Negafibonacci Numbers via Matrices

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In this paper, negafibonacci numbers are generated by means of matrix methods. A 2×2 matrix is used to obtain some properties of negafibonacci numbers; on the other hand, families of tridiagonal matrices are introduced to generate negafibonacci numbers through determinants.

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1. Introduction

The Fibonacci sequence \( \{f_n\} \) is defined by the following recurrence relation

\[ f_{n+1} = f_n + f_{n-1}, \text{ for } n \geq 1, \]

with \( f_0 = 0, f_1 = 1 \). The Fibonacci numbers have been widely studied, and the different ways to generate those numbers have gained continued interest, among them matrix methods [10], determinants [5], permanents [6], Pascal’s triangle [9], binomial coefficients [3], and many others [8].

An interesting connection between Fibonacci numbers and matrices, introduced in [4], is given by the matrix \( Q = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \), known as Fibonacci Q-matrix [7] or Fibonacci’s matrix [11], such that

\[ \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n = \begin{bmatrix} f_{n+1} & f_n \\ f_n & f_{n-1} \end{bmatrix}. \]

In [1] two tridiagonal Toeplitz matrices were presented

\[ H_n = \begin{bmatrix} 1 & i & i & \cdots & i \\ i & 1 & i & \cdots & i \\ i & i & 1 & \cdots & i \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ i & \cdots & i & 1 & \end{bmatrix}_{n \times n}, \quad D_n = \begin{bmatrix} 1 & -1 & \cdots & \cdots & -1 \\ 1 & 1 & \cdots & \cdots & 1 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & 1 & 1 \\ 1 & 1 & \cdots & \cdots & 1 \end{bmatrix}_{n \times n}. \]

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such that $\det(H_n) = \det(D_n) = f_{n+1}$.

By the relation $f_{-n} = (-1)^{n+1}f_n$, where $n$ is any positive integer, Fibonacci numbers can be extended to negative index [2], terms in this sequence are called negafibonacci numbers. Since $f_{n+1} = f_n + f_{n-1}$, it is easy to check that $f_{-(n+1)} = -f_{-n} + f_{-(n-1)}$; some negafibonacci numbers are $f_{-1} = 1$, $f_{-2} = -1$, $f_{-3} = 2$, $f_{-4} = -3$, $f_{-5} = 5$. In this paper negafibonacci numbers are generated by means of matrices, and some identities are proved by matrix methods.

2. Negafibonacci identities by matrix methods

Motivated by the Fibonacci $Q$-matrix, the matrix $N = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}$ is presented, the following proposition shows a connection between $N$ and negafibonacci numbers.

**Proposition 2.1:** 

$$ \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}^n = \begin{bmatrix} f_{-(n+1)} & f_{-n} \\ f_{-n} & f_{-(n-1)} \end{bmatrix}. $$

**Proof:** Since $\begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} f_{-2} & f_{-1} \\ f_{-1} & f_0 \end{bmatrix}$, the proposition is true for $n = 1$. Assuming that $\begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}^n = \begin{bmatrix} f_{-(n+1)} & f_{-n} \\ f_{-n} & f_{-(n-1)} \end{bmatrix}$, $\begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}$ is calculated as follows:

$$ \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} f_{-(n+1)} & f_{-n} \\ f_{-n} & f_{-(n-1)} \end{bmatrix} = \begin{bmatrix} -f_{-(n+1)} + f_{-n} & -f_{-n} + f_{-(n-1)} \\ f_{-(n+1)} & f_{-n} \end{bmatrix} = \begin{bmatrix} f_{-(n+2)} & f_{-(n+1)} \\ f_{-(n+1)} & f_{-n} \end{bmatrix}. $$

The above proposition is useful to prove some identities about negafibonacci numbers.

**Proposition 2.2:** For all $n, k \geq 0$:

$$ f_{-(n+k+1)} = f_{-(n+1)}f_{-(k+1)} + f_{-n}f_{-k} \quad f_{-(n+k)} = f_{-(n+1)}f_{-k} + f_{-n}f_{-(k+1)} $$

$$ f_{-(n+k)} = f_{-n}f_{-(k+1)} + f_{-(n-1)}f_{-k} \quad f_{-(n+k+1)} = f_{-n}f_{-k} + f_{-(n-1)}f_{-(k+1)}. $$

**Proof:** Since $\begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}^n = \begin{bmatrix} f_{-(n+1)} & f_{-n} \\ f_{-n} & f_{-(n-1)} \end{bmatrix}$, by Proposition 2.1, then

$$ \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}^{n+k} = \begin{bmatrix} f_{-(n+k+1)} & f_{-(n+k)} \\ f_{-(n+k)} & f_{-(n+k-1)} \end{bmatrix}. $$
On the other hand,
\[
\begin{bmatrix}
-1 & 1 \\
1 & 0
\end{bmatrix}^{n+k} = \begin{bmatrix}
-1 & 1 \\
1 & 0
\end{bmatrix}^n \begin{bmatrix}
-1 & 1 \\
1 & 0
\end{bmatrix}^k
= \begin{bmatrix}
f_{-(n+1)} & f_{-n} \\
f_{-n} & f_{-(n-1)}
\end{bmatrix} \begin{bmatrix}
f_{-(k+1)} & f_{-k} \\
f_{-k} & f_{-(k-1)}
\end{bmatrix}
= \begin{bmatrix}
f_{-(n+1)}f_{-(k+1)} + f_{-n} f_{-k} & f_{-(n+1)} f_{-k} + f_{-n} f_{-(k-1)} \\
f_{-n} f_{-k} + f_{-(n-1)} f_{-k} & f_{-(n-1)} f_{-k} + f_{-(n-1)} f_{-(k-1)}
\end{bmatrix}.
\]

Thus obtaining the desired equalities.

From Proposition 2.2, we may immediately deduce the following identities.

**Corollary 2.3:** For all \( n \geq 0 \):

1. \( f_{-2n} = f_{-n} f_{-(n+1)} + f_{-(n-1)} f_{-n} \).
2. \( f_{-(n+2)} = 2f_{-n} - f_{-(n-1)} \).

The following result can be called Cassini’s formula for negafibonacci numbers, the reader is referred to [12] for more Cassini-like formulas.

**Proposition 2.4:** \( f_{-(n+1)} f_{-(n-1)} - f_{-2n}^2 = (-1)^n \)

**Proof:** Let \( N = \begin{bmatrix}
-1 & 1 \\
1 & 0
\end{bmatrix} \) then \( \det(N) = -1 \). Since \( N^n = \begin{bmatrix}
f_{-(n+1)} & f_{-n} \\
f_{-n} & f_{-(n-1)}
\end{bmatrix} \), then \( \det(N^n) = f_{-(n+1)} f_{-(n-1)} - f_{-2n}^2 \). On the other hand, \( \det(N^n) = (\det(N))^n = (-1)^n \); therefore \( f_{-(n+1)} f_{-(n-1)} - f_{-2n}^2 = (-1)^n \).

3. Negafibonacci numbers as tridiagonal matrix determinants

In this section, we present the matrices \( G_n \) and \( K_n \) defined as follows:

\[
G_n = \begin{bmatrix}
-1 & -1 & \cdots & -1 \\
1 & -1 & \cdots & -1 \\
\vdots & \ddots & \ddots & \vdots \\
(-1)^n & \cdots & (-1)^n & -1
\end{bmatrix}_{n \times n} \quad \quad K_n = \begin{bmatrix}
-1 & -i & \cdots & -i \\
-i & -1 & \cdots & -i \\
\vdots & \ddots & \ddots & \vdots \\
(-1)^n i & \cdots & (-1)^n i & -1
\end{bmatrix}_{n \times n}
\]

The following proposition shows a connection between negafibonacci numbers and the determinants of a family of tridiagonal matrices.

**Proposition 3.1:** For all \( n > 0 \):

1. \( \det(G_n) = f_{-(n+1)} \).
2. \( \det(K_n) = f_{-(n+1)} \).

**Proof:** Here we prove (1); (2) can be similarly proved.

We argue by induction on \( n \). Clearly \( G_1 = -1 = f_{-2} \) and \( \det(G_2) = 2 = f_{-3} \).
Let $G_{n+2}$ be the matrix

$$
G_{n+2} = \begin{bmatrix}
G_n & 0_{n-1 \times 1} & 0_{n-1 \times 1} \\
0_{1 \times n-1} & (-1)^n+1 & 0 \\
0_{1 \times n-1} & 0 & (-1)^{n+2} -1
\end{bmatrix}.
$$

Assuming that the determinant $\det(G_k) = f-(k+1)$ for all $k \leq n$, we aim to show that $\det(G_{n+2}) = f-(n+3)$. Assuming that $n$ is odd

$$
\begin{bmatrix}
G_n & 0_{n-1 \times 1} & 0_{n-1 \times 1} \\
-1 & 0 & 1 \\
0_{1 \times n-1} & -2 & 0
\end{bmatrix}
$$

Applying the column operation $C_{n+1} - C_{n+2}$, we obtain

$$
\begin{bmatrix}
G_n & 0_{n-1 \times 1} & 0_{n-1 \times 1} \\
-1 & 0 & 1 \\
0_{1 \times n-1} & 0 & -1
\end{bmatrix}.
$$

Since the above row and column elementary operations do not change the value of the determinant [11], we have

$$
\det(G_{n+2}) = [-1] \left[ (-1)^{2n+1} [-2] \det(G_n) + (-1)^{2n+1} [1] \det(B_n) \right]
$$

where $B_n = \begin{bmatrix} G_{n-1} & 0_{n-1 \times 1} \\ 0_{1 \times n-2} & -1 \end{bmatrix}$. Applying the column operation $C_{n-1} - C_n$, we obtain the equivalent matrix

$$
\begin{bmatrix}
G_{n-1} & 0_{n-1 \times 1} \\
0_{1 \times n-1} & -1
\end{bmatrix}.
$$

Therefore $\det(B_n) = [-1] \det(G_{n-1}) = -f-n$. By replacing in equation (3.1) we obtain

$$
\det(G_{n+2}) = [-1] \left[ [(-2)[f-(n+1)] + (-1)^{2n+1}[1][-f-n] \right] = 2f-(n+1) - f-n.
$$

Thus, by corollary 2.3, $\det(G_{n+2}) = f-(n+3)$ for $n$ odd. Similarly, if $n$ is even $G_{n+2}$ is given by

$$
G_{n+2} = \begin{bmatrix}
G_n & 0_{n-1 \times 1} & 0_{n-1 \times 1} \\
1 & 0 & -1 \\
0_{1 \times n-1} & -1 & -1
\end{bmatrix}.
$$
Applying, first the row operation $R_{n+1} - R_{n+2}$, and after the column operation $C_{n+1} + C_{n+2}$, we obtain the equivalent matrix

$$
\begin{bmatrix}
G_n & 0_{n-1 \times 1} & 0_{n-1 \times 1} \\
0_{1 \times n-1} & -1 & -2 \\
0_{1 \times n-1} & 0 & 0 & -1
\end{bmatrix}.
$$

Therefore,

$$
det(G_{n+2}) = [-1] \left( (-1)^{2n+2} [-2] det(G_n) + (-1)^{2n+1} [-1] det(C_n) \right), \quad (3.2)
$$

where $C_n$ is given by

$$
C_n = \begin{bmatrix} G_{n-1} & 0_{n-1 \times 1} \\ 0_{1 \times n-2} & 1 & 1 \end{bmatrix}.
$$

Hence, $det(C_n) = det(G_{n-1}) = f_{-n}$. By replacing in equation (3.2) we obtain

$$
det(G_{n+2}) = [-1] \left( [-2] (f_{-(n+1)}) + (f_{-n}) \right) = 2f_{-(n+1)} - f_{-n}.
$$

Thus, by Corollary 2.3, $det(G_{n+2}) = f_{-(n+3)}$ for $n$ even. □

Given a matrix that generates Fibonacci numbers we can obtain a matrix for negafibonacci numbers and vice-versa, as shows the following proposition.

**Proposition 3.2:** $F_n$ is a matrix such that $det(F_n) = f_{n+1}$ if and only if $N_n = -F_n$ is such that $det(N_n) = f_{-(n+1)}$.

**Proof:** Assuming that $det(F_n) = f_{n+1}$, for the matrix $N_n = -F_n$ we have $det(N_n) = det(-F_n) = (-1)^n det(F_n) = (-1)^n f_{n+1} = (-1)^{n+2} f_{n+1} = f_{-(n+1)}$.

Assuming that $det(N_n) = f_{-(n+1)}$, for the matrix $F_n = -N_n$ we have $det(F_n) = det(-N_n) = (-1)^n det(N_n) = (-1)^n f_{-(n+1)} = (-1)^n((-1)^{n+2} f_{n+1}) = f_{n+1}$. □

By Proposition 3.2, we have the following result.

**Corollary 3.3:** For all $n \geq 0$:

1. $det(-G_n) = det(-K_n) = f_{n+1}$.
2. $det(-D_n) = det(-H_n) = f_{-(n+1)}$.

**References**


