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On Number of Optimal Solutions in some Scheduling Problems

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Some special cases of single-machine problems is considered. In these cases there are many optimal solutions. It is given the formulas of quantity of optimal solutions and calculated the probability of the event that an arbitrary schedule is optimal; the sufficient conditions to increase the value of this probability is given and the corresponding optimal full completion time is calculated.

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1. Introduction

A combinatorial optimization problem is characterized by a set of feasible solutions that contains one or more optimal solution. A combinatorial optimization problem, in fact, may possess a large number of optimal solutions, and there may exist special cases of that problem with even larger number of optimal solutions. Defining the whole set of optimal solutions may not be easy even for a polynomially solvable problem, and of course, this is always the case for an NP-hard one. Even finding the number of optimal solutions might be useful. Indeed, this may allow us to find the probability of an event that a randomly selected solution is optimal. In this paper we study a strongly NP-hard single-machine scheduling problem in which jobs have release and delivery times and the objective is to minimize the maximum job completion time. We identify the quantity of optimal solutions for a number of special cases of the scheduling problem and calculate the probability of the event that a randomly created schedule is optimal. We derive sufficient conditions under which this probability is very close to 1. For each of these cases we give an explicit formula for calculating the optimal objective value.

We first describe our scheduling problem that is commonly abbreviated as $1|r_i, q_i|C_{max}$ (the three-field notation introduced by Graham et al. [2]). We have a single machine and n jobs $J_1, J_2, \ldots, J_n$. Each job J_i becomes available at its release time r_i, it needs continuous processing time p_i on the machine, and an additional delivery time or tail q_i after the completion of processing of the job J_i on

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the machine for its full completion (the delivery of job J is machine-independent and requires no further resource (the job is delivered by an independent agent). The release times and delivery times are non-negative real numbers, while a processing time is a positive number. A feasible schedule S assigns every job J_i to the machine in time interval $[t_i(S), t_i(S)+p_i]$ on a non-negative time axes so that $t_i(S) \geq r_i$ and this interval has no intersection with the interval of any other job (we consider half-open time intervals), where $t_i(S)$ is the starting time of job J_i and $t_i(S)+p_i = c_i$ is the completion time of that job in schedule S. The full completion time of job J_i in schedule S, $C_i(S) = c_i(S)+q_i$. The objective is to find an optimal schedule, a feasible one S minimizing the maximum job full completion time $C_{max}(S) = \max_{i \leq n} C_i(S)$. So the optimal objective value is $C_{opt} = \min_{S \in \pi(J(n))} \max_{i \leq n} C_i(S) = \min_{S \in \pi(J(n))} C_{max}(S)$, where $\pi(J(n))$ is the set of all feasible schedules.

Jackson [3] has proposed an efficient heuristic method for the version of the above scheduling problem without job release time, and later it was extended by Schrage [5] for the case when the job release times exist. The extended Jackson's heuristic (J-heuristic for short) iteratively, determines the current scheduling time as the maximum between the minimum release time of yet unscheduled job and the completion time of the latest so far assigned job. Iteratively, among the jobs released by time t, it schedules one with the largest delivery time. As the number of scheduling times is $O(n)$ and at each time search for a minimal/maximal element in an ordered list is accomplished, the time complexity of the heuristic is $O(n \log n)$. Jackson's heuristic is also referred to as LTD-heuristic (Largest Delivery Time) heuristic.

In the special cases of problem $1|r_i, q_i|C_{max}$ when all job release times or delivery times are equal, J-heuristic gives an optimal solution, and with integer release times and unit processing times J-heuristic gives also an optimal solution. Vakhania [6] has proposed an $O(n^2 \log n)$ algorithm for the minimization version with two possible job processing times. Chinos and Vakhania [6] have shown that even in the case, when we have only two release times and only two delivery times, the problem $1|r_1, r_2, q_1, q_2|C_{max}$ is NP-hard. Recently, Reynoso and N. Vakhania [4] have proposed some heuristics of optimal solutions to some special cases of problem $1|r_1, r_2, q_1, q_2|C_{max}$.

In this paper we develop this line of investigation for problem $1|r_i, q_i|C_{max}$ based on the observation that, in a number of special cases of that problem, the total amount of optimal solutions is large. Hence, it is meaningful to calculate the probability of the event that the stochastically chosen schedule is optimal. We find sufficient conditions which provide a very high probability that a randomly chosen schedule is optimal. In the following Section 2 we study the special case of the scheduling problem with equal job release times, and in Section 3 we consider another special case with two allowable job release and delivery times.

2. Equal job release times

Throughout this section, we deal with the special case $1|q_i|C_{max}$ of our scheduling problem in which the release time of all jobs is the same, i.e., $r_1 = r_2 = \cdots = r_n = r$. Without loss of generality, we assume that the numbering of the jobs is such that $q_1 \geq q_2 \geq \cdots \geq q_n$.

Theorem 2.1 : *Suppose for problem $1|q_i|C_{max}$ there is an optimal schedule σ such that*

$$C_{opt}(\sigma) = r + p_1 + p_2 + \cdots + p_k + q_k,$$

and

$$q_k + p_k \geq q_1 \tag{1}$$

and

$$q_k \geq p_{k+1} + \cdots + p_n + q_{k+1}. \tag{2}$$

Then for all permutations of numbers $1, 2, \ldots, k-1, i_1, i_2, \ldots, i_{k-1}$ and permutations of numbers $k+1, k+2, \ldots, n, i_{k+1}, \ldots, i_n$, the schedule

$$J_{i_1}, J_{i_2}, \ldots, J_{i_{k-1}}, J_k, J_{i_{k+1}}, \ldots, J_{i_n}$$

is optimal; the number of the optimal schedules equals to $(k-1)!(n-k)!$. The probability of the event that stochastically chosen schedule is optimal is $\frac{(k-1)!(n-k)!}{n!}$.

Proof : As

$$C_{opt}(\sigma) = r + p_1 + p_2 + \cdots + p_k + q_k,$$

for any schedule $C_{max}(L) \geq C_{opt}(\sigma)$. Consider any schedule S with the sequence $J_{i_1}, J_{i_2}, \ldots, J_{i_{k-1}}, J_k, J_{i_{k+1}}, \ldots, J_{i_n}$, where $J_{i_1}, J_{i_2}, \ldots, J_{i_{k-1}}$ is any permutation of jobs $J_1, J_2, \ldots, J_{k-1}$ and $J_{i_{k+1}}, \ldots, J_{i_n}$ is any permutation of jobs $J_{k+1}, \ldots, J_n$. We show that this schedule is optimal by computing the value of $C_m(S)$ for all $m \leq n$.

If $m = n$, it is easy to see that $C_m(S) = C_{opt}(\sigma)$.

If $m < n$, then by condition (1)

$$C_m(S) \leq r + p_1 + \cdots + p_{k-1} + q_1 \leq r + p_1 + \cdots + p_{k-1} + p_k + q_k = C_{opt}(\sigma).$$

If $m > n$, then by condition (2)

$$C_m(S) \leq r + p_1 + \cdots + p_k + p_{k+1} \cdots + p_n + q_{k+1} \leq r + p_1 + \cdots + p_k + q_k = C_{opt}(\sigma).$$

Therefore, for this schedule $C_m(S) = C_{opt}(\sigma)$.

The number of all permutations $J_{i_1}, J_{i_2}, \ldots, J_{i_{k-1}}$ is $(k-1)!$ and the number of all permutations $J_{i_{k+1}}, \ldots, J_{i_n}$ is $(n-k)!$. Therefore, as the number of all feasible schedule is $n!$, the probability of the event that the stochastically chosen schedule is optimal equal to $\frac{(k-1)!(n-k)!}{n!}$. □

Remark 1 : From the optimal schedule σ with $C_{opt}(\sigma) = C_k(\sigma)$, if it is constructed a feasible schedule S in which job $J_m, m < k$ is inserted in position $l > k$, then this schedule is not optimal. Because, for this schedule we have

$$r + p_1 + \cdots + p_{m-1} + p_{m+1} + \cdots + p_{k-1} + p_k + \cdots + p_{l-1} + p_m + q_m > C_k(\sigma)$$

since $q_k < q_m$ (and $p_{k+1} > 0$. Also, if in any schedule L the job $J_m, m > k$, is inserted in position $l < k$, then

$$r + p_1 + \cdots + p_{l-1} + p_m + p_l + \cdots + p_k + q_k > C_k(\sigma).$$

Therefore, the full completion time of last mentioned schedule becomes greater then C_{opt}. Hence, if the condition $C_{opt}(\sigma) = C_k(\sigma)$ is fulfilled, only schedules with the following types $J_{l_1}, \dots, J_{l_{k-1}}, J_k, J_{l_{k+1}}, \dots, J_{l_n}$ can be optimal, where $J_{l_1}, \dots, J_{l_{k-1}}$ is any permutation of jobs $J_1, \dots, J_{k-1}$ from schedule σ and $J_{l_{k+1}}, \dots, J_{l_n}$ is any permutation of jobs $J_{k+1}, \dots, J_n$ from schedule σ. Since the number of such permutations equals to $(k-1)!(n-k)!$.

In the conditions of Theorem 2.1, the probability of the event that an optimal schedule has the form $J_{l_1}, \dots, J_{l_{k-1}}, J_k, J_{l_{k+1}}, \dots, J_{l_n}$ is 1.

Remark 2 : It is possible to generalize the conditions (1) and (2) of the Theorem 2.1 by the conditions

$$q_s + p_s \geq q_l \tag{1'}$$

for any $1 \leq l < s \leq k-1$ and

$$q_r \geq p_{r+1} + \cdots + p_t + q_{r+1} \tag{2'}$$

for any $k \leq r < t \leq n$.

Then we obtain the following strengthened result from Theorem 2.1:

Theorem 2.2 : *Suppose that for problem $1|q_j|C_{max}$ there is an optimal schedule such that $C_{opt}(\sigma) = r + p_1 + p_2 + \cdots + p_k + q_k$ for any $0 < k \leq n$.*

If the conditions $(1')$ and $(2')$ are fulfilled then for all permutations of numbers $l, l+1, \dots, s-1, s$; $i_l, i_{l+1}, \dots, i_{s-1}, i_s$ and for all permutations of numbers $r, r+1, \dots, t-1, t$; $i_r, i_{r+1}, \dots, i_{t-1}, i_t$, a schedule of the form

$$J_1, J_2, \dots, J_{l-1}, J_{i_l}, J_{i_{l+1}}, \dots, J_{i_{s-1}}, J_{i_s}, J_{i_{s+1}}, \dots$$

$$\dots, J_k, J_{k+1}, \dots, J_{r-1}, J_{i_r}, J_{i_{r+1}}, \dots, J_{i_t}, J_{t+1}, \dots, J_n$$

is optimal; probability of the event that the stochastically chosen schedule is optimal is greater or equal to $\frac{(s-l+1)!(t-r+1)!}{(k-1)!(n-k)!}$. For the conditions 1 and 2 of Theorem 2.1 ($l = 1$, $s = k-1$ and $r = k+1$, $t = n$) this probability equals to 1.

Proof: Denote the above mentioned schedule by S. It is easy to see, that $C_k(S) = C_k(\sigma) = C_{opt}$. We show that $C_m(S) \leq C_k(S)$ for all $1 \leq m \leq n$. It is obvious if $m \leq l-1$ and $m > t$. Let $l \leq m \leq s$, then

$$C_m(S) = r + p_1 + \cdots + p_{l-1} + p_{i_l} + \cdots + p_{i_m} + q_{i_m} \leq$$

$$\leq r + p_1 + \cdots + p_{l-1} + p_{i_l} + \cdots + p_{i_m} + \cdots + p_{s-1} + q_l \leq$$

$$\leq r + p_1 + \cdots + p_{s-1} + p_s + q_s = C_s(\sigma) \leq C_k(\sigma) = C_k(S).$$

Let now $r \leq m \leq t$. Then

$$C_m(S) = r + p_1 + \cdots + p_k + p_{k+1} + \cdots + p_{r-1} + p_{i_r} + \cdots + p_{i_m} + q_{i_m} \leq$$

$$\leq r + p_1 + \cdots + p_{r-1} + p_{i_r} + \cdots + p_{i_m} + \cdots + p_t + q_{r+1} \leq$$

$$\leq r + p_1 + \cdots + p_r + q_r = C_r(S) \leq C_k(\sigma) = C_{opt}.$$

The number of all permutations $J_{i_l}, J_{i_{l+1}}, \ldots, J_{i_s}$ is $(s-l+1)!$ and the number of all permutations $J_{i_r}, J_{i_{r+1}}, \ldots, J_{i_t}$ is $(t-r+1)!$. According to Remark 1, the number of feasible schedules is $(k-1)!(n-k)!$. Therefore, probability of the event that stochastically chosen schedule is optimal is greater or equal to

$$Pr(S_{opt}) \geq \frac{(s-l+1)!(t-r+1)!}{(k-1)!(n-k)!}.$$

The conditions $(1')$ and $(2')$ will be satisfy another part of jobs, by this reason we use relation "greater or equal". □

Proposition 2.3: *Suppose that for the problem* $1|q_j|C_{max}$, $q_i = q_{i+1} + p_{i+1}$, $i = 1, 2, \ldots, n-1$, *then schedule* σ *is an unique optimal schedule and*

$$C_{opt}(\sigma) = p_1 + q_1 = C_1(\sigma) = C_2(\sigma) = \cdots = C_n(\sigma).$$

Proof: It is easy to see that

$$C_{opt}(\sigma) = p_1 + q_1 = C_1(\sigma) = p_1 + p_2 + q_2 =$$

$$= C_2(\sigma) = \cdots = p_1 + p_2 + \cdots + p_{n-1} + q_{n-1} = C_n(\sigma).$$

Consider any schedule S, different from schedule σ. If the job in position l in schedule σ appears in position $m > l$ in schedule S then schedule S cannot be optimal, since

$$C_m(S) = p_1 + \cdots + p_{l-1} + p_{l+1} + \cdots + p_{m-1} + p_l + q_l =$$

$$= p_1 + \cdots + p_{m-1} + q_l > C_m(\sigma) = p_1 + \cdots + p_m + q_m,$$

as $q_l = p_{l+1} + p_{l+2} + q_{l+2} = \cdots = p_{l+1} + \cdots + p_m + q_m$. Hence schedule σ is unique optimal schedule. □

3. Two allowable job release and delivery times

In this section we deal with an NP-hard version $1|r_i \in \{r_1, r_2\},\ q_i \in \{q_1, q_2\}|C_{max}$ (see [1]). Without loss of generality, let $r_1 < r_2$ and $q_1 < q_2$. We aim at finding the number of optimal solutions; calculate the number of feasible solutions and based on which we calculate the probability of the event that a stochastically chosen schedule is optimal as a derivation of number of optimal solutions to the number of feasible schedules.

We need the following notations: denote by $J(r_i, q_j)$ the set of jobs with release time r_i and delivery time q_j and by $J(r_i)$ ($J(q_j)$ respectively) the set of jobs with release time r_i (delivery time q_j); for any set of jobs A, denote by $P(A)$ the sum of the processing times of jobs from the set A. Denote by $J(r_1, q_2, r_2 - r_1)$ the minimal subset of set $J(r_1, q_2)$ ordered in non increasing order with $P(J(r_1, q_2, r_2 - r_1)) \geq r_2 - r_1$, i.e., by removing any element from this set, the above inequality will not hold any more. It is clear, that such a subset is not unique. Denote by $A(r_1, q_2, r_2 - r_1)$ the collection of sets such that:

$$A(r_1, q_2, r_2 - r_1) = \{J(r_1, q_2, r_2 - r_1), J(r_1, q_2, r_2 - r_1) \subset J(r_1, q_2)\}.$$

We similarly denote by $J(r_1, r_2 - r_1)$ the minimal subset of the set $J(r_1)$ ordered in non increasing order with $P(J(r_1, r_2 - r_1)) \geq r_2 - r_1$ and denote by $A(r_1, r_2 - r_1)$ the collection of such sets:

$$A(r_1, r_2 - r_1) = \{J(r_1, r_2 - r_1), J(r_1, r_2 - r_1) \subset J(r_1)\}.$$

The corresponding complement sets are:

$$J^c(r_1, q_2, r_2 - r_1) = J(r_1, q_2) \setminus J(r_1, q_2, r_2 - r_1)$$

and

$$J^c(r_1, r_2 - r_1) = J(r_1) \setminus J(r_1, r_2 - r_1).$$

We will use $N(A)!$ for the factorial of the number of elements of the set A. As it is shown in [4], schedule σ is optimal for problem $1|r_i \in \{r_1, r_2\},\ q_i \in \{q_1, q_2\}|C_{max}$ if the condition

$$r_1 + P(J(r_1, q_2)) \geq r_2 \tag{3}$$

is satisfied. By adding additional conditions, we can obtain the following results.

Proposition 3.1: *If in the problem $1|r_i \in \{r_1, r_2\},\ q_i \in \{q_1, q_2\}|C_{max}$ the condition (3) is satisfied then the schedule*

$$S = J(r_1, q_2, r_2 - r_1)J(r_2, q_2)J^c(r_1, q_2, r_2 - r_1)J(r_2, q_1)J(r_1, q_1)$$

Is optimal, if

$$q_2 > P(J(q_1)) + q_1, \tag{4}$$

then

$$C_{opt} = r_1 + P(J(r_1, q_2)) + P(J(r_2, q_2)) + q_2;$$

if

$$q_2 < P(J(q_1)) + q_1, \tag{5}$$

then

$$C_{opt} = r_1 + P(J(r_1, q_2)) + P(J(r_2, q_2)) + P(J(q_1)) + q_1.$$

The total number of optimal solutions is greater or equal to

$$N_{opt} = \sum_{A(r_1,q_2,r_2-r_1)} N(J(r_1, q_2, r_2 - r_1))!N(J^c(r_1, q_2, r_2 - r_1) \cup J(r_2, q_2))!N(J(q_1))!.$$

The probability of the event that the stochastically chosen schedule is optimal is greater or equal to $Pr(S_{opt}) \geq \frac{N_{opt}}{N_{feas.}}$*, where*

$$N_{feas.} = \sum_{A(r_1,r_2-r_1)} N(J(r_1, r_2 - r_1))!N(J^c(r_1, r_2 - r_1) \cup J(r_2))!.$$

Proof: It is easy to see, that S is the optimal schedule: we can construct an optimal solution combining in order: first, any permutation formed from the elements of subset $J(r_1, q_2, r_2 - r_1)$ of set $J(r_1, q_2)$; further, any permutation of the rest of the elements from the set $J(q_2)$, and at last, any permutation formed from the jobs from the set $J(q_1)$. The number of such schedules is given by the formula of N_{opt}. The total amount of all feasible schedules is similarly calculated. As $J(r_1, q_2, r_2-r_1)$ and $J(r_1, r_2 - r_1)$ are not unique, we have sum by such sets of jobs in formulas N_{opt} and $N_{feas.}$. The value of C_{opt} is easily calculated according of inequalities (4) and (5). □

Proposition 3.2: *If in the problem* $1|r_i \in \{r_1, r_2\}, q_i \in \{q_1, q_2\}|C_{max}$ *the condition*

$$r_1 + P(J(r_1)) \leq r_2 \tag{6}$$

is satisfied, then $C_{opt} = r_2 + P(J(r_2)) + q_2$*.*

The number of optimal schedules is $N_{opt} = N(J(r_1))!N(J(r_2))!$ *and the number of feasible schedules is same –* $N_{feas.} = N(J(r_1))!N(J(r_2))!$*.*

Thus, probability of the event that stochastically chosen schedule is optimal equals to 1.

Proof: The inequality (6) obligates to schedule firstly jobs from $J(r_1)$ and after possible gap, to schedule the jobs from $J(r_2)$. From this easily follows the formula of the number of optimal solutions, which is equal to the number of feasible schedules. The value of C_{opt} is easily calculated too. □

Now we consider more interesting and non-trivial cases when the following condition is satisfied:

$$r_1 + P(J(r_1, q_2)) < r_2 < P(J(r_1)). \tag{7}$$

As in [4], denote by $L = (J_1, J_2, \ldots, J_k)$ the sequence of all jobs from the set $J(r_1, q_1)$ sorted in non-increasing order of their processing times. Denote by $J_1(r_1, q_1)$ any subset of set $J(r_1, q_1)$ such that

$$P(J_1(r_1, q_1)) = \max J \subset J(r_1, q_1)P(J),\ \ P(J) \leq r_2 - r_1 - P(J(r_1, q_2))$$

and by $J_2(r_1, q_1)$ any subset of $J(r_1, q_1)$ such that

$$P(J_2(r_1, q_1)) = \min J \subset J(r_1, q_1)P(J),\ \ P(J) \geq r_2 - r_1 - P(J(r_1, q_2)).$$

As we have a finite number of Jobs, the above defined sets exist but may be there are more than one subset of each type (in fact, it is NP-hard to find such a subset).

Given one of the above defined subsets, we will construct feasible schedules composed of the permutations from the specified subsets in which the jobs in each subset are included in non-increasing order of their processing times. For example, the schedule

$$S = (J(r_1, q_2), J_1(r_1, q_1), J(r_2, q_2), J(r_2, q_1), J_1^c(r_1, q_1))$$

is composed of the jobs of the specified five subsets, the jobs of each subset being included in non-increasing order of their processing times, whereas the partial order of the jobs of different subsets is determined by the order in which the subsets appear in the expression.

Consider now two schedules

$$S' = (J(r_1, q_2), J_1(r_1, q_1), J(r_2, q_2), J(r_2, q_1), J_1^c(r_1, q_1))$$

and

$$S'' = (J(r_1, q_2), J_2(r_1, q_1), J(r_2, q_2), J(r_2, q_1), J_2^c(r_1, q_1)).$$

Now we describe the conditions under which one of them is optimal.

Proposition 3.3: *Let the condition (7) be satisfied. If*

$$q_2 \geq P(J_1^c(r_1, q_1)) + P(J(r_2, q_1)) + q_1 \tag{8}$$

then the schedule

$$S' = (J(r_1, q_2), J_1(r_1, q_1), J(r_2, q_2), J(r_2, q_1), J_1^c(r_1, q_1))$$

is optimal, $C_{opt} = r_2 + P(J(r_2, q_2)) + q_2$. The total amount of all optimal schedules is

$$N_{opt} = N(J(r_1, q_2) \cup J_1(r_1, q_1))!N(J(r_2, q_2))!N(J(r_2, q_1) \cup J_1^c(r_1, q_1))!.$$

The number of feasible schedules is

$$N_{feas.} = N(J(r_1) \setminus J_1^c(r_1, q_1))!N(J(r_2) \cup J_1^c(r_1, q_1))!.$$

Accordingly, probability of the event that stochastically chosen schedule is optimal, equals to $Pr(S_{opt}) = \frac{N_{opt}}{N_{feas.}}$.

Proof: It is clear that $C_{max}(S') \geq r_2 + P(J(r_2, q_2)) + q_2$. By the condition (8) we have

$$r_2 + P(J(r_2, q_2)) + q_2 \geq r_2 + P(J(r_2, q_2)) + P(J_1^c(r_1, q_1)) + P(J(r_2 + q_1)) + q_1.$$

Therefore, $C_{max}(S') = r_2 + P(J(r_2, q_2)) + q_2$. As

$$C_{max}(S') = r_2 + P(J(r_2, q_2)) + q_2 \leq$$

$$r_1 + P(J(r_1, q_2)) + P(J_2(r_1, q_1)) + P(J(r_2, q_2)) + q_2 \leq$$

$$\leq \max\{r_1 + P(J(r_1, q_2)) + P(J_2(r_1, q_1)) + P(J(r_2, q_2)) + q_2,$$

$$r_1 + P(J(r_1, q_2)) + P(J_2(r_1, q_1)) + P(J(r_2, q_2)) +$$

$$+P(J(r_2, q_1)) + P(J_2^c(r_1, q_1)) + q_1\} = C_{max}(S'');$$

$$C_{opt} = C_{max}(S') = r_2 + P(J(r_2, q_2)) + q_2.$$

Further, the sum of the processing times for any permutation of jobs from the set $J(r_1, q_2) \cup J_1(r_1, q_1)$ is equal to the sum of the processing times of jobs $J(r_1, q_2)J_1(r_1, q_1)$ (here jobs are ordered according to there processing times in non-increasing order). As well the sum of the processing times for any permutation of jobs from the set $J(r_2, q_1) \cup J_2^c(r_2, q_1)$ is equal to the sum of the processing times of jobs from the set $J(r_2, q_1)J_2^c(r_2, q_1)$. The number of permutations of jobs in these sets of jobs is $N(J(r_1, q_2) \cup J_1(r_1, q_1))!N(J(r_2, q_1) \cup J_1^c(r_2, q_1))!$ and we obtain the formula of N_{opt}. In regard to the value $N_{feas.}$, the set $J(r_1) \setminus J_2^c(r_1, q_1)$ is same as the set $J(r_1, q_2) \cup J_2(r_1, q_1)$ and it is feasible to schedule any of permutations of jobs $J(r_2) \cup J_2^c(r_1, q_1)$ after the time moment r_2 (To receive high probability of number of optimal schedules, we shortened the huge number of schedules selected them naturally). □

Further, denote $\varepsilon_1 \equiv r_1 + P(J(r_1, q_2)) + P(J_2(r_1, q_1)) - r_2 \geq 0$. It is clear, that $\varepsilon = 0$ if and only if, when $P(J_1(r_1, q_2)) = P(J_2(r_1, q_1))$. Denote also $\varepsilon_2 \equiv P(J_1^c(r_1, q_1)) + P(J(r_2, q_1)) + q_1 - q_2$. Then we have

Proposition 3.4: *Let the condition (7) be satisfied. If*

$$P(J_1^c(r_1,q_1)) + P(J(r_2,q_1)) + q_1 \leq q_2 \leq \varepsilon_2 + q_2, \tag{9}$$

Then:

if $\varepsilon_1 - \varepsilon_2 < 0$*, the schedule*

$$S'' = (J(r_1,q_2), J_2(r_1,q_1), J(r_2,q_2), J(r_2,q_1), J_2^c(r_1,q_1))$$

is optimal,

$$C_{opt} = r_1 + P(J(r_1,q_2)) + P(J_2(r_1,q_1)) + P(J(r_2,q_2)) + q_2,$$

$$N_{opt} = N(J(r_1,q_2) \cup J_2(r_1,q_1))!N(J(r_2,q_2))!N(J(r_2,q_1) \cup J_2^c(r_1,q_1))!, \tag{10}$$

$$N_{feas.} = N(J(r_1) \setminus J_2^c(r_1,q_1))!N(J(r_2) \cup J_2^c(r_1,q_1))!; \tag{11}$$

if $\varepsilon_1 - \varepsilon_2 > 0$*, the schedule*

$$S' = (J(r_1,q_2), J_1(r_1,q_1), J(r_2,q_2), J(r_2,q_1), J_1^c(r_1,q_1))$$

is optimal,

$$C_{opt} = r_2 + P(J(r_2)) + P(J_1^c(r_1,q_1)) + q_1,$$

$$N_{opt} = N(J(r_1,q_2) \cup J_1(r_1,q_1))!N(J(r_2,q_2))!N(J(r_2,q_1) \cup J_1^c(r_1,q_1))!, \tag{12}$$

$$N_{feas.} = N(J(r_1) \setminus J_1^c(r_1,q_1))!N(J(r_2) \cup J_1^c(r_1,q_1))!. \tag{13}$$

Proof: From the left inequality of (9) it follows that

$$C_{max}(S'') = r_1 + P(J(r_1,q_2)) + P(J_2(r_1,q_1)) + P(J(r_2,q_2)) + q_2;$$

$$C_{max}(S') = \max\{r_2 + P(J(r_2,q_2)) + q_2,$$

$$r_2 + P(J(r_2,q_2)) + P(J(r_2,q_1)) + P(J_1^c(r_1,q_1)) + q_2\}.$$

From the right inequality of (9) it follows that

$$C_{max}(S') = r_2 + P(J(r_2,q_2)) + P(J(r_2,q_1)) + P(J_1^c(r_1,q_1)) + q_1.$$

It is easy to see that $C_{max}(S'') - C_{max}(S') = \varepsilon_1 - \varepsilon_2$. Therefore, if $\varepsilon_1 - \varepsilon_2 < 0$ then $C_{max}(S'') = C_{opt}$. The schedule S'' is optimal. The schedule, which is received firstly from any permutation of the set $J(r_1) \setminus J_2^c(r_1, q_1))$ and after, from any permutation of the set $J(r_2, q_2))$ and at last from any permutation of the set $J(r_2, q_1) \cup J_2^c(r_1, q_1)$ is again an optimal schedule, which gives the number of N_{opt}. We can reject some feasible schedules and consider only schedules firstly from the set $J(r_1) \setminus J_2^c(r_1, q_1))$ and after from the set $J(r_2) \cup J_2^c(r_1, q_1)$. The quantity of such feasible schedules is given by formula (11). Analogously, If $\varepsilon_1 - \varepsilon_2 > 0$ then $C_{max}(S') = C_{opt}$. The schedule S' is optimal. The schedule, which is received from any permutation of the set $J(r_1) \setminus J_1^c(r_1, q_1))$ and from any permutation of the set $J(r_2, q_2))$ and from any permutation of the set $J(r_2, q_1) \cup J_1^c(r_1, q_1)$ is again an optimal schedule, which gives the quantity of N_{opt}. We can obtain formulas (12) and (13) similarly as in the case S''. □

Proposition 3.5: *Let the condition (7) be satisfied. If*

$$q_2 \leq P(J_2^c(r_1, q_1)) + P(J(r_2, q_1)) + q_1 \tag{14}$$

then the schedule

$$S'' = (J(r_1, q_2), J_2(r_1, q_1), J(r_2, q_2), J(r_2, q_1), J_2^c(r_1, q_1)) \tag{15}$$

(in all of partial sets jobs are ordered according to their processing times in non-increasing order) is optimal,

$$C_{opt} = r_1 + P(J(r_1)) + P(J(r_2)) + q_1, \tag{16}$$

$$N_{opt} = N(J(r_1) \setminus J_2^c(r_1, q_1))!N(J(r_2, q_2))!N(J(r_2, q_1) \cup J_2^c(r_1, q_1))!. \tag{17}$$

The number of feasible schedules is

$$N_{feas.} = N(J(r_1) \setminus J_2^c(r_1, q_1))!N(J(r_2) \cup J_2^c(r_1, q_1))!, \tag{18}$$

probability of the event that stochastically chosen schedule is optimal, equals to $Pr(S_{opt}) = \frac{N_{opt}}{N_{feas.}}$.

Proof: It is easy to see that

$$C_{max}(S'') = r_1 + P(J(r_1, q_2)) + P(J_2(r_1, q_1)) +$$

$$+ P(J(r_2, q_2)) + P(J_2(r_2, q_1)) + P(J_2^c(r_1, q_1)) + q_1$$

and

$$C_{max}(S') = r_1 + P(J(r_1, q_2)) + P(J_1(r_1, q_1)) +$$

$$+ P(J(r_2, q_2)) + P(J(r_2, q_1)) + P(J_1^c(r_1, q_1)) + q_1 + \varepsilon,$$

Where $\varepsilon \geq 0$ is the gap after scheduling of jobs from $J_1(r_1, q_1)$ before job release time moment r_2. Therefore, $C_{max}(S') \geq C_{max}(S'')$ and $C_{max}(S'') = C_{opt}$, which is represented by formula (16). We receive every optimal schedule if we schedule any permutation of jobs from the set $J(r_1, q_2) \cup J_2(r_1, q_1)$, after schedule any permutation of jobs from the set $J(r_2, q_2)$ and finally we schedule any permutation of jobs from the set $J(r_2, q_1) \cup J_2^c(r_1, q_1)$. According of this, the formula of N_{opt} is given by formula (17). In this case the number of feasible schedules is given by the formula (18). □

When we introduced above the sets of schedules $J_1(r_1, q_1)$ and $J_2(r_1, q_1)$, we mentioned, that to find these sets is NP-hard problem. Therefore, to find schedules in Proposition 3.3-Proposition 3.5 is NP-hard problem. For this reason in [4] are introduced heuristics concerning the problems of these propositions.

At last we consider the heuristic of the schedule σ and calculate the number of such heuristics S which gives the same to the heuristic of the schedule σ full completion time $C_{max}(S) = C_{max}((H(\sigma))$.

As it is mentioned above, in the cases (3) and (6) the schedule σ is optimal and in Propositions 3.1 and 3.2 is given the formulas of the number of optimal solutions. For the case (7), we consider again the set $L = (J_1, J_2, \ldots, J_k)$ the sequence of all jobs from the set $J(r_1, q_1)$ sorted in non-increasing order of their processing times. Denote by J_H the subset of the set L, $J_H = (J_1, J_2, \ldots, J_m)$, $m \leq k$, such that $r_1 + P(J_H) \geq r_2$, but $r_1 + P(J_H \setminus \{J_m\}) < r_2$. Denote also by J_H^c the set $L \setminus J_H$. In case of (7), according of the σ-heuristic, we have the following result:

Proposition 3.6: *If the condition (7) is satisfied, then the schedule σ is*

$$\sigma = (J(r_1, q_2), J_H, J(r_2, q_2), J(r_2, q_1), J_H^c) \tag{19}$$

(in all of partial sets jobs are ordered according to there processing times in non-increasing order). If

$$q_2 \geq P(J_H^c) + P(J(r_2, q_1)) + q_1, \tag{20}$$

then

$$C_{max}(\sigma) = r_1 + P(J(r_1, q_2)) + P(J_H) + P(J(r_2, q_2)) + q_2. \tag{21}$$

The number of heuristic schedules S with $C_{max}(\sigma) = C_{max}(S)$ is

$$N_H = N(J(r_1, q_2) \cup J_H)!N(J(r_2, q_2))!N(J(r_2, q_1) \cup J_H^c)!. \tag{22}$$

The number of feasible schedules is

$$N_{feas.} = N(J(r_1, q_2) \cup J_H)!N(J(r_2) \cup J_H^c)!. \tag{23}$$

If

$$q_2 < P(J_H^c) + P(J(r_2, q_1)) + q_1, \tag{24}$$

then

$$C_{max}(\sigma) = r_1 + P(J(r_1, q_2)) + P(J_H) + P(J(r_2)) + P(J_H^c) + q_1. \quad (25)$$

And all other values are same as the above considered case.

Proof: It is easy to see that formula (19) gives the schedule σ. By the condition (20) we have

$$C_{max}(\sigma) = \max\{r_1 + P(J(r_1, q_2)) + P(J_H) + P(J(r_2, q_2)) + q_2,$$

$$r_1 + P(J(r_1, q_2)) + P(J_H) + P(J(r_2, q_2)) + P(J(r_2, q_1)) + P(J_H^c(r_1, q_1)) + q_1\} =$$

$$= r_1 + P(J(r_1, q_2)) + P(J_H) + P(J(r_2, q_2)) + q_2.$$

The schedule, which contains any permutation of jobs $(J(r_1, q_2) \cup J_H)$, and after, any permutations of jobs $J(r_2, q_2)$ and, at last, any permutations of jobs $(J(r_2, q_1) \cup J_H^c)$ has full completion time equal to $C_{max}(\sigma)$, therefore the number of such heuristic schedules is given by formula (22). Formula (23) gives the number of shortened feasible schedules as it is obvious, that among such schedules is our heuristic schedules. It is easy to see, that the condition (24) gives the value of full completion time by the formula (25). □

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