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On One Special Case of Internal Boundary Value Problem of Elasticity for the Domain Bounded by Hyperbolas

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An analytical (exact) solution of two-dimensional problems of elasticity in the domain bounded by hyperbolas is constructed in the elliptic coordinates. A special kind of internal boundary value problem is set and solved in the area bounded by hyperbolas when both parts of one hyperbola (border) are lines and non-homogeneous symmetry or antisymmetry conditions are given on it, while non-homogeneous conditions, such as stresses or displacements, are given on another hyperbola. Exact solution is obtained using the method of separation of variables. The graphs for the numerical results of some test problems are presented.

Keywords: Internal boundary value problem, Hyperbolic boundary, Method of separation of variables, Homogeneous isotropic body

AMS Subject Classification: 74B05.

1. Introduction

The solution of boundary value and boundary-contact problems in the areas with curvilinear border, is simplified if examining such problems in the appropriate curvilinear coordinate system. For instance, the problems for the areas bounded by a circle or its parts are studied in the polar coordinates [1-3], while the problems for the areas bounded by the circles with different centers and radii are studied in the bipolar coordinates [4-6]. The problems for the areas bounded by an ellipse or its parts are studied in the elliptic coordinates [7-14], and the problems for the areas with parabolic boundaries are considered in the parabolic coordinates [15-17].

The problems I consider do not coincide with the above-mentioned ones. This work deals with mathematical modeling of the stress-strain state of a homogeneous isotropic body with a hyperbolic crack and the corresponding boundary value problems are solved analytically.

The current paper examines the internal boundary value problems (BVP) for the area introduced in Fig. 1 in elliptic coordinates ξ, η ($-\infty < \xi < \infty$, $0 \leq \eta \leq \pi$, $x = c\cosh\xi\cos\eta$, $y = c\sinh\xi\sin\eta$, where x, y $(-\infty < x < \infty,\ -\infty < y < \infty)$ are Cartesian coordinates, c is a scale factor. In the present article, we take $c = 1$) [18,19]. In particular, a special kind of internal BVP is set and solved in the area $\Omega = \left\{-\xi_1 < \xi < \xi_1,\ 0 < \eta < \eta_1 = \frac{\pi}{2}\right\}$ (see Fig. 1), when both parts of the

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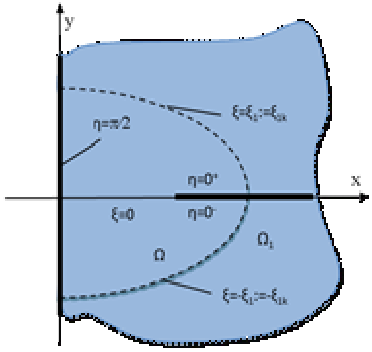

Figure 1. The area $\Omega_1 = \{-\infty < \xi < \infty,\ 0 < \eta < \eta_1\}$ (and $\Omega = \{-\xi_1 < \xi < \xi_1,\ 0 < \eta < \eta_1\}$) bounded by hyperbolas when both parts of the hyperbola $\eta = 0$ are lines.

hyperbola $\eta = 0$ are lines and non-homogeneous symmetry or antisymmetry conditions are given on it, while such non-homogeneous conditions, as stresses or displacements, are given on $\eta = \frac{\pi}{2}$. The analytical (exact) solution of two-dimensional problems of elasticity is constructed in the area bounded by the coordinate lines of the elliptic coordinate system [11, 13]. The analytical solution is obtained with the method of separation of variables.

Using the MATLAB software, the numerical results are obtained of some test problems and relevant graphs are presented.

2. Principal equalities in elliptic coordinate system

The paper deals with the homogeneous isotropic elastic body free of volume forces, to which the following area (see Fig. 1) corresponds:

$$\Omega = \left\{-\xi_1 < \xi < \xi_1,\ \ 0 < \eta < \frac{\pi}{2}\right\} \tag{1}$$

(Selection of $\xi_1 := \xi_{1k}$ see Appendix B).

2.1. *Equilibrium equations and Hooke's law*

In the elliptic coordinates the equilibrium equations with respect to the function $D,\ K,\ u,\ v$ can be writen as [10,20]:

$$\begin{aligned} &\text{a) } D_{,\xi} - K_{,\eta} = 0, \qquad &\text{c) } \bar{u}_{,\xi} + \bar{v}_{,\eta} = h_0^2 B \\ &\text{b) } D_{,\eta} + K_{,\xi} = 0, \qquad &\text{d) } \bar{v}_{,\xi} - \bar{u}_{,\eta} = h_0^2 R, \end{aligned} \tag{2}$$

and Hooke's law as follows:

$$
\begin{aligned}
\frac{h_0^2}{2\mu}\sigma_{\xi\xi} &= \frac{h_0^2}{2\mu}D - \bar{v}_{,\eta} - \frac{L_1}{h_0^2},\\
\frac{h_0^2}{2\mu}\sigma_{\eta\eta} &= \frac{h_0^2}{2\mu}D - \bar{u}_{,\xi} + \frac{L_1}{h_0^2},\\
\frac{h_0^2}{2\mu}\tau_{\xi\eta} &= \frac{h_0^2}{2\mu}K - \bar{u}_{,\eta} - \frac{L_2}{h_0^2},
\end{aligned} \tag{3}
$$

where $L_1 = \sinh(2\xi)\,\bar{u} - \sin(2\eta)\,\bar{v}$, $L_2 = \sin(2\eta)\,\bar{u} + \sinh(2\xi)\,\bar{v}$, $\bar{u} = \frac{2hu}{c^2}$, $\bar{v} = \frac{2hv}{c^2}$, $h_0 = \sqrt{\cosh(2\xi) - \cos(2\eta)}$; $h := h_\xi = h_\eta = \frac{c}{\sqrt{2}}h_0$ are Lammé coefficients [19]; $u,\ v$ are the components of the displacement vector along the tangents of $\eta,\ \xi$ lines; $B = \frac{\kappa-2}{\kappa\mu}D$ is the divergence of the displacement vector; $R = \frac{1}{\mu}K$ is the rotor component of the displacement vector; $\mu = \frac{E}{2(1-\nu)}$, $\kappa = 4(1-\nu)$; ν is Poisson's ratio and E is the modulus of elasticity; $\sigma_{\xi\xi}$, $\sigma_{\eta\eta}$ and $\tau_{\xi\eta} = \tau_{\eta\xi}$ are normal and tangential stresses.

2.2. *Analytical solution of BVP*

Let us find the solution of system (2) in class $C^2(\Omega)$ (Ω area see Fig. 1). The solution is presented by two harmonic φ_1 and φ_2 functions (see Appendix A). From formulas (A.11)-(A.13), after inserting $\alpha = \eta_1$ and making simple transformations, we will obtain:

$$
\begin{aligned}
\bar{u} &= \left[F_1 \sin^2\eta_1 \cdot \cot\eta + (\kappa-1)\,\varphi_2\right]\sin\eta\cdot\cosh\xi\\
&\quad - \left[F_2 \cos^2\eta_1 \cdot \tan\eta + (\kappa-1)\,\varphi_1\right]\cos\eta\cdot\sinh\xi,\\
\bar{v} &= \left[F_1 \cos^2\eta_1 \cdot \tan\eta + (\kappa-1)\,\varphi_2\right]\cos\eta\cdot\sinh\xi\\
&\quad + \left[F_2 \sin^2\eta_1 \cdot \cot\eta + (\kappa-1)\,\varphi_1\right]\sin\eta\cdot\cosh\xi.
\end{aligned} \tag{4}
$$

$$
\Delta\varphi_i = \tfrac{1}{h^2}\left(\varphi_{i,\xi\xi} + \varphi_{i,\eta\eta}\right) = 0, \quad i = 1,2. \tag{5}
$$

$$
\begin{aligned}
D &= -\frac{\kappa\mu}{\cosh(2\xi) - \cos(2\eta)}\left[F_1\cos\eta\cdot\sinh\xi - F_2\sin\eta\cdot\cosh\xi\right],\\
K &= \frac{\kappa\mu}{\cosh(2\xi) - \cos(2\eta)}\left[F_1\sin\eta\cdot\cosh\xi - F_2\cos\eta\cdot\sinh\xi\right],
\end{aligned}
$$

where $F_1 = (\varphi_{1,\xi} - \varphi_{2,\eta})$, $F_2 = (\varphi_{1,\eta} + \varphi_{2,\xi})$.

The stress tensor components will be written as follows:

$$
\begin{aligned}
\frac{h_0^2}{\mu}\sigma_{\eta\eta} = & -\left[F_3 \sin^2\eta_1 \cdot \cot\eta - \kappa\varphi_{1,\eta} + (\kappa-2)\,\varphi_{2,\xi}\right] \sin\eta\cosh\xi \\
& + \left[F_4 \cos^2\eta_1 \cdot \tan\eta + (\kappa-2)\,\varphi_{1,\xi} + \kappa\varphi_{2,\eta}\right] \cos\eta \cdot \sinh\xi + F_5, \\
\frac{h_0^2}{\mu}\tau_{\xi\eta} = & \left[F_3 \cos^2\eta_1 \cdot \tan\eta - \kappa\varphi_{1,\eta} + (\kappa-2)\,\varphi_{2,\xi}\right] \cos\eta\sinh\xi \\
& + \left[F_4 \sin^2\eta_1 \cdot \cot\eta + (\kappa-2)\,\varphi_{1,\xi} + \kappa\varphi_{2,\eta}\right] \sin\eta \cdot \cosh\xi + F_6, \\
\frac{h_0^2}{\mu}\sigma_{\xi\xi} = & \left[F_3 \sin^2\eta_1 \cdot \cot\eta - (\kappa-2)\,\varphi_{1,\eta} + \kappa\varphi_{2,\xi}\right] \sin\eta\cosh\xi \\
& - \left[F_4 \cos^2\eta_1 \cdot \tan\eta + \kappa\varphi_{1,\xi} + (\kappa-2)\,\varphi_{2,\eta}\right] \cos\eta \cdot \sinh\xi - F_5,
\end{aligned} \tag{6}
$$

where

$$
\begin{aligned}
& F_3 = 2\left(\varphi_{1,\xi\xi} - \varphi_{2,\xi\eta}\right), \quad F_4 = 2\left(\varphi_{1,\xi\eta} + \varphi_{2,\xi\xi}\right), \\
& F_5 = \frac{4\sin(\eta_1+\eta)\cdot\sin(\eta_1-\eta)}{\cosh(2\xi)-\cos(2\eta)}\left[F_1 \sinh\xi\cdot\cos\eta - F_2\cosh\xi\cdot\sin\eta\right], \\
& F_6 = \frac{4\sin(\eta_1+\eta)\cdot\sin(\eta_1-\eta)}{\cosh(2\xi)-\cos(2\eta)}\left[F_1 \cosh\xi\cdot\sin\eta - F_2\sinh\xi\cdot\cos\eta\right].
\end{aligned}
$$

From (5), by using the method of separation of variables, we will obtain

$$
\varphi_i = \sum_{n=1}^{\infty} \varphi_{in}, \quad i = 1, 2, \tag{7}
$$

where

$$
\varphi_{1n} = A_{1n}\cosh(n\eta)\cdot\sin(n\xi), \qquad \varphi_{2n} = A_{2n}\sinh(n\eta)\cdot\cos(n\xi)
$$

or

$$
\varphi_{1n} = A_{1n}\sinh(n\eta)\cdot\cos(n\xi), \qquad \varphi_{2n} = A_{2n}\cosh(n\eta)\cdot\sin(n\xi).
$$

We are introducing the following assumptions: (a) ξ_1 is a sufficiently great positive number (see Appendix B); (b) the boundary conditions given on $\eta = \eta_1$, i.e. stresses or displacements equal zero at $\tilde{\xi}_1 < \xi < \xi_1$ interval; (c) when stresses are given on $\eta = \eta_1$, the main vector and main moment equal zero.

If $\bar{u}$ and $\bar{v}$ are given on $\eta = \eta_1$, it is purposeful to take the following equivalent expressions instead of them [13]:

$$
\begin{aligned}
& \frac{2}{h_0^2}\left(\cosh\xi\cdot\sin\eta_1\bar{u} + \sinh\xi\cdot\cos\eta_1\bar{v}\right) \\
& \qquad = \frac{1}{2}\sin(2\eta_1)\left(\varphi_{1,\xi} - \varphi_{2,\eta}\right) + (\kappa-1)\,\varphi_2, \\
& \frac{2}{h_0^2}\left(\sinh\xi\cdot\cos\eta_1\bar{u} - \cosh\xi\cdot\sin\eta_1\bar{v}\right) \\
& \qquad = -\frac{1}{2}\sin(2\eta_1)\left(\varphi_{1,\eta} + \varphi_{2,\xi}\right) - (\kappa-1)\,\varphi_1.
\end{aligned} \tag{8}
$$

And when $\frac{h_0^2}{2\mu}\sigma_{\eta\eta}$ and $\frac{h_0^2}{2\mu}\sigma_{\xi\eta}$ are given on $\eta = \eta_1$, it is purposeful to take the following equivalent expressions instead of them [13]:

$$\begin{aligned} &\frac{2}{\mu}\left(\cosh\xi\cdot\sin\eta_1\sigma_{\eta\eta} - \sinh\xi\cdot\cos\eta_1\tau_{\xi\eta}\right) = -\sin\left(2\eta_1\right)\left(\varphi_{1,\xi\xi} - \varphi_{2,\xi\eta}\right) \\ &\qquad\qquad +\kappa\varphi_{1,\eta} - \left(\kappa-2\right)\varphi_{2,\xi}, \\ &\frac{2}{\mu}\left(\sinh\xi\cdot\cos\eta_1\sigma_{\eta\eta} - \cosh\xi\cdot\sin\eta_1\tau_{\xi\eta}\right) = \sin\left(2\eta_1\right)\left(\varphi_{1,\xi\eta} + \varphi_{2,\xi\xi}\right) \\ &\qquad\qquad + \left(\kappa-2\right)\varphi_{1,\xi} + \kappa\varphi_{2,\eta}. \end{aligned} \tag{9}$$

Using the homogeneous boundary conditions of the posed specific problem, we will paste functions φ_1 and φ_2 selected from (7) in the right parts of equations (8) or (9) and will decompose the left parts of the equations into Fourier trigonometric series. We will equate the expressions in both parts at the trigonometric functions of the same name and will receive an infinite system of linear algebraic equations with rspect to unknown coefficients A_{1n} and A_{2n}. The major matrix of this system is of a block-diagonal kind. The dimensions of each block are 2×2 and the determinant does not equal zero, while the determinant in the infinity tends to a finite number different from zero.

Convergence of functional series corresponding to expressions (4) and (6) in the area $\bar{\Omega} = \{-\xi_1 \le \xi \le \xi_1,\ \ 0 \le \eta \le \eta_1\}$ are easily proved by constructing the relevant majorizing uniformly convergent numerical series.

3. Special case of internal BVP

3.1. *Setting and solving problems*

Let us set and solve a special kind of internal problems in the area $\Omega = \left\{-\xi_1 < \xi < \xi_1,\ \ 0 < \eta < \frac{\pi}{2}\right\}$, when both parts of the contour $\eta = 0$ are lines (see Fig. 1). So, let us find the solution of the system of equilibrium equations (2) of a homogeneous isotropic elastic body in the area Ω, which meets the following boundary conditions: it meets non-homogeneous conditions of antisymmetry or symmetry on $\eta = 0$, which are common boundary conditions and not their combination and stresses or displacements given on $\eta = \eta_1 = \frac{\pi}{2}$.

So, we have the following conditions:

$$\begin{aligned} &\text{for } \xi = 0: && \text{a) } \bar{v} = 0,\ \ \bar{u}_{,\xi} = 0 \ \text{ or } \ \text{b) } \bar{u} = 0,\ \ \bar{v}_{,\xi} = 0, \\ &\text{for } \eta = 0^+,\ \xi > 0: && \text{a) } \bar{u} = f_1(\xi),\ \bar{v}_{,\eta} = f_2(\xi) \ \text{ or} \\ & && b)\ -\bar{\text{v}} = f_3(\xi) \ \text{ or } \bar{u}_{,\eta} = f_4(\xi), \\ &\text{for } \eta = 0^-,\ \xi < 0: && \text{a) } \bar{u} = -f_1(\xi),\ \bar{v}_{,\eta} = f_2(\xi) \ \text{ or} \\ & && b)\ \bar{\text{v}} = -f_3(\xi) \ \text{ or } \bar{u}_{,\eta} = f_4(\xi), \end{aligned} \tag{10}$$

$$\begin{aligned} &\text{for } \eta = \eta_1 = \tfrac{\pi}{2}: \quad a) \quad \tfrac{h_0^2}{\mu}\sigma_{\eta\eta} = Q_1(\xi),\ \tfrac{h_0^2}{\mu}\tau_{\xi\eta} = Q_2(\xi) \ \text{ or} \\ & \qquad\qquad\qquad\ \ b) \quad \bar{u} = H_1(\xi),\ \bar{v} = H_2(\xi), \end{aligned} \tag{11}$$

where Q_i $(i=1,2)$, together with its first order derivatives, and H_i together with its first and second order derivatives are decomposed into absolutely and uniformly convergent trigonometric Fourier series [21].

When $\bar{u}$ and $\bar{v}$ are given on $\eta=\eta_1$, it is purposeful to take their equivalent expressions (8) instead of them, and if $\frac{h_0^2}{2\mu}\sigma_{\eta\eta}$ and $\frac{h_0^2}{2\mu}\sigma_{\xi\eta}$ are given on $\eta=\eta_1$, then it is purposeful to take their equivalent expressions (9) instead of them.

From formula (7), functions φ_1 and φ_2 will be selected with boundary conditions (10).

1) Taking into account (10a), φ_{1n} and φ_{2n} we will have:

$$\varphi_{1n}=A_{1n}\cosh(n\eta)\cdot\sin(n\xi), \qquad \varphi_{2n}=A_{2n}\sinh(n\eta)\cdot\cos(n\xi). \tag{12}$$

System (8) will be as follows:

$$\begin{aligned}
&\frac{2}{h_0^2}\left(\cosh\xi\cdot\sin\eta_1\bar{u}+\sinh\xi\cdot\cos\eta_1\bar{v}\right)=\sum_{n=1}^{\infty}(\kappa-1)\,A_{2n}\sinh(n\eta)\cos(n\xi),\\
&\frac{2}{h_0^2}\left(\sinh\xi\cdot\cos\eta_1\bar{u}-\cosh\xi\cdot\sin\eta_1\bar{v}\right)=-\sum_{n=1}^{\infty}(\kappa-1)\,A_{1n}\cosh(n\eta)\sin(n\xi).
\end{aligned}$$

System (9) will be as follows:

$$\begin{aligned}
&\frac{2}{\mu}\left(\cosh\xi\cdot\sin\eta_1\sigma_{\eta\eta}-\sinh\xi\cdot\cos\eta_1\tau_{\xi\eta}\right)\\
&\qquad=\sum_{n=1}^{\infty}n\left[\kappa A_{1n}+(\kappa-2)\,A_{2n}\right]\sinh(n\eta)\sin(n\xi),\\
&\frac{2}{\mu}\left(\sinh\xi\cdot\cos\eta_1\sigma_{\eta\eta}-\cosh\xi\cdot\sin\eta_1\tau_{\xi\eta}\right)\\
&\qquad=\sum_{n=1}^{\infty}n\left[(\kappa-2)\,A_{1n}+\kappa A_{2n}\right]\cosh(n\eta)\cos(n\xi).
\end{aligned} \tag{13}$$

As a result of inserting expressions (7), (12) in (4), the following formulas for displacements are obtained:

$$\begin{aligned}
\bar{u}=&\sum_{n=1}^{\infty}\{[(A_{1n}-A_{2n})\,n\cdot\cos\eta\cosh(n\eta)+(\kappa-1)\,A_{2n}\sin\eta\sinh(n\eta)]\\
&\cdot\cosh\xi\cos(n\xi)-(\kappa-1)\,A_{1n}\cos\eta\cosh(n\eta)\sinh\xi\sin(n\xi)\},\\
\bar{v}=&\sum_{n=1}^{\infty}\{[(A_{1n}-A_{2n})\,n\cdot\cos\eta\sinh(n\eta)+(\kappa-1)\,A_{1n}\sin\eta\cosh(n\eta)]\\
&\cdot\cosh\xi\sin(n\xi)+(\kappa-1)\,A_{2n}\cos\eta\sinh(n\eta)\sinh\xi\cos(n\xi)\}.
\end{aligned} \tag{14}$$

After inserting (7), (12) in formulas (6), the following expressions are obtained for stresses:

$$
\begin{aligned}
&\frac{h_0^2}{2\mu}\sigma_{\eta\eta} \\
&=\sum_{n=1}^{\infty}\left\{\left[2n^2\left(A_{1n}-A_{2n}\right)\cos\eta\cosh\left(n\eta\right)+\kappa nA_{1n}\sin\eta\sinh\left(n\eta\right)\right.\right. \\
&\left.+\left(\kappa-2\right)nA_{2n}\sin\eta\sinh\left(n\eta\right)\right]\cosh\xi\sin\left(n\xi\right) \\
&+\left[\left(\kappa-2\right)A_{1n}+\kappa A_{2n}\right]n\cos\eta\cosh\left(n\eta\right)\sin\xi\cos\left(n\xi\right) \\
&+\frac{4\cos^2\eta}{\cosh\left(2\xi\right)-\cos\left(2\eta\right)}n\left(A_{1n}-A_{2n}\right)\left[\cos\eta\cosh\left(n\eta\right)\sinh\xi\cos\left(n\xi\right)\right. \\
&\left.\left.-\sin\eta\sinh\left(n\eta\right)\cosh\xi\sin\left(n\xi\right)\right]\right\}, \\
&\frac{h_0^2}{2\mu}\tau_{\xi\eta} \\
&=\sum_{n=1}^{\infty}\left\{\left[2n^2\left(A_{1n}-A_{2n}\right)\cos\eta\sinh\left(n\eta\right)+\left(\kappa-2\right)nA_{1n}\sin\eta\cosh\left(n\eta\right)\right.\right. \\
&\left.+\kappa nA_{2n}\sin\eta\cosh\left(n\eta\right)\right]\cosh\xi\cos\left(n\xi\right) \\
&-\left[\kappa A_{1n}+\left(\kappa-2\right)A_{2n}\right]n\cos\eta\sinh\left(n\eta\right)\sinh\xi\sin\left(n\xi\right) \\
&+\frac{4\cos^2\eta}{\cosh\left(2\xi\right)-\cos\left(2\eta\right)}n\left(A_{1n}-A_{2n}\right)\left[\sin\eta\cosh\left(n\eta\right)\cosh\xi\cos\left(n\xi\right)\right. \\
&\left.\left.-\cos\eta\sinh\left(n\eta\right)\sinh\xi\sin\left(n\xi\right)\right]\right\}, \\
&\frac{h_0^2}{2\mu}\sigma_{\xi\xi} \\
&=\sum_{n=1}^{\infty}\left\{\left[-2n^2\left(A_{1n}-A_{2n}\right)\cos\eta\cosh\left(n\eta\right)-\left(\kappa-2\right)nA_{1n}\sin\eta\sinh\left(n\eta\right)\right.\right. \\
&\left.-\kappa nA_{2n}\sin\eta\sinh\left(n\eta\right)\right]\cosh\xi\sin\left(n\xi\right) \\
&-\left[\kappa A_{1n}+\left(\kappa-2\right)A_{2n}\right]n\cos\eta\cosh\left(n\eta\right)\sinh\xi\cos\left(n\xi\right) \\
&-\frac{4\cos^2\eta}{\cosh\left(2\xi\right)-\cos\left(2\eta\right)}n\left(A_{1n}-A_{2n}\right)\left[\cos\eta\cosh\left(n\eta\right)\sinh\xi\cos\left(n\xi\right)\right. \\
&\qquad\left.\left.-\sin\eta\sinh\left(n\eta\right)\cosh\xi\sin\left(n\xi\right)\right]\right\}.
\end{aligned} \tag{15}
$$

2) By considering (10b), we will obtain:

$$
\varphi_{1n}=A_{1n}\sinh\left(n\eta\right)\cdot\cos\left(n\xi\right), \qquad \varphi_{2n}=A_{2n}\cosh\left(n\eta\right)\cdot\sin\left(n\xi\right). \tag{16}
$$

System (8) will be as follows:

$$\frac{2}{h_0^2}\left(\cosh\xi\cdot\sin\eta_1\bar{u}+\sinh\xi\cdot\cos\eta_1\bar{v}\right)=\sum_{n=1}^{\infty}(\kappa-1)\,A_{2n}\cosh(n\eta)\sin(n\xi)\,,$$
$$\frac{2}{h_0^2}\left(\sinh\xi\cdot\cos\eta_1\bar{u}-\cosh\xi\cdot\sin\eta_1\bar{v}\right)=-\sum_{n=1}^{\infty}(\kappa-1)\,A_{1n}\sinh(n\eta)\cos(n\xi)\,.$$

System (9) will be as follows:

$$\begin{aligned}&\frac{2}{\mu}\left(\cosh\xi\cdot\sin\eta_1\sigma_{\eta\eta}-\sinh\xi\cdot\cos\eta_1\tau_{\xi\eta}\right)\\&\qquad=\sum_{n=1}^{\infty}n\left[\kappa A_{1n}-(\kappa-2)\,A_{2n}\right]\cosh(n\eta)\cos(n\xi)\,,\\&\frac{2}{\mu}\left(\sinh\xi\cdot\cos\eta_1\sigma_{\eta\eta}-\cosh\xi\cdot\sin\eta_1\tau_{\xi\eta}\right)\\&\qquad=-\sum_{n=1}^{\infty}n\left[(\kappa-2)\,A_{1n}-\kappa A_{2n}\right]\sinh(n\eta)\sin(n\xi)\,.\end{aligned}\tag{17}$$

As a result of inserting expressions (7), (16) in formulas (4), the following expressions for displacements are obtained:

$$\begin{aligned}\bar{u}=\sum_{n=1}^{\infty}&\{[-(A_{1n}+A_{2n})\,n\cdot\cos\eta\sinh(n\eta)+(\kappa-1)\,A_{2n}\sin\eta\cosh(n\eta)]\\&\cdot\cosh\xi\sin(n\xi)-(\kappa-1)\,A_{1n}\cos\eta\sinh(n\eta)\sinh\xi\cos(n\xi)\}\,,\\\bar{v}=\sum_{n=1}^{\infty}&\{[(A_{1n}+A_{2n})\,n\cdot\cos\eta\cosh(n\eta)+(\kappa-1)\,A_{1n}\sin\eta\sinh(n\eta)]\\&\cdot\cosh\xi\cos(n\xi)+(\kappa-1)\,A_{2n}\cos\eta\cosh(n\eta)\sinh\xi\sin(n\xi)\}\,.\end{aligned}\tag{18}$$

After inserting expressions (7), (16) in (6), the following formulas are obtained for stresses:

$$
\begin{aligned}
&\frac{h_0^2}{2\mu}\sigma_{\eta\eta} \\
&=\sum_{n=1}^{\infty}\left\{\left[2n^2\left(A_{1n}+A_{2n}\right)\cos\eta\sinh\left(n\eta\right)+\kappa nA_{1n}\sin\eta\cosh\left(n\eta\right)\right.\right. \\
&-\left(\kappa-2\right)nA_{2n}\sin\eta\cosh\left(n\eta\right)]\cosh\xi\cos\left(n\xi\right) \\
&-\left[\left(\kappa-2\right)A_{1n}-\kappa A_{2n}\right]n\cos\eta\sinh\left(n\eta\right)\sin\xi\sin\left(n\xi\right) \\
&-\frac{4\cos^2\eta}{\cosh\left(2\xi\right)-\cos\left(2\eta\right)}n\left(A_{1n}+A_{2n}\right)\left[\cos\eta\sinh\left(n\eta\right)\sinh\xi\sin\left(n\xi\right)\right. \\
&\left.\left.+\sin\eta\cosh\left(n\eta\right)\cosh\xi\cos\left(n\xi\right)\right]\right\}, \\
&\frac{h_0^2}{2\mu}\tau_{\xi\eta} \\
&=\sum_{n=1}^{\infty}\left\{\left[-2n^2\left(A_{1n}+A_{2n}\right)\cos\eta\cosh\left(n\eta\right)-\left(\kappa-2\right)nA_{1n}\sin\eta\sinh\left(n\eta\right)\right.\right. \\
&+\kappa nA_{2n}\sin\eta\sinh\left(n\eta\right)]\cosh\xi\sin\left(n\xi\right) \\
&-\left[\kappa A_{1n}+\left(\kappa-2\right)A_{2n}\right]n\sin\eta\cosh\left(n\eta\right)\cosh\xi\cos\left(n\xi\right) \\
&+\frac{4\cos^2\eta}{\cosh\left(2\xi\right)-\cos\left(2\eta\right)}n\left(A_{1n}+A_{2n}\right)\left[\sin\eta\sinh\left(n\eta\right)\cosh\xi\sin\left(n\xi\right)\right. \\
&\left.\left.-\cos\eta\cosh\left(n\eta\right)\sinh\xi\cos\left(n\xi\right)\right]\right\}, \\
&\frac{h_0^2}{2\mu}\sigma_{\xi\xi} \\
&=\sum_{n=1}^{\infty}\left\{\left[-2n^2\left(A_{1n}+A_{2n}\right)\cos\eta\sinh\left(n\eta\right)-\left(\kappa-2\right)nA_{1n}\sin\eta\cosh\left(n\eta\right)\right.\right. \\
&+\kappa nA_{2n}\sin\eta\cosh\left(n\eta\right)]\cosh\xi\cos\left(n\xi\right) \\
&+\left[\kappa A_{1n}-\left(\kappa-2\right)A_{2n}\right]n\cos\eta\sinh\left(n\eta\right)\sinh\xi\sin\left(n\xi\right) \\
&+\frac{4\cos^2\eta}{\cosh\left(2\xi\right)-\cos\left(2\eta\right)}n\left(A_{1n}+A_{2n}\right)\left[\cos\eta\sinh\left(n\eta\right)\sin\xi\sin\left(n\xi\right)\right. \\
&\left.\left.+\sin\eta\cosh\left(n\eta\right)\cosh\xi\cos\left(n\xi\right)\right]\right\}.
\end{aligned}
\tag{19}
$$

3.2. *Test problems*

1) Let us find numerical solutions of problem (2), (10a) (11a) when ***a)*** $Q_1(\xi) = P$ and $Q_2(\xi) = 0$, i.e. normal load $\frac{2}{\mu}\sigma_{\eta\eta} = \frac{2P}{h_0^2}$ is given on the boundary $\eta = \eta_1 = \frac{\pi}{2}$, and tangential stress equals zero, and ***b)*** $Q_1(\xi) = 0$ and $Q_2(\xi) = P$, i.e. tangential stress $\frac{2}{\mu}\tau_{\xi\eta} = \frac{2P}{h_0^2}$ is given on the boundary $\eta = \eta_1 = \frac{\pi}{2}$, and normal stress equals zero.
a) System (13) will be as follows:

$$\sum_{n=1}^{\infty} n\left[\kappa A_{1n} + (\kappa - 2)\, A_{2n}\right]\sinh(n\eta_1)\sin(n\xi) = \frac{2P}{\cosh(2\xi) - \cos(2\eta_1)}\cosh\xi,$$
$$\sum_{n=1}^{\infty} n\left[(\kappa - 2)\, A_{1n} + \kappa A_{2n}\right]\cosh(n\eta_1)\cos(n\xi) = 0.$$

Following the expansion of the right side of the equations into Fourier series, an infinite system of linear algebraic equations is obtained in relation to unknown A_{1n} and A_{2n} coefficients

$$n\left[\kappa A_{1n} + (\kappa - 2)\, A_{2n}\right]\sinh(n\eta_1) = \tilde{F}_{1n}, (\kappa - 2)\, A_{1n} + \kappa A_{2n} = 0 \qquad n = 1, 2, \ldots \tag{20}$$

where $\tilde{F}_{1n} = \frac{2}{\pi}\int_0^{\pi} f_1(\xi)\sin(n\xi)\, d\xi$ is the coefficient of expansion of the function

$$f_1(\xi) = \frac{2P}{\cosh(2\xi) - \cos(2\eta_1)}\cosh\xi$$

into the Fourier series $F_1(\xi) = \sum\limits_{n=1}^{\infty} \tilde{F}_{1n}\sin(n\xi)$.

From (20):

$$A_{1n} = \frac{\kappa \tilde{F}_{1n}}{4n(\kappa - 1)\sinh(n\eta_1)}, \qquad A_{2n} = -\frac{(\kappa - 2)\tilde{F}_{1n}}{4n(\kappa - 1)\sinh(n\eta_1)}.$$

Then, by putting A_{1n} and A_{2n} in formulas (14) and (15), we will obtain the values of displacements and stresses at any point of the considered area.
b) System (13) will be as follows:

$$\sum_{n=1}^{\infty} n\left[\kappa A_{1n} + (\kappa - 2)\, A_{2n}\right]\sinh(n\eta_1)\sin(n\xi) = 0,$$
$$\sum_{n=1}^{\infty} n\left[(\kappa - 2)\, A_{1n} + \kappa A_{2n}\right]\cosh(n\eta_1)\cos(n\xi) = \sum_{n=1}^{\infty} \tilde{F}_{1n}\cos(n\xi),$$

where $\tilde{F}_{1n}$ is the coefficient of expansion of function

$$f_1(\xi) = -\frac{2P}{\cosh(2\xi) - \cos(2\eta_1)}\cosh\xi$$

into Fourier series.

$$\kappa A_{1n} + (\kappa - 2)\, A_{2n} = 0,$$
$$n\left[(\kappa - 2)\, A_{1n} + \kappa A_{2n}\right] \cosh(n\eta_1) = \tilde{F}_{1n}, \qquad n = 1, 2, \ldots$$

$$A_{1n} = \frac{\kappa \tilde{F}_{1n}}{4n\,(\kappa - 1)\cosh(n\eta_1)}, \qquad A_{2n} = -\frac{(\kappa - 2)\,\tilde{F}_{1n}}{4n\,(\kappa - 1)\cosh(n\eta_1)}.$$

Further, by putting A_{1n} and A_{2n} in formulas (14) and (15), we will obtain the values of displacements and stresses at any point of the considered domain.

2) Let us find numerical solutions of problems (2), (10b),(11a) when ***a)*** $Q_1(\xi) = P$ and $Q_2(\xi) = 0$, i.e. normal load $\frac{2}{\mu}\sigma_{\eta\eta} = \frac{2P}{h_0^2}$ is given on the boundary $\eta = \eta_1 = \frac{\pi}{2}$, and the tangential stress equals zero, and ***b)*** $Q_1(\xi) = 0$ and $Q_2(\xi) = P$, i.e. the tangential stress $\frac{2}{\mu}\tau_{\xi\eta} = \frac{2P}{h_0^2}$ is given on the boundary $\eta = \eta_1 = \frac{\pi}{2}$, and normal stress equals zero.

a) System (17) will be as follows:

$$\sum_{n=1}^{\infty} n\left[\kappa A_{1n} - (\kappa - 2)\, A_{2n}\right] \cosh(n\eta) \cos(n\xi) = \sum_{n=1}^{\infty} \tilde{F}_{1n} \cos(n\xi),$$

$$\sum_{n=1}^{\infty} -n\left[(\kappa - 2)\, A_{1n} - \kappa A_{2n}\right] \sinh(n\eta) \sin(n\xi) = 0,$$

where $\tilde{F}_{1n}$ is the coefficient of expansion of the function

$$f_1(\xi) = \frac{2P}{\cosh(2\xi) - \cos(2\eta_1)} \cosh\xi$$

into the Fourier series.

$$n\left[\kappa A_{1n} - (\kappa - 2)\, A_{2n}\right] \cosh(n\eta_1) = \tilde{F}_{1n}, (\kappa - 2)\, A_{1n} - \kappa A_{2n} = 0,$$
$$n = 1, 2, \ldots,$$

$$A_{1n} = \frac{\kappa \tilde{F}_{1n}}{4n\,(\kappa - 1)\cosh(n\eta_1)}, \qquad A_{2n} = \frac{(\kappa - 2)\,\tilde{F}_{1n}}{4n\,(\kappa - 1)\cosh(n\eta_1)}.$$

By putting A_{1n} and A_{2n} in formulas (18) and (19), we will obtain the values of displacements and stresses at any point of the considered area.

b) System (17) will be as follows:

$$\sum_{n=1}^{\infty} n\left[\kappa A_{1n}-(\kappa-2)\,A_{2n}\right]\cosh(n\eta)\cos(n\xi)=0,$$

$$\sum_{n=1}^{\infty} -n\left[(\kappa-2)\,A_{1n}-\kappa A_{2n}\right]\sinh(n\eta)\sin(n\xi)=\sum_{n=1}^{\infty}\tilde{F}_{1n}\sin(n\xi)\,,$$

where $\tilde{F}_{1n}$ is the coefficient of expansion of the function

$$f_1(\xi)=-\frac{2P}{\cosh(2\xi)-\cos(2\eta_1)}\cosh\xi$$

into the Fourier series.

$$\kappa A_{1n}-(\kappa-2)\,A_{2n}=0,$$
$$n\left[(\kappa-2)\,A_{1n}-\kappa A_{2n}\right]\sinh(n\eta)=\tilde{F}_{1n},\qquad n=1,2,\ldots.$$

$$A_{1n}=-\frac{(\kappa-2)\,\tilde{F}_{1n}}{4n\,(\kappa-1)\sinh(n\eta_1)},\qquad A_{2n}=-\frac{\kappa\tilde{F}_{1n}}{4n\,(\kappa-1)\sinh(n\eta_1)}.$$

Next, by putting A_{1n} and A_{2n} in formulas (18) and (19), we will obtain the values of displacements and stresses at any point of the considered domain.

3.3. *Results and discussion*

Numerical values of stresses and displacements are obtained for all four instances mentioned above at the points of the area bounded by lines $\eta=0$, $\eta=\eta_1$ and $\xi=\pm\xi_1$ (see Fig. 1), and appropriate 3D graphs are constructed. The numerical values are obtained for the following data: $\nu=0.3,\ E=2\cdot10^6 kg/cm^2$, $P=-10kg/cm^2$, $\xi_1=3.5$, $\eta_1=\pi/2$.

Fig. 2 and Fig. 3 show the distribution of stresses and displacements in the area bounded by lines $\eta=0$, $\eta=\eta_1$ and $\xi=\pm\xi_1$, when conditions (10a) are fulfilled and normal stress is applied to $\eta=\eta_1$, and tangential stress equals zero (See Fig. 1). Following conditions (10a), $\sigma_{\xi\xi}$, $\sigma_{\eta\eta}$ stresses and v displacement are antisymmetric to axis ox, and $\tau_{\xi\eta}$ and u are symmetric, what is seen from Fig. 2 and Fig. 3, too. When (ξ,η) tends to points $(0,0)$ or $\left(0,\frac{\pi}{2}\right)$, then stresses and normal displacement tend to a sufficiently large number with an absolute value and tangential displacement tends to sufficiently large number with an absolute value, when (ξ,η) tends to a point $\left(0,\frac{\pi}{2}\right)$.

Fig. 4 and Fig. 5 show the distribution of stresses and displacements in the area bounded by lines $\eta=0$, $\eta=\eta_1$ and $\xi=\pm\xi_1$ when conditions (10a) are satisfied and tangential stress is applied to $\eta=\eta_1$, and normal stress equals zero (see Fig. 1). Following conditions (10a), $\sigma_{\xi\xi}$, $\sigma_{\eta\eta}$ stresses and v displacement are antisymmetric to axis ox, and $\tau_{\xi\eta}$ and u are symmetric, what is seen in Fig. 4 and Fig. 5, too. When (ξ,η) tends to point$(0,0)$, then, the stresses tend to sufficiently large numbers

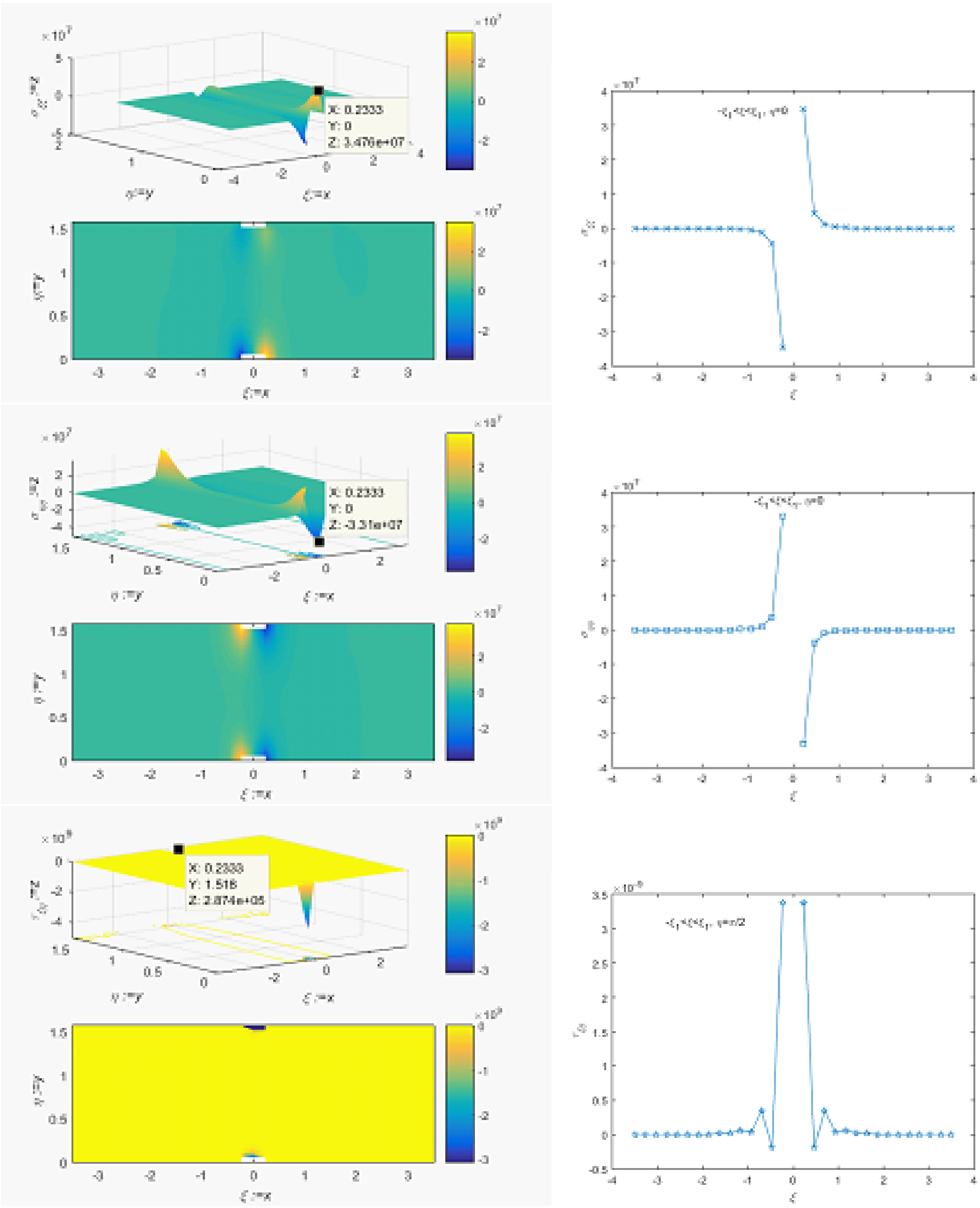

Figure 2. Distribution of stresses in the area bounded by lines $\eta = 0$, $\eta = \eta_1$ and $\xi = \pm\xi_1$ when conditions (10a) are valid (normal stress is applied to $\eta = \eta_1$. See Fig. 1).

with an absolute values (for example, $\sigma_{\eta\eta}(0.2333, 0) = -3.196 \cdot 10^7$), when (ξ, η) tends to points $(0, 0)$ or $\left(0, \frac{\pi}{2}\right)$, then normal displacement tends to sufficiently large number with an absolute value, and tangential displacement tends to sufficiently large number with an absolute value when (ξ, η) tends to point $\left(0, \frac{\pi}{2}\right)$.

The results presented on Figures 2,3 and 4,5, except for $\tau_{\xi\eta}$, differ from each other by the number signs, for example, at Fig.2 $\sigma_{\xi\xi}(0.2333, 0) = 3.476 \cdot 10^7$, and at Fig.4 $\sigma_{\xi\xi}(0.2333, 0) = -3.3636 \cdot 10^7$.

Fig. 6 and Fig. 7 show the distribution of stresses and displacements in the area bounded by lines $\eta = 0$, $\eta = \eta_1$ and $\xi = \pm\xi_{1k}$, when conditions (10b) are

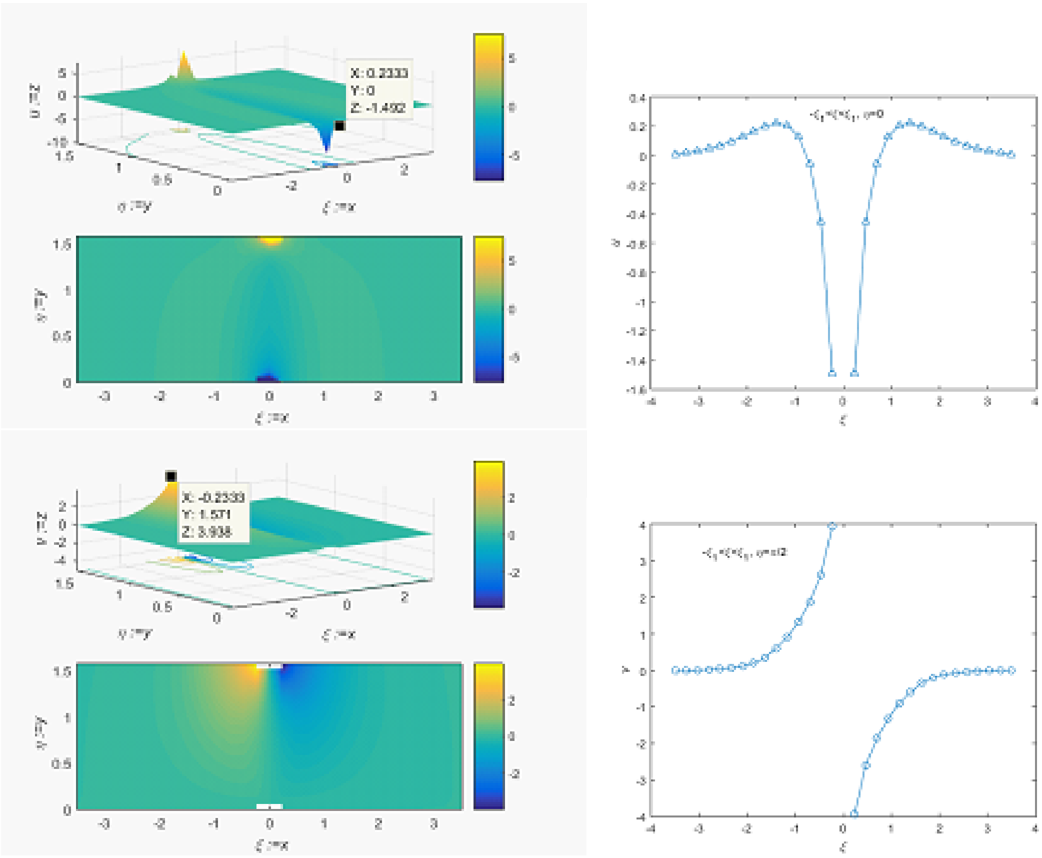

Figure 3. Distribution of displacements in the area bounded by lines $\eta = 0$, $\eta = \eta_1$ and $\xi = \pm\xi_1$ when conditions (10a) are valid (normal stress is applied to $\eta = \eta_1$. See Fig. 1).

satisfied and normal stress is applied to $\eta = \eta_1$, and tangential stress equals zero (see Fig. 1). Following conditions (10b), $\sigma_{\xi\xi}$, $\sigma_{\eta\eta}$ stresses and v displacement are symmetric to axis ox, and $\tau_{\xi\eta}$ and u are antisymmetric, what is seen in Fig. 6 and Fig. 7, too. When (ξ, η) tends to points $(0, 0)$ or $\left(0, \frac{\pi}{2}\right)$, then normal stresses and tangential displacement tend to a sufficiently large number with an absolute value), and tangential stress and normal displacement tend to a sufficiently large number with an absolute value when (ξ, η) tends to point $\left(0, \frac{\pi}{2}\right)$.

Fig. 8 and Fig. 9 show the distribution of stresses and displacements in the area bounded by lines $\eta = 0$, $\eta = \eta_1$ and $\xi = \pm\xi_1$, when conditions (10b) are fulfilled and tangential stress is applied to $\eta = \eta_1$, and normal stress equals zero (see Fig. 1). Following conditions (10b), $\sigma_{\xi\xi}$, $\sigma_{\eta\eta}$ stresses and v displacement are symmetric to axis ox, and $\tau_{\xi\eta}$ and u are symmetric, what is seen in Fig. 8 and Fig. 9, too. When (ξ, η) tends to points $(0, 0)$ or $\left(0, \frac{\pi}{2}\right)$, then stresses and tangential displacement tend to a sufficiently large number with an absolute value, and normal displacement tends to a sufficiently large number with an absolute value, when (ξ, η) tends to point $\left(0, \frac{\pi}{2}\right)$.

4. Conclusion

The paper deals the internal BVP for the domain introduced in Fig.1 in elliptic coordinates. In particular, the special kind of internal BVP is set and solved in

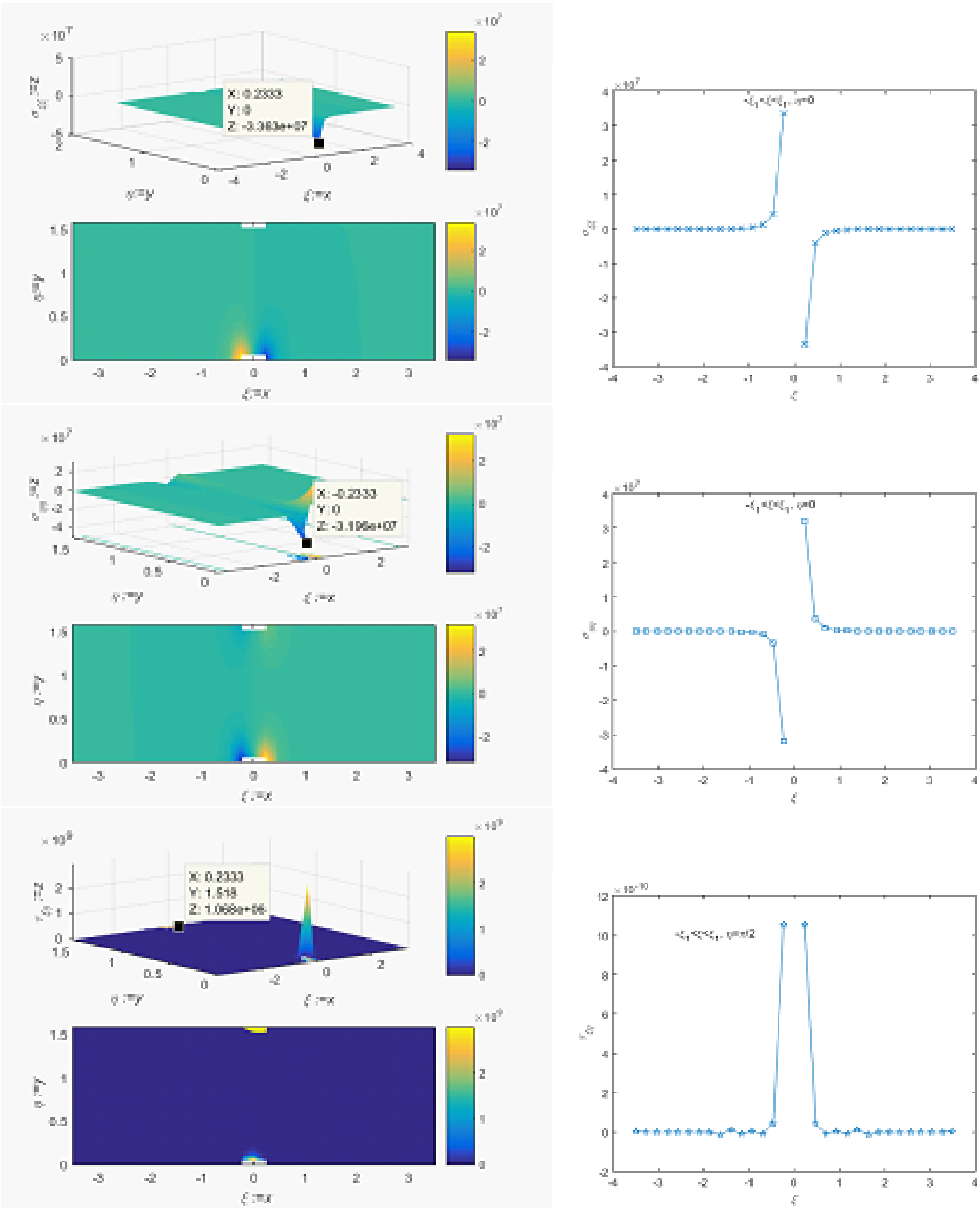

Figure 4. Distribution of stresses in the area bounded by lines $\eta = 0$, $\eta = \eta_1$ and $\xi = \pm\xi_1$, when conditions (10a) are valid (tangential stress is applied to $\eta = \eta_1$. See Fig. 1).

the area $\Omega = \left\{-\xi_1 \leq \xi \leq \xi_1\ ,\ 0 \leq \eta \leq \frac{\pi}{2}\right\}$, when both parts of the hyperbola $\eta = 0$ (border) are lines and non-homogeneous symmetry or antisymmetry conditions are given on it, while non-homogeneous conditions, such as stresses or displacements, are given on another hyperbola $\eta = \frac{\pi}{2}$. This problem is a mathematical model of the stress strain state of a homogeneous isotropic body with a hyperbolic crack.

The analytical solution of 2D problems of elasticity in the area bounded by hyperbolas is written in the elliptic coordinates. The analytical solutions are derived by the method of separation of variables, which is represented by means of two harmonic functions. Numerical values of stresses and displacements are obtained

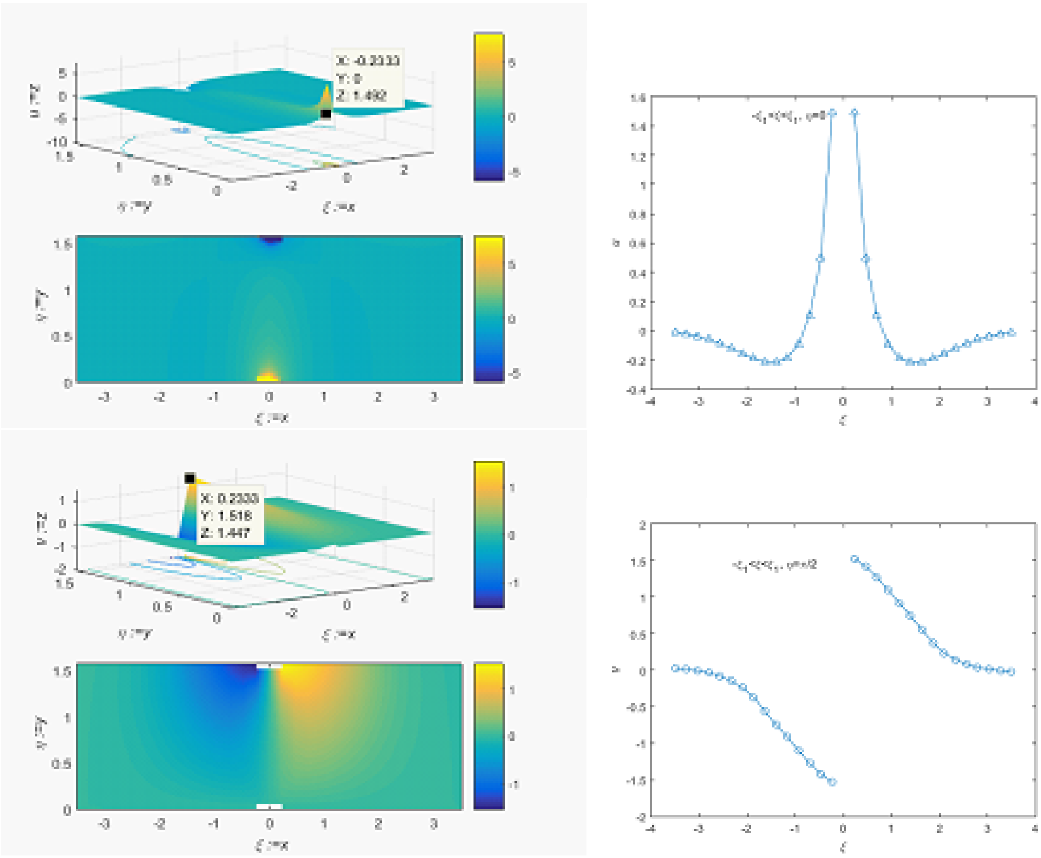

Figure 5. Distribution of displacements in the area bounded by lines $\eta = 0$, $\eta = \eta_1$ and $\xi = \pm\xi_1$, when condition (10a) is valid (tangential stress is applied to $\eta = \eta_1$. See Fig. 1).

for four test problems at the points of the area bounded by lines $\eta = 0$, $\eta = \eta_1$ and $\xi = \pm\xi_1$ (see Fig. 1), and appropriate graphs are presented and discussed. The computation and graphs were made by using MATLAB software.

Bodies with hyperbolic cracks are frequently applied in practice, for instance, in building, mining mechanics, mechanical engineering, biology, medicine, etc. The investigation of the stress-strain state of such bodies is relevant and thus, in my point of view, setting the problems considered in the work and method to solve them is interesting from the practical point of view.

Appendix A: Solution partial differential equations

Let us solve of partial differential equations(2).

Let us introduce the harmonic function φ_1, and if we take

$$\begin{aligned} &\text{a)}\quad D = \frac{\mu}{h_0^2}\left(\cosh\xi\sin\eta\frac{\partial\varphi_1}{\partial\eta} - \sinh\xi\cos\eta\frac{\partial\varphi_1}{\partial\xi}\right),\\ &\text{b)}\quad K = \frac{\mu}{h_0^2}\left(\cosh\xi\sin\eta\frac{\partial\varphi_1}{\partial\eta} + \sinh\xi\cos\eta\frac{\partial\varphi_1}{\partial\xi}\right), \end{aligned} \tag{A.1}$$

then equations (2a) and (2b) will be satisfied identically and equations (2c) and

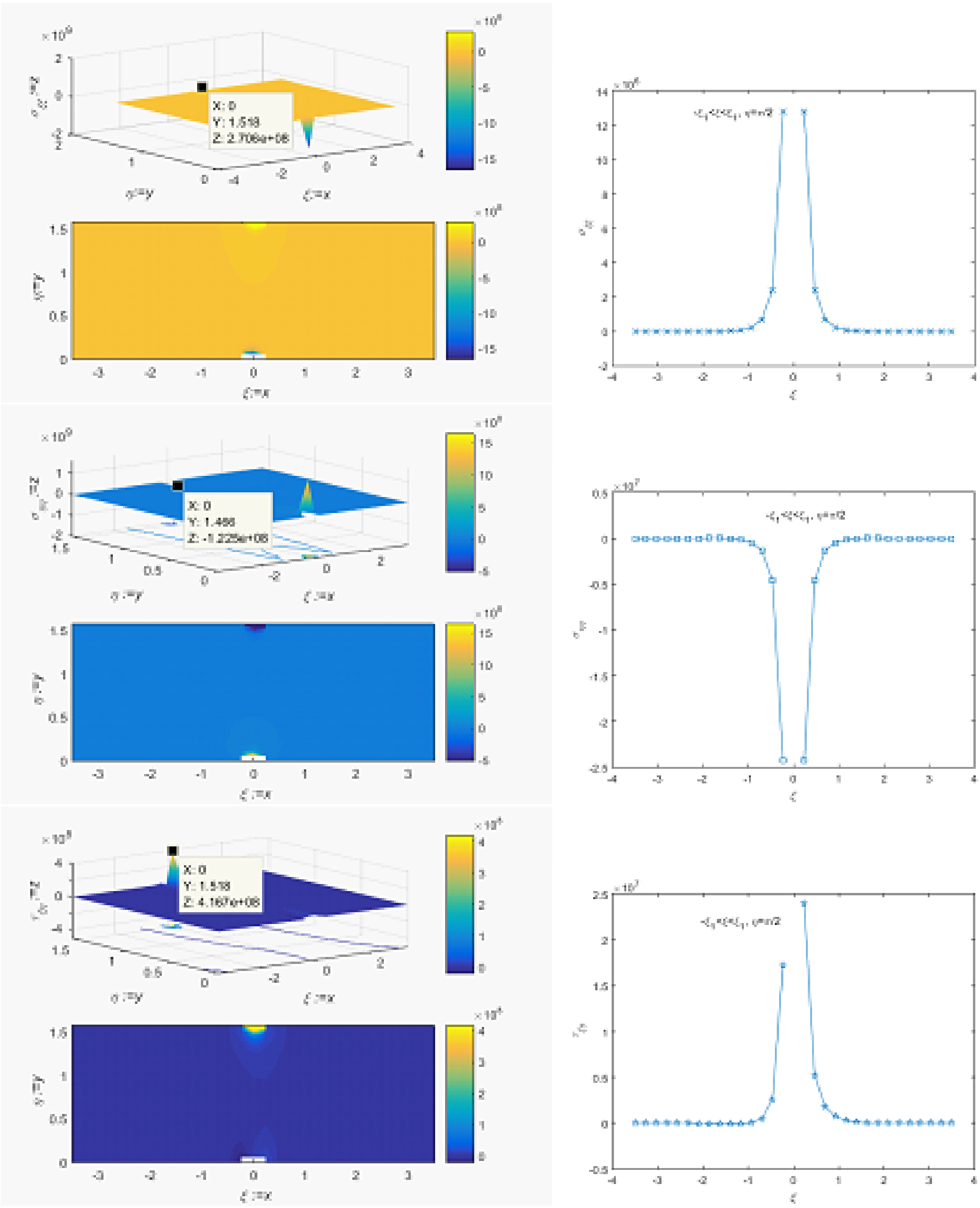

Figure 6. Distribution of stresses in the area bounded by lines $\eta = 0$, $\eta = \eta_1$ and $\xi = \pm\xi_1$ when conditions (10b) are valid (normal stress is applied to $\eta = \eta_1$. See Fig. 1).

(2d) will be as follows

$$\begin{aligned} &\text{a)}\quad \frac{\partial \bar{u}}{\partial \xi} + \frac{\partial \bar{v}}{\partial \eta} = \frac{(\kappa-2)}{\kappa}\left(\cosh\xi \sin\eta \frac{\partial \varphi_1}{\partial \eta} - \sinh\xi \cos\eta \frac{\partial \varphi_1}{\partial \xi}\right), \\ &\text{b)}\quad \frac{\partial \bar{v}}{\partial \xi} - \frac{\partial \bar{u}}{\partial \eta} = \cosh\xi \sin\eta \frac{\partial \varphi_1}{\partial \xi} + \sinh\xi \cos\eta \frac{\partial \varphi_1}{\partial \eta}. \end{aligned} \tag{A.2}$$

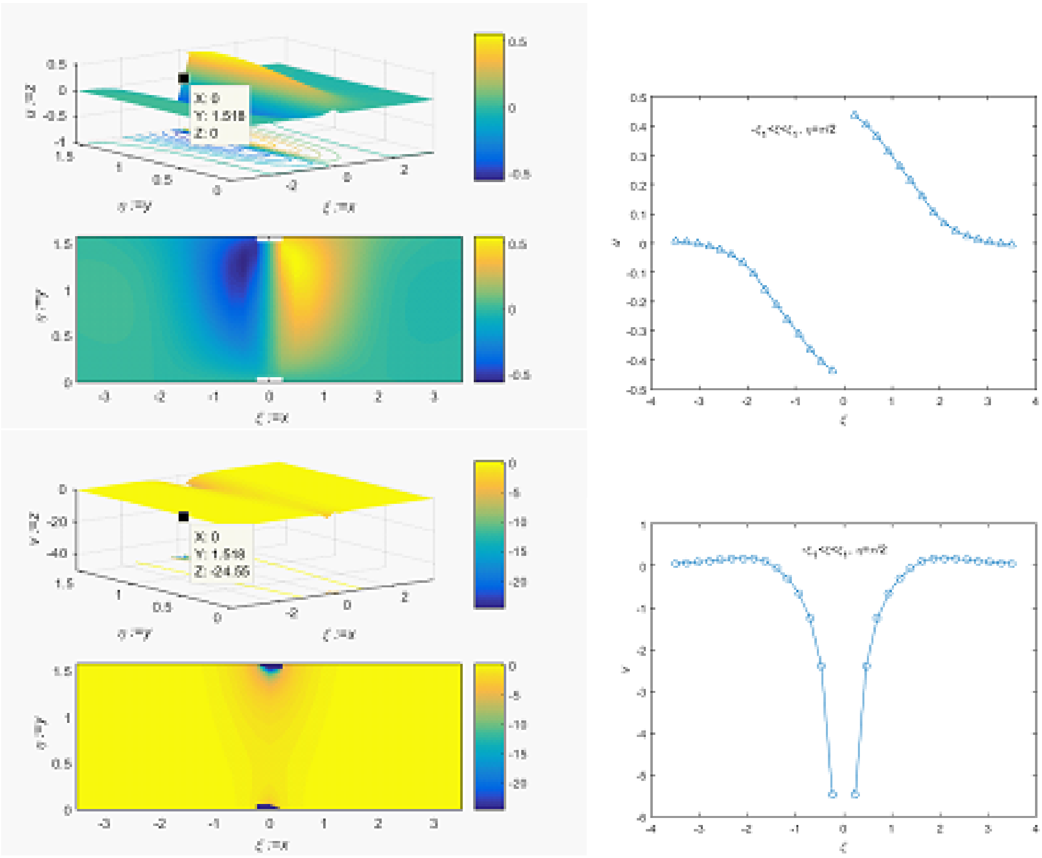

Figure 7. Distribution of displacements in the area bounded by lines $\eta = 0$, $\eta = \eta_1$ and $\xi = \pm\xi_1$, when conditions (10b) are valid (normal stress is applied to $\eta = \eta_1$. See Fig. 1).

From here

$$\begin{aligned} &\text{a)} \quad \frac{\partial \bar{u}}{\partial \xi} + \frac{\partial \bar{v}}{\partial \eta} = \frac{(\kappa-2)}{\kappa}\left(\cosh\xi \sin\eta \frac{\partial \varphi_1}{\partial \eta} - \sinh\xi \cos\eta \frac{\partial \varphi_1}{\partial \xi}\right), \\ &\text{b)} \quad \frac{\partial}{\partial \xi}\left(\bar{v} - \sinh\xi \sin\eta \frac{\partial \varphi_1}{\partial \xi}\right) = \frac{\partial}{\partial \eta}\left(\bar{u} + \sinh\xi \sin\mu \frac{\partial \varphi_1}{\partial \eta}\right). \end{aligned} \tag{A.3}$$

From (A.3b), it follows that there is a certain harmonic function ψ, for which the following condition is met

$$\bar{u} = \frac{\partial \psi}{\partial \xi} - \sinh\xi \sin\eta \frac{\partial \varphi_1}{\partial \eta}, \qquad \bar{v} = \frac{\partial \psi}{\partial \eta} + \sinh\xi \sin\eta \frac{\partial \varphi_1}{\partial \xi}, \tag{A.4}$$

and using (A.4), it follows from that (A.3a)

$$h^2 \Delta \psi = \frac{\partial^2 \psi}{\partial \xi^2} + \frac{\partial^2 \psi}{\partial \eta^2} = \frac{2(\kappa-1)}{\kappa}\left(\cosh\xi \sin\eta \frac{\partial \varphi_1}{\partial \eta} - \sinh\xi \cos\eta \frac{\partial \varphi_1}{\partial \xi}\right) \tag{A.5}$$

General solution of system (A.2) will be presented as

$$\bar{u} = \chi_1, \quad \bar{v} = \chi_2,$$

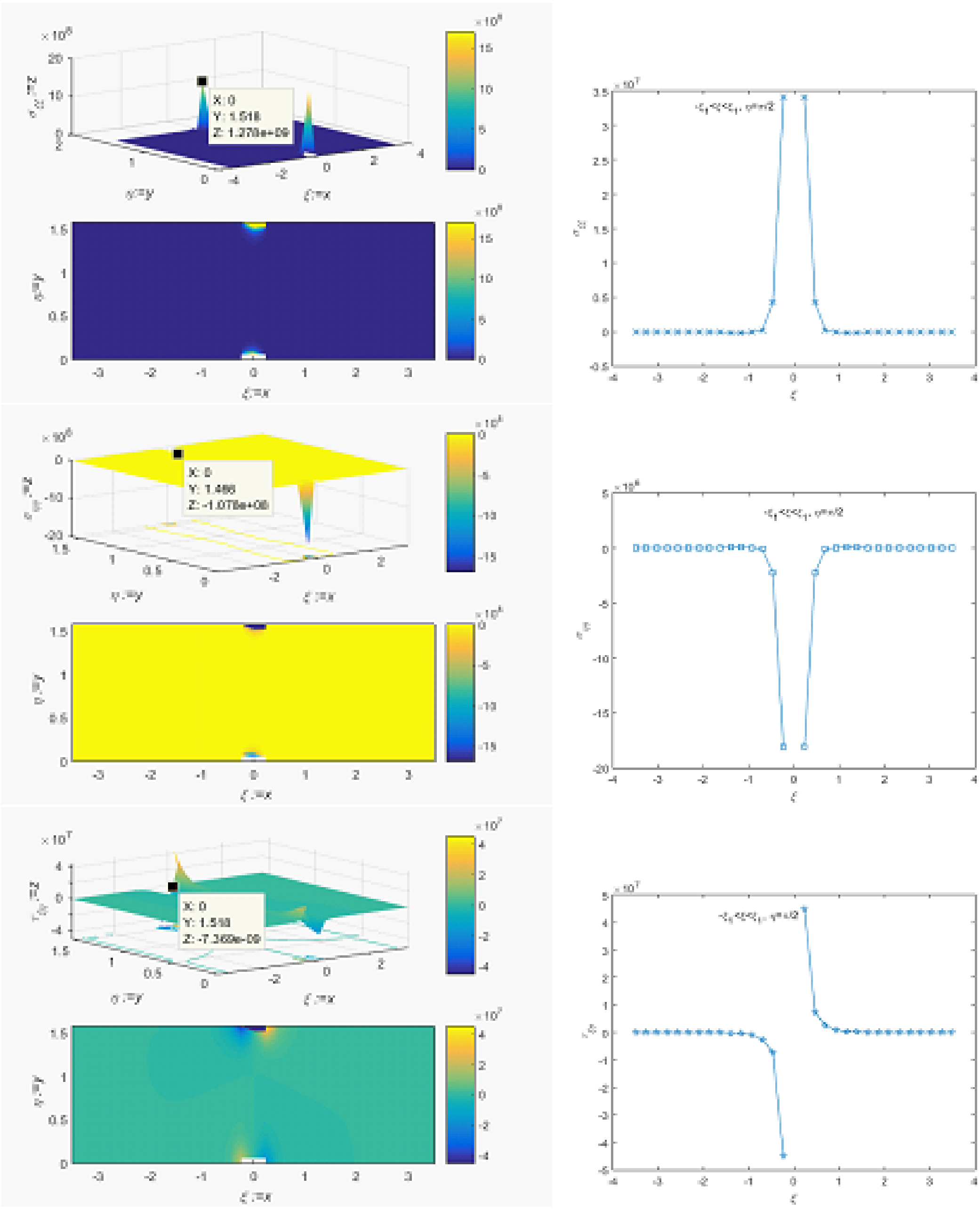

Figure 8. Distribution of stresses in the area bounded by lines $\eta = 0$, $\eta = \eta_1$ and $\xi = \pm\xi_1$, when conditions (10b) are valid (tangential stress is applied to $\eta = \eta_1$. See Fig. 1).

where

$$\frac{\partial \chi_1}{\partial \xi} + \frac{\partial \chi_2}{\partial \eta} = 0, \qquad \frac{\partial \chi_2}{\partial \xi} - \frac{\partial \chi_1}{\partial \eta} = 0.$$

The full solution of the equation system (A.2) will be written as follows:

$$\bar{u} = \frac{\partial \psi}{\partial \xi} - \sinh \xi \sin \eta \frac{\partial \varphi_1}{\partial \eta} + \chi_1, \qquad \bar{v} = \frac{\partial \psi}{\partial \eta} + \sinh \xi \sin \eta \frac{\partial \varphi_1}{\partial \xi} + \chi_2 \tag{A.6}$$

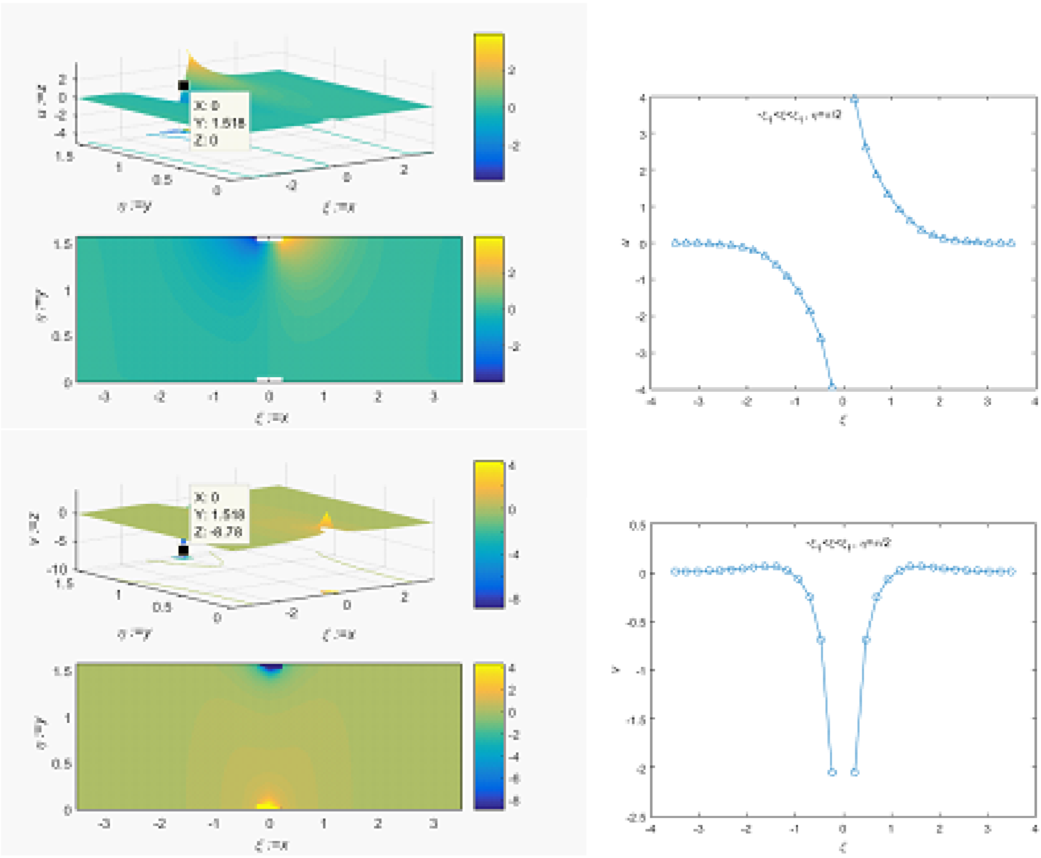

Figure 9. Distribution of displacements in the area bounded by lines $\eta = 0$, $\eta = \eta_1$ and $\xi = \pm\xi_1$, when conditions (10b) are valid (tangential stress is applied to $\eta = \eta_1$. See Fig. 1).

where ψ is the particular solution of (A.5).

Taking $\kappa = const$, we get

$$\psi = -\frac{\kappa - 1}{\kappa} \sinh \xi \cos \eta \cdot \varphi_1$$

and formulas (A.6) will be as follows

$$\bar{u} = \tfrac{1}{\kappa} \cosh \xi \cos \eta \tfrac{\partial \varphi_1}{\partial \xi} - \tfrac{\kappa-1}{\kappa} \sinh \xi \cos \eta \cdot \varphi_1$$
$$- \left(\cosh \xi \cos \eta \tfrac{\partial \varphi_1}{\partial \xi} + \sinh \xi \sin \eta \tfrac{\partial \varphi_1}{\partial \eta} \right) + \chi_1,$$
$$\bar{v} = \tfrac{1}{\kappa} \cosh \xi \cos \eta \tfrac{\partial \varphi_1}{\partial \eta} + \tfrac{\kappa-1}{\kappa} \cosh \xi \sin \eta \cdot \varphi_1$$
$$- \left(\cosh \xi \cos \eta \tfrac{\partial \varphi_1}{\partial \eta} - \sinh \xi \sin \eta \tfrac{\partial \varphi_1}{\partial \xi} \right) + \chi_2.$$

Without loss of generality the expression in brackets can be taken to be zero, because we already have in $\bar{u}$ and $\bar{v}$ of the solutions Laplacian (we mean χ_1 and χ_2). Therefore, if in (A.1) and (A.7), instead of the function φ_1, we insert the function

$\kappa\varphi_1$, the solution of system (2) will be written as follows

$$\begin{aligned}
&\text{a)}\quad h_0^2 D = \kappa\mu\left(\cosh\xi\sin\eta\frac{\partial\varphi_1}{\partial\eta} - \sinh\xi\cos\eta\frac{\partial\varphi_1}{\partial\xi}\right),\\
&\text{b)}\quad h_0^2 K = \kappa\mu\left(\sinh\xi\cos\eta\frac{\partial\varphi_1}{\partial\xi} + \sinh\xi\cos\eta\frac{\partial\varphi_1}{\partial\eta}\right),\\
&\text{c)}\quad \bar{u} = \cosh\xi\cos\eta\frac{\partial\varphi_1}{\partial\xi} - (\kappa-1)\sinh\xi\cos\eta\cdot\varphi_1 + \chi_1,\\
&\text{d)}\quad \bar{v} = \cosh\xi\cos\eta\frac{\partial\varphi_1}{\partial\eta} + (\kappa-1)\cosh\xi\sin\eta\cdot\varphi_1 + \chi_2.
\end{aligned} \tag{A.7}$$

Now, let us write out three variants to present functions χ_1 and χ_2. In the first variant

$$\begin{aligned}
\chi_1 &= \frac{\partial\bar{\varphi}_1}{\partial\eta} + \frac{\partial\tilde{\varphi}_1}{\partial\eta} + \frac{\partial\varphi_2}{\partial\eta},\\
\chi_2 &= \frac{\partial\bar{\varphi}_1}{\partial\xi} + \frac{\partial\tilde{\varphi}_1}{\partial\xi} + \frac{\partial\varphi_2}{\partial\xi},
\end{aligned} \tag{A.8}$$

$\bar{\varphi}_1, \tilde{\varphi}_1, \varphi_2$ are harmonic functions, herewith, $\bar{\varphi}_1, \tilde{\varphi}_1$ are selected in the way ensuring that the following equations are valid at $\eta = \alpha$, when $\alpha = \eta_1$ or $\alpha = \eta_2$

$$\begin{aligned}
&\cosh\xi\cos\eta\frac{\partial\varphi_1}{\partial\xi} - (\kappa-1)\sinh\xi\cos\eta\cdot\varphi_1 + \frac{\partial\bar{\varphi}_1}{\partial\xi} + \frac{\partial\tilde{\varphi}_1}{\partial\xi} = 0,\\
&\cosh\xi\cos\eta\frac{\partial\varphi_1}{\partial\eta} + (\kappa-1)\cosh\xi\sin\eta\cdot\varphi_1 + \frac{\partial\bar{\varphi}_1}{\partial\eta} + \frac{\partial\tilde{\varphi}_1}{\partial\eta} = 0
\end{aligned}$$

In variant two

$$\begin{aligned}
\chi_1 = &-\cos\alpha\left(\cosh\xi\cos(\eta-\alpha)\frac{\partial\varphi_1}{\partial\xi} + \sinh\xi\sin(\eta-\alpha)\frac{\partial\varphi_1}{\partial\eta}\right)\\
&+\ \sinh\xi\cos\eta\frac{\partial\varphi_2}{\partial\xi} + \cosh\xi\sin\eta\frac{\partial\varphi_2}{\partial\eta},\\
\chi_2 = &-\cos\alpha\left(\cosh\xi\cos(\eta-\alpha)\frac{\partial\varphi_1}{\partial\eta} - \sinh\xi\sin(\eta-\alpha)\frac{\partial\varphi_1}{\partial\xi}\right)\\
&+\ \sinh\xi\cos\eta\frac{\partial\varphi_2}{\partial\eta} - \cosh\xi\sin\eta\frac{\partial\varphi_2}{\partial\xi},
\end{aligned} \tag{A.9}$$

where φ_2 is a harmonic function.

In variant three

$$\begin{aligned}
\chi_1 = &-\cos^2\alpha\left(\cosh\xi\cos(\eta-\alpha)\frac{\partial\varphi_1}{\partial\xi} + \sinh\xi\sin(\eta-\alpha)\frac{\partial\varphi_1}{\partial\eta}\right)\\
&+\sinh\xi\cos\eta\frac{\partial\varphi_2}{\partial\xi} + \cosh\xi\sin\alpha\frac{\partial\varphi_2}{\partial\eta},\\
\chi_2 = &\cos^2\alpha\left(\sinh\xi\sin(\eta-\alpha)\frac{\partial\varphi_1}{\partial\xi} - \cosh\xi\cos(\eta-\alpha)\frac{\partial\varphi_1}{\partial\eta}\right)\\
&+\sinh\xi\cos\eta\frac{\partial\varphi_2}{\partial\eta} - \cosh\xi\sin\alpha\frac{\partial\varphi_2}{\partial\xi}
\end{aligned} \tag{A.10}$$

Let us insert (A.8) in (A.7c, d). We will obtain

$$\begin{aligned} &\text{a)}\ \bar{u} = \cosh\xi\cos\alpha\frac{\partial\varphi_1}{\partial\xi} - (\kappa-1)\sinh\xi\cos\eta\cdot\varphi_1 \\ &\quad+\frac{\partial\bar{\varphi}_1}{\partial\xi} + \frac{\partial\tilde{\varphi}_1}{\partial\xi} + \frac{\partial\varphi_2}{\partial\xi}, \\ &\text{b)}\ \bar{v} = \cosh\xi\cos\alpha\frac{\partial\varphi_1}{\partial\eta} + (\kappa-1)\cosh\xi\sin\eta\cdot\varphi_1 \\ &\quad+\frac{\partial\bar{\varphi}_1}{\partial\eta} + \frac{\partial\tilde{\varphi}_1}{\partial\eta} + \frac{\partial\varphi_2}{\partial\eta.} \end{aligned} \tag{A.11}$$

Inserting (A.9) in (A.7c, d), we will get

$$\begin{aligned} &\text{a)}\ \bar{u} = (\cos\xi - \cos(\xi-\alpha))\cosh\eta\frac{\partial\varphi_1}{\partial\xi} - \cos\alpha\sin(\xi-\alpha)\sinh\eta\frac{\partial\varphi_1}{\partial\eta} \\ &\quad-(\kappa-1)\sinh\xi\cos\eta\cdot\varphi_1 + \sinh\xi\cos\eta\frac{\partial\varphi_2}{\partial\xi} + \cosh\xi\sin\eta\frac{\partial\varphi_2}{\partial\eta}, \\ &\text{b)}\ \bar{v} = (\cos\xi - \cos(\xi-\alpha))\cosh\eta\frac{\partial\varphi_1}{\partial\eta} + \cos\alpha\sin(\xi-\alpha)\sinh\eta\frac{\partial\varphi_1}{\partial\xi} \\ &\quad+(\kappa-1)\cosh\xi\sin\eta\cdot\varphi_1 + \sinh\xi\cos\eta\frac{\partial\varphi_2}{\partial\eta} - \cosh\xi\sin\eta\frac{\partial\varphi_2}{\partial\xi}. \end{aligned} \tag{A.12}$$

Inserting (A.10) in (A.7c, d), we will have

$$\begin{aligned} &\text{a)}\ \bar{u} = \sin^2\alpha\cosh\xi\cos\eta\frac{\partial\varphi_1}{\partial\xi} - \cos^2\alpha\sinh\xi\sin\eta\frac{\partial\varphi_1}{\partial\eta} \\ &\quad-(\kappa-1)\sinh\xi\cos\eta\varphi_1 + \sinh\xi\cos\eta\frac{\partial\varphi_2}{\partial\xi} + \cosh\xi\sin\eta\frac{\partial\varphi_2}{\partial\eta}, \\ &\text{b)}\ \bar{v} = \cos^2\alpha\sinh\xi\sin\eta\frac{\partial\varphi_1}{\partial\xi} - \sin^2\alpha\cosh\xi\cos\eta\frac{\partial\varphi_1}{\partial\eta} \\ &\quad+(\kappa-1)\cosh\xi\sin\eta\cdot\varphi_1 + \sinh\xi\cos\eta\frac{\partial\varphi_2}{\partial\eta} - \cosh\xi\sin\eta\frac{\partial\varphi_2}{\partial\xi}. \end{aligned} \tag{A.13}$$

Appendix B: Finding of ξ_1

After the BVP with relevant boundary conditions on $\xi = \xi_1 = \xi_{11}$ is solved, the following condition is examined:

$$\frac{F_{11}}{F_{10}} < \varepsilon.$$

ε is a sufficiently small positive number given in advance ($\varepsilon = 0,001 - 0,0001$).

$$F_{11} = \left[\int_0^{\eta_1} (|\sigma_{\xi\xi}| + |\sigma_{\eta\eta}| + |\tau_{\xi\eta}|)\, hd\eta\right]_{\xi=\xi_1},$$

$$F_{10} = \left[\int_0^{\eta_1} (|\sigma_{\xi\xi}| + |\sigma_{\eta\eta}| + |\tau_{\xi\eta}|)\, hd\eta\right]_{\xi=g\tilde{\xi}_1}.$$

g number will be selected so that on the boundary $\eta = \eta_1$, point $M(g\tilde{\xi}_1, \eta_1)$ should correspond to the highest value of expression$[\sigma_{\eta\eta}(g\tilde{\xi}_1, \eta_1)t]^2 + [\tau_{\xi\eta}(g\tilde{\xi}_1, \eta_1)]^2$ (when stresses are given) or to the highest value of expression $[\bar{u}(g\tilde{\xi}_1,\eta_1)t]^2+[\bar{v}(g\tilde{\xi}_1,\eta_1)]^2$ (when displacements are given).

If condition $\frac{F_{11}}{F_{10}}<\varepsilon$ is not valid for $\xi_1=\xi_{11}$, the same problem will be solved at the beginning, but $\xi_1=\xi_{12}$ will be used instead of $\xi_1=\xi_{11}$. In addition, $\xi_{12}>\xi_{11}$. Then, if condition $\frac{F_{12}}{F_{10}}<\varepsilon$ is not still valid, we will continue with the boundary problem, where $\xi_1=\xi_{13}$; besides, $\xi_{13}>\xi_{12}>\xi_{11}$, and we will examine condition $\frac{F_{13}}{F_{10}}<\varepsilon$. The process will be over at the k^{th} stage, if condition $\frac{F_{1k}}{F_{10}}<\varepsilon$ is valid.

Finding such $\xi_1=\xi_{1k}$, for which $\frac{F_{1k}}{F_{10}}<\varepsilon$.

Distance l between surfaces $\xi=\xi_1$ and $\xi=\tilde{\xi}_1$, which gives the guarantee for condition $\frac{F_{1k}}{F_{10}}<\varepsilon$ to be valid in the elliptic coordinate system, will be taken along a small or large axis of the ellipse. In the former case, the following expression will be obtained:

$$\cosh\xi_1=\frac{l}{c}+\cosh\tilde{\xi}_1,$$

and the following expression will be obtained in the latter case:

$$\sinh\xi_1=\frac{l}{c}+\sinh\tilde{\xi}_1.$$

By relying on the known solutions of the relevant plain problems of elasticity, it is purposeful to admit that:

$$\frac{l}{c}=4,5,6,\dots,$$

what allows finding ξ_1 from the relevant equation. Let us note that when $\frac{l}{c}=4$, we will denote value ξ_1 by ξ_{11}, when $\frac{l}{c}=5$, by ξ_{12}, when $\frac{l}{c}=6$, byξ_{13}, etc.

If after selecting $\xi_1=\xi_{1k}$, inequality $\frac{F_{1k}}{F_{10}}<\varepsilon$ is valid, in order to check the righteousness of the selection, it is necessary to once again make sure that together with condition $\frac{F_{1k}}{F_{10}}<\varepsilon$, condition $\varepsilon>\frac{F_{1k}}{F_{10}}>\frac{F_{1k+1}}{F_{10}}>\frac{F_{1k+2}}{F_{10}}>\dots$ is valid, too.

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