm{orb}(C, \psi) = (\frac{x^2}{4} - \frac{d^2}{dx^2} + \frac{1}{2})^\infty \mathrm{orb}(C, \psi_j)$$

$$= (j\psi_j, \sqrt{j+1}(j+1)\psi_{j+1}, \cdots, \sqrt{j+1}\sqrt{j+2}\cdots\sqrt{j+n}(j+n)\psi_{j+n}, \cdots).$$

Problem 2. It is not known whether the locally convex space $D(A^\infty)$ is nuclear and countable-Hilbert.

Theorem 3.1: *For the commutator $[A^\infty, C^\infty] = A^\infty C^\infty - C^\infty A^\infty$, where C^∞ and A^∞ are defined, respectivaly, by equalities (11) and (12), the following relations hold:*

a. If $(\varphi_0, \cdots, \varphi_n, \cdots) \in D([A^\infty, C^\infty])$, then

$$[A^\infty, C^\infty](\varphi_0, \cdots, \varphi_n, \cdots) = (\varphi_0, \cdots, \varphi_n, \cdots).$$

b. If $\mathrm{orb}(A, \varphi) \in D(C^\infty A^\infty)$ and $\mathrm{orb}(C, \psi) \in D(A^\infty C^\infty)$, then

$$(A^\infty C^\infty \mathrm{orb}(C, \varphi) - C^\infty A^\infty \mathrm{orb}(A, \varphi)$$
$$= (I\varphi, AC^2\varphi - CA^2\varphi, \cdots, AC^{n+1}\varphi - CA^{n+1}\varphi, \cdots).$$

c. If $\mathrm{orb}(A, \varphi) \in D(A^\infty, C^\infty)$ and $\mathrm{orb}(C, \varphi) \in D(C^\infty A^\infty)$, then

$$A^\infty C^\infty \mathrm{orb}(A, \varphi) - C^\infty A^\infty \mathrm{orb}(C, \varphi)$$
$$= (I\varphi, ACA\varphi - CAC\varphi, \cdots, ACA^n\varphi - CAC^n\varphi, \cdots).$$

The statement a. gives us the direct generalization of canonical commutation relation. The statements b. and c. also are generalization of canonical commutation relation.

Corollary 3.2: *From the proposition a. of Theorem 3.1 it follows that*

a. If $\mathrm{orb}(A, \varphi) \in D([A^\infty, C^\infty])$, then

$$[A^\infty, C^\infty]\mathrm{orb}(A, \varphi) = \mathrm{orb}(A, \varphi).$$

b. If $\mathrm{orb}(C, \varphi) \in D([A^\infty, C^\infty])$, then

$$[A^\infty, C^\infty]\mathrm{orb}(C, \varphi) = \mathrm{orb}(C, \varphi).$$

c.$[N^\infty, C^\infty] = C^\infty$ and $[N^\infty, A^\infty] = A^\infty$.

Really, according to the distributivity property, we have

$$[N^\infty, C^\infty] = [C^\infty A^\infty, C^\infty] = C^\infty[A^\infty, C^\infty] + [C^\infty, C^\infty]A^\infty = C^\infty.$$

As well

$$[N^\infty, A^\infty] = [C^\infty A^\infty, A^\infty] = C^\infty[A^\infty, A^\infty] + [C^\infty, A^\infty]A^\infty = -A^\infty.$$

References

[1] B.C. Hall. *Quantum theory for mathematicians*, Springer, Graduate texts in mathematics, **267**, 2013

[2] D. Ugulava, D. Zarnadze. *On a central algorithm for calculation of the inverse of the harmonic oscillator in the spaces of orbits*, Journal of Complexity. **68** (2022), https:// doi.org. 10.1016/j.jco.2022.101599

[3] D. Ugulava, D. Zarnadze. *Ill-posed problems and associated with them spaces of orbits and orbital operators*, Rep. Enlarged Sess. Semin. I. Vekua Appl. Math., **328** (2018) 79-82

[4] D. Zarnadze, S. Tsotniashvili. *Selfadjoint operators and generalized central algorithms in Frechet spaces*, Georgian Math. J., **13** (2006) 363-382

[5] S. Rolevich. *On orbits of elements*, Studia Mathematica, **XXXII** (1969), 17-22.

[6] S.A. Wegner. *Universal extrapolation spaces for C_0-semigroups*, Ann. Univ. Ferrara Sez. VII Sci. Mat., **60**, 2 (2014), 447-463

[7] D. Ugulava, D. Zarnadze. *About the Concept of Orbital Quantum Mechanics*, XI International Conference of the Georgian Mathematical Union, Batumi, August **23-28** (2021), p. 180

[8] J. Becnel, A. Sengupta. *The Schwartz Space: Tools for Quantum Mechanics and Infinite Dimensional Analysis*, Mathematics **3** (2015), 527-562; doi:10.3390/math3020527

[9] H. Triebel. *Interpolation Theory, Function Spaces, Differential Operators*, Veb Deutscher Verlag, Berlin, 1978

[10] M. Reed, B. Simon. *Methods of modern mathematical physics*, **1** (1972), **2** (1975), Academic Press

[11] S. Dierolf, K. Floret. *Uber die fortsetzbarkeit of stetiger normen.*, Arch. Math. (Basel), **35** (1980), 149-154

[12] L.A. Lusternik, V.I. Sobolev. *Elements of Functional Analysis*, New York, 1974. Translation from Russian, Moskow, Nauka, 1965