On Initial-Boundary Value Problem with Mixed Boundary Conditions for One System of Nonlinear Partial Integro-Differential Equations with Source Terms

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Initial-boundary value problem with mixed boundary conditions for one system of nonlinear partial integro-differential equations with source terms are studied. Fully discrete finite difference scheme is constructed and its stability and convergence is proved. Compared to previous researches, in this note more general case for nonlinear coefficient of terms with high order derivatives is considered.

Keywords: System of nonlinear partial differential equations, Source terms, Finite difference scheme, Convergence.

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1. Introduction

In the cylinder $Q = [0, 1] \times [0, \infty)$ let us consider the following initial-boundary value problem:

$$\frac{\partial U}{\partial t} - \frac{\partial}{\partial x} \left[ a \left( \int_0^t \left( \frac{\partial U}{\partial x} \right)^2 + \left( \frac{\partial V}{\partial x} \right)^2 \right) dt \right] \frac{\partial U}{\partial x} + g(U) = f_1(x,t),$$

$$\frac{\partial V}{\partial t} - \frac{\partial}{\partial x} \left[ a \left( \int_0^t \left( \frac{\partial U}{\partial x} \right)^2 + \left( \frac{\partial V}{\partial x} \right)^2 \right) dt \right] \frac{\partial V}{\partial x} + g(V) = f_2(x,t),$$

$$U(0,t) = V(0,t) = \frac{\partial U(x,t)}{\partial x} \bigg|_{x=1} = \frac{\partial V(x,t)}{\partial x} \bigg|_{x=1} = 0,$$

$$U(x,0) = U_0(x), \quad V(x,0) = V_0(x),$$

where $a = a(S)$, $g$, $f_1$, $f_2$, $U_0$ and $V_0$ are given functions of their arguments.

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System (1) is obtained by adding the source term to the following integro-differential vector equation [5]:

$$\frac{\partial H}{\partial t} = \text{rot} \left[ a \left( \int_0^t |\text{rot}H|^2 \, d\tau \right) \text{rot}H \right],$$  \hspace{1cm} (4)

where $H = (H_1, H_2, H_3)$ is a vector of the magnetic field and function $a = a(S)$ is defined for $S \in [0, \infty)$. If the vector of the magnetic field has the form $H = (0, U, V)$, where $U = U(x, t)$, $V = V(x, t)$, then from (4) we get the system of nonlinear parabolic integro-differential equations (1) without the source terms. Vector equation (4) itself comes from the system of Maxwell equation [13]:

$$\frac{\partial H}{\partial t} = -\text{rot}(\nu_m \text{rot}H),$$

$$c \frac{\partial \theta}{\partial t} = \nu_m (\text{rot}H)^2,$$

(5)

where again $H = (H_1, H_2, H_3)$ is a vector of the magnetic field, $\theta$ is temperature, $c$ and $\nu_m$ characterize the thermal heat capacity and electroconductivity of the substance. Note that reduction of system (5) to the integro-differential form (4) at first, as we already mentioned above, was given in [5]. Later a number of scientists studied proposed above integro-differential models for different cases of magnetic field and different kind of diffusion coefficient (see, for example, [1] - [12], [14], [15], [17] - [19] and references therein).

2. Uniqueness and large time behavior of solution

Here we give the main identity for proving the uniqueness of solution of problem (1) - (3). Let us assume that there exist two different solutions of problem (1) - (3): $(U_1, V_1)$ and $(U_2, V_2)$. To show that $U_2 - U_1 = V_2 - V_1 \equiv 0$ the methodology of proving the convergence theorem, which is given in the next section, monotone growth and positiveness features of function $g$ and the following identity are mainly used:

$$\left\{ a \left( \int_0^t \left[ \left( \frac{\partial U_2}{\partial x} \right)^2 + \left( \frac{\partial V_2}{\partial x} \right)^2 \right] \, d\tau \right) \frac{\partial U_2}{\partial x} \right\}$$

$$- a \left( \int_0^t \left[ \left( \frac{\partial U_1}{\partial x} \right)^2 + \left( \frac{\partial V_1}{\partial x} \right)^2 \right] \, d\tau \right) \frac{\partial U_1}{\partial x} \right\} \left( \frac{\partial U_2}{\partial x} - \frac{\partial U_1}{\partial x} \right)$$

$$+ \left\{ a \left( \int_0^t \left[ \left( \frac{\partial U_2}{\partial x} \right)^2 + \left( \frac{\partial V_2}{\partial x} \right)^2 \right] \, d\tau \right) \frac{\partial V_2}{\partial x} \right\}$$

$$- a \left( \int_0^t \left[ \left( \frac{\partial U_1}{\partial x} \right)^2 + \left( \frac{\partial V_1}{\partial x} \right)^2 \right] \, d\tau \right) \frac{\partial V_1}{\partial x} \right\} \left( \frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial x} \right)$$
\[
\frac{1}{a} \int_0^1 \frac{d}{d\mu} a \left( \int_0^t \left\{ \left[ \frac{\partial U_1}{\partial x} + \mu \left( \frac{\partial U_2}{\partial x} - \frac{\partial U_1}{\partial x} \right) \right]^2 + \left[ \frac{\partial V_1}{\partial x} + \mu \left( \frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial x} \right) \right]^2 \right\} d\tau \right) \times \left[ \frac{\partial U_1}{\partial x} + \mu \left( \frac{\partial U_2}{\partial x} - \frac{\partial U_1}{\partial x} \right) \right] d\mu \left( \frac{\partial U_2}{\partial x} - \frac{\partial U_1}{\partial x} \right) + \\
\frac{1}{a} \int_0^1 \frac{d}{d\mu} a \left( \int_0^t \left\{ \left[ \frac{\partial U_1}{\partial x} + \mu \left( \frac{\partial U_2}{\partial x} - \frac{\partial U_1}{\partial x} \right) \right]^2 + \left[ \frac{\partial V_1}{\partial x} + \mu \left( \frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial x} \right) \right]^2 \right\} d\tau \right) \times \left[ \frac{\partial V_1}{\partial x} + \mu \left( \frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial x} \right) \right] d\mu \left( \frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial x} \right) \right). \]

For obtaining stabilization of solution the method of a-priori estimates based on analogical methodology given in [11] is used and large time behavior of solution is obtained.

Thus, combining the above said the following statement takes place.

**Theorem 2.1:** If \( a = a(S) \geq a_0 = \text{Const} > 0, \ a'(S) \geq 0, \ a''(S) \leq 0, \ g \) is a monotonically increased and positively defined function, \( U_0, V_0 \in H^1(0, 1), \ U_0(0) = V_0(0) = \frac{dU_0(x)}{dx} |_{x=1} = \frac{dV_0(x)}{dx} |_{x=1} = 0, \ f_1, f_2, \frac{\partial f_1}{\partial x}, \frac{\partial f_2}{\partial x} \in L_2(Q) \) and problem (1) - (3) has a solution then it is unique and exponential stabilization of solution as \( t \to \infty \) takes place.

Here we use usual \( L_2 \) and Sobolev \( H^1 \) spaces.

### 3. Finite difference scheme and its convergence

In the rectangle \( Q_T = [0, 1] \times [0, T] \), where \( T \) is a positive constant, let us consider again problem (1) - (3). On \( Q_T \) let us introduce a net with mesh points denoted by \((x_i, t_j) = (ih, j\tau)\), where \( i = 0, 1, ..., M; \ j = 0, 1, ..., N \) with \( h = 1/M, \ \tau = T/N \). Denote discrete approximation at \((x_i, t_j)\) by \((u^j_i, v^j_i)\) and the exact solution to the problem (1) - (3) by \((U^j_i, V^j_i)\). Below, in this section, we will use the following known notations [20] of forward and backward derivatives:

\[
r_{ij}^x = \frac{r_{i+1}^j - r_{i}^j}{h}, \quad r_{ij}^x = \frac{r_{i}^j - r_{i-1}^j}{h}, \quad r_{ij}^t = \frac{r_{i+1}^j - r_{i}^j}{h},
\]

and inner products and norms:

\[
(r^j, y^j) = h \sum_{i=1}^{M-1} r_{i}^j y_{i}^j, \quad (r^j, y^j) = h \sum_{i=1}^{M} r_{i}^j y_{i}^j, \quad \|r^j\| = (r^j, r^j)^{1/2}, \quad \|r^j\| = (r^j, r^j)^{1/2}.
\]

For problem (1) - (3) let us consider the following finite difference scheme:
\[ \frac{u_{j+1}^i - u_j^i}{\tau} - \left\{ a \left( \tau \sum_{k=1}^{j+1} \left[ (u_{j+1}^k)^2 + (v_{j+1}^k)^2 \right] \right)^i_x \right\} + g(u_{j+1}^i) = f_{1,i}^j, \]

\[ \frac{v_{j+1}^i - v_j^i}{\tau} - \left\{ a \left( \tau \sum_{k=1}^{j+1} \left[ (u_{j+1}^k)^2 + (v_{j+1}^k)^2 \right] \right)^i_x \right\} + g(v_{j+1}^i) = f_{2,i}^j, \]  \hspace{1cm} (6)

Multiplying equations in (6) scalarly by \( u_{j+1}^i \) and \( v_{j+1}^i \) respectively, it is not difficult to get the inequalities:

\[ \|u^n\|^2 + \sum_{j=1}^{n} \|u_{j,x}^n\|^2 \tau < C, \quad \|v^n\|^2 + \sum_{j=1}^{n} \|v_{j,x}^n\|^2 \tau < C, \quad n = 1, 2, ..., N, \]  \hspace{1cm} (9)

where here and below \( C \) is a positive constant independent from \( \tau \) and \( h \).

On the basis of one variant of the well known Brouwer fixed point theorem (see, [16], pp. 53-54) using the a priori estimate (9) the solvability of the scheme (6) - (8) is easily derived. The technique used to prove convergence theorem below, can be applied to prove the stability and uniqueness of the solution of the scheme (6) - (8) too.

The main statement of the present section can be stated as follows.

**Theorem 3.1:** If \( a = a(S) \geq a_0 = \text{Const} > 0, a'(S) \geq 0, a''(S) \leq 0, g \) is monotonically increased and positively defined function and problem (1) - (3) has a sufficiently smooth solution \((U(x,t), V(x,t))\), then the solution \( \psi = (u_1, u_2, ..., u_{M-1}) \), \( \psi = (v_1, v_2, ..., v_{M-1}) \), \( j = 1, 2, ..., N \) of the difference scheme (6) - (8) tends to the solution of continuous problem (1) - (3) \((U_1, U_2, ..., U_{M-1}), V_1, V_2, ..., V_{M-1}) \), \( j = 1, 2, ..., N \) as \( \tau \to 0, h \to 0 \) and the following estimates are true:

\[ \max_{j=1,2,...,N} \|u^j - U^j\| \leq C(\tau + h), \quad \max_{j=1,2,...,N} \|v^j - V^j\| \leq C(\tau + h). \]  \hspace{1cm} (10)

**Proof:** Introducing the differences \( z_j^i = U^j_i - U^j_i \) and \( w_j^i = v^j_i - V^j_i \) we get the following relations:
\[ z_{i,1}^{j+1} = \left\{ a \left( \tau \sum_{k=1}^{j+1} \left[ (v_{x,i}^k)^2 + (v_{z,i}^k)^2 \right] \right) u_{x,i}^{j+1} \right\} + g(u_i^{j+1}) - g(U_i^{j+1}) = -\psi_{1,i}^j, \]

\[ w_{i,1}^{j+1} = \left\{ a \left( \tau \sum_{k=1}^{j+1} \left[ (u_{x,i}^k)^2 + (v_{z,i}^k)^2 \right] \right) v_{z,i}^{j+1} \right\} + g(v_i^{j+1}) - g(V_i^{j+1}) = -\psi_{2,i}^j, \]

where \(\psi_{1,i}^j\) and \(\psi_{2,i}^j\) are approximation errors of scheme (6) and

\[ \psi_{k,i}^j = O(\tau + h), \quad k = 1, 2. \]

Multiplying the first equation of system (11) scalarly by the grid function \(\tau z_{i,1}^{j+1} = \tau(z_1^{j+1}, z_2^{j+1}, \ldots, z_{M-1}^{j+1})\) and using the boundary conditions (12) we get

\[ \| z^{j+1} \| - (z^{j+1}, z^j) + \tau h \sum_{i=1}^M \left\{ a \left( \tau \sum_{k=1}^{j+1} \left[ (u_{x,i}^k)^2 + (v_{z,i}^k)^2 \right] \right) u_{x,i}^{j+1} \right\} w_{x,i}^{j+1} \]

\[ -a \left( \tau \sum_{k=1}^{j+1} \left[ (U_{x,i}^k)^2 + (V_{z,i}^k)^2 \right] \right) U_{x,i}^{j+1} \]

\[ + \left( g(u_i^{j+1}) - g(U_i^{j+1}), u^{j+1} - U^{j+1} \right) = -\tau(\psi_{1,i}^j, z^{j+1}). \]

Analogously,

\[ \| w^{j+1} \| - (w^{j+1}, w^j) + \tau h \sum_{i=1}^M \left\{ a \left( \tau \sum_{k=1}^{j+1} \left[ (u_{x,i}^k)^2 + (v_{z,i}^k)^2 \right] \right) u_{x,i}^{j+1} \right\} w_{z,i}^{j+1} \]

\[ -a \left( \tau \sum_{k=1}^{j+1} \left[ (U_{z,i}^k)^2 + (V_{z,i}^k)^2 \right] \right) V_{z,i}^{j+1} \]

\[ + \left( g(v_i^{j+1}) - g(V_i^{j+1}), v^{j+1} - V^{j+1} \right) = -\tau(\psi_{2,i}^j, w^{j+1}). \]

Taking into account monotonicity and positiveness of the function \(g\), from the last two equalities we have
\[ \| z^{j+1} \| ^2 - (z^{j+1}, z^j) + \| w^{j+1} \| ^2 - (w^{j+1}, w^j) \]

\[ + \tau h \sum_{i=1}^{M} \left\{ a \left( \tau \sum_{k=1}^{j+1} \left[ (u_{x,i}^k)^2 + (v_{x,i}^k)^2 \right] \right) u_{x,i}^{j+1} \right\} \]

\[ - a \left( \tau \sum_{k=1}^{j+1} \left[ (U_{x,i}^k)^2 + (V_{x,i}^k)^2 \right] \right) U_{x,i}^{j+1} \]

\[ + \tau h \sum_{i=1}^{M} \left\{ a \left( \tau \sum_{k=1}^{j+1} \left[ (u_{x,i}^k)^2 + (v_{x,i}^k)^2 \right] \right) v_{x,i}^{j+1} \right\} \]

\[ - a \left( \tau \sum_{k=1}^{j+1} \left[ (U_{x,i}^k)^2 + (V_{x,i}^k)^2 \right] \right) V_{x,i}^{j+1} = -\tau (\psi_1^j, z^{j+1}) - \tau (\psi_2^j, w^{j+1}). \]

Note that, using the Hadamard formula

\[ \varphi(y) - \varphi(z) = \int_{0}^{1} \frac{d}{d\mu} \varphi(z + \mu(y - z)) d\mu, \]

below we prove one of the main inequalities to estimate terms with the nonlinear diffusion coefficient \( a(S) \)

\[ \left\{ a \left( \tau \sum_{k=1}^{j+1} \left[ (u_{x,i}^k)^2 + (v_{x,i}^k)^2 \right] \right) u_{x,i}^{j+1} - a \left( \tau \sum_{k=1}^{j+1} \left[ (U_{x,i}^k)^2 + (V_{x,i}^k)^2 \right] \right) U_{x,i}^{j+1} \right\} \]

\[ \times \left( u_{x,i}^{j+1} - U_{x,i}^{j+1} \right) + \left\{ a \left( \tau \sum_{k=1}^{j+1} \left[ (u_{x,i}^k)^2 + (v_{x,i}^k)^2 \right] \right) v_{x,i}^{j+1} \right\} \]

\[ - a \left( \tau \sum_{k=1}^{j+1} \left[ (U_{x,i}^k)^2 + (V_{x,i}^k)^2 \right] \right) V_{x,i}^{j+1} \]

\[ \left\{ v_{x,i}^{j+1} - V_{x,i}^{j+1} \right\} \]

\[ = \int_{0}^{1} \frac{d}{d\mu} a \left( \tau \sum_{k=1}^{j+1} \left\{ \left[ U_{x,i}^{j+1} + \mu (U_{x,i}^k - U_{x,i}^k) \right] \left[ U_{x,i}^{j+1} + \mu (U_{x,i}^k - U_{x,i}^k) \right] \right\} \right) \]

\[ \times \left[ U_{x,i}^{j+1} + \mu (U_{x,i}^{j+1} - U_{x,i}^{j+1}) \right] d\mu \left( u_{x,i}^{j+1} - U_{x,i}^{j+1} \right) \]

\[ + \int_{0}^{1} \frac{d}{d\mu} a \left( \tau \sum_{k=1}^{j+1} \left\{ \left[ U_{x,i}^{j+1} + \mu (U_{x,i}^k - U_{x,i}^k) \right] \left[ U_{x,i}^{j+1} + \mu (U_{x,i}^k - U_{x,i}^k) \right] \right\} \right) \]

\[ \times \left[ V_{x,i}^{j+1} + \mu (V_{x,i}^{j+1} - V_{x,i}^{j+1}) \right] d\mu \left( v_{x,i}^{j+1} - V_{x,i}^{j+1} \right) \]
\[
= 2 \int_0^1 a' \left( \frac{3}{j+1} \left\{ \left[ U^k_{x,i} + \mu(u^k_{x,i} - U^k_{x,i}) \right] + \left[ V^k_{x,i} + \mu(v^k_{x,i} - V^k_{x,i}) \right] \right\} \right) \]
\[
\times \frac{3}{j+1} \sum_{k=1}^{j+1} \left\{ \left[ U^k_{x,i} + \mu(u^k_{x,i} - U^k_{x,i}) \right] + \left[ V^k_{x,i} + \mu(v^k_{x,i} - V^k_{x,i}) \right] \right\} \]
\[
+ \int_0^1 a \left( \frac{3}{j+1} \left\{ \left[ U^k_{x,i} + \mu(u^k_{x,i} - U^k_{x,i}) \right] + \left[ V^k_{x,i} + \mu(v^k_{x,i} - V^k_{x,i}) \right] \right\} \right) \]
\[
\times \left( u^j_{x,i} - U^j_{x,i} \right) d\mu \left( u^j_{x,i} - U^j_{x,i} \right) \]
\[
+ \int_0^1 a' \left( \frac{3}{j+1} \left\{ \left[ U^k_{x,i} + \mu(u^k_{x,i} - U^k_{x,i}) \right] + \left[ V^k_{x,i} + \mu(v^k_{x,i} - V^k_{x,i}) \right] \right\} \right) \]
\[
\times \left( v^j_{x,i} - V^j_{x,i} \right) d\mu \left( v^j_{x,i} - V^j_{x,i} \right) \]
\[
= 2 \int_0^1 a' \left( \frac{3}{j+1} \left\{ \left[ U^k_{x,i} + \mu(u^k_{x,i} - U^k_{x,i}) \right] + \left[ V^k_{x,i} + \mu(v^k_{x,i} - V^k_{x,i}) \right] \right\} \right) \]
\[
\times \left( u^j_{x,i} - U^j_{x,i} \right) \]
\[
+ \int_0^1 a \left( \frac{3}{j+1} \left\{ \left[ U^k_{x,i} + \mu(u^k_{x,i} - U^k_{x,i}) \right] + \left[ V^k_{x,i} + \mu(v^k_{x,i} - V^k_{x,i}) \right] \right\} \right) \]
\[
\times \left( v^j_{x,i} - V^j_{x,i} \right) d\mu \left( v^j_{x,i} - V^j_{x,i} \right) \]
\[
= 2 \int_0^1 a' \left( \frac{3}{j+1} \left\{ \left[ U^k_{x,i} + \mu(u^k_{x,i} - U^k_{x,i}) \right] + \left[ V^k_{x,i} + \mu(v^k_{x,i} - V^k_{x,i}) \right] \right\} \right) \]
\[
\times \xi^j_{x,i} (\mu) \xi^j_{x,i} (\mu) d\mu \]
+ \int_0^1 a \left( \tau \sum_{k=1}^{j+1} \left\{ \left[ U_{x,i}^k + \mu (u_{x,i}^k - U_{x,i}^k) \right]^2 + \left[ V_{x,i}^k + \mu (v_{x,i}^k - V_{x,i}^k) \right]^2 \right\} \right)
\times \left[ \left( u_{x,i}^{j+1} - U_{x,i}^{j+1} \right)^2 + \left( v_{x,i}^{j+1} - V_{x,i}^{j+1} \right)^2 \right] \, d\mu,

where

\xi_{s_i}^{j+1}(\mu) = \tau \sum_{k=1}^{j+1} \left\{ \left[ U_{x,i}^k + \mu (u_{x,i}^k - U_{x,i}^k) \right] \left( u_{x,i}^{j+1} - U_{x,i}^{j+1} \right) \right. 
\left. + \left[ V_{x,i}^k + \mu (v_{x,i}^k - V_{x,i}^k) \right] \left( v_{x,i}^{j+1} - V_{x,i}^{j+1} \right) \right\},

\xi_0(\mu) = 0,

and therefore,

\xi_{t, i}^j(\mu) = \left[ U_{x,i}^{j+1} + \mu (u_{x,i}^{j+1} - U_{x,i}^{j+1}) \right] \left( u_{x,i}^{j+1} - U_{x,i}^{j+1} \right) 
\left[ V_{x,i}^{j+1} + \mu (v_{x,i}^{j+1} - V_{x,i}^{j+1}) \right] \left( v_{x,i}^{j+1} - V_{x,i}^{j+1} \right).

Introducing the following notation

\xi_{s_i}^{j+1}(\mu) = \tau \sum_{k=1}^{j+1} \left\{ \left[ U_{x,i}^k + \mu (u_{x,i}^k - U_{x,i}^k) \right]^2 + \left[ V_{x,i}^k + \mu (v_{x,i}^k - V_{x,i}^k) \right]^2 \right\},

from the previous equality we have

\left\{ a \left( \tau \sum_{k=1}^{j+1} \left[ (u_{x,i}^k)^2 + (v_{x,i}^k)^2 \right] \right) u_{x,i}^{j+1} 
- a \left( \tau \sum_{k=1}^{j+1} \left[ (U_{x,i}^k)^2 + (V_{x,i}^k)^2 \right] \right) U_{x,i}^{j+1} \right\} \left( u_{x,i}^{j+1} - U_{x,i}^{j+1} \right) 
+ \left\{ a \left( \tau \sum_{k=1}^{j+1} \left[ (u_{x,i}^k)^2 + (v_{x,i}^k)^2 \right] \right) v_{x,i}^{j+1} \right\} \left( v_{x,i}^{j+1} - V_{x,i}^{j+1} \right) 
- a \left( \tau \sum_{k=1}^{j+1} \left[ (U_{x,i}^k)^2 + (V_{x,i}^k)^2 \right] \right) V_{x,i}^{j+1} \right\} \left( v_{x,i}^{j+1} - V_{x,i}^{j+1} \right)
= 2 \int_0^1 a' (s_{i}^{j+1}(\mu)) s_{i}^{j+1} s_{t, i}^j \, d\mu
+ \int_0^1 a (s_{i}^{j+1}(\mu)) \left[ (u_{x,i}^{j+1} - U_{x,i}^{j+1})^2 + (v_{x,i}^{j+1} - V_{x,i}^{j+1})^2 \right] \, d\mu,
After substituting this equality in (14) we get
\[
\|z^{j+1}\|^{2} - (z^{j+1}, z^{j}) + \|w^{j+1}\|^{2} - (w^{j+1}, w^{j})
+ 2\tau h \sum_{i=1}^{M} \int_{0}^{1} a'(s_{i}^{j+1}(\mu)) \xi_{i}^{j+1} \xi_{i}^{j} d\mu
\]
\[+ \tau h \sum_{i=1}^{M} \int_{0}^{1} a(s_{i}^{j+1}(\mu)) \left[ (w_{x,i}^{j+1} - U_{x,i}^{j+1})^2 + (v_{x,i}^{j+1} - V_{x,i}^{j+1})^2 \right] d\mu\]
\[= -\tau(\psi_{1}, z^{j+1}) - \tau(\psi_{2}, w^{j+1}).\]

Taking into account that \(a(S) \geq a_{0} = \text{Const} > 0\) and relations \(s^{j+1}(\mu) \geq 0\),
\[
(r^{j+1}, r^{j}) = \frac{1}{2} \|r^{j+1}\|^{2} + \frac{1}{2} \|r^{j}\|^{2} - \frac{1}{2} \|r^{j+1} - r^{j}\|^{2},
\]
\[\tau \xi_{i}^{j+1} \xi_{i}^{j} = \frac{1}{2} (\xi_{i}^{j+1})^2 - \frac{1}{2} (\xi_{i}^{j})^2 + \frac{\tau^2}{2} (\xi_{i}^{j})^2
\]
from (15) we have
\[
\|z^{j+1}\|^{2} - \frac{1}{2} \|z^{j+1}\|^{2} - \frac{1}{2} \|z^{j}\|^{2} + \frac{1}{2} \|z^{j+1} - z^{j}\|^{2} + \|w^{j+1}\|^{2} - \frac{1}{2} \|w^{j+1}\|^{2}
\]
\[- \frac{1}{2} \|w^{j}\|^{2} + \frac{1}{2} \|w^{j+1} - w^{j}\|^{2} + h \sum_{i=1}^{M} \int_{0}^{1} a'(s_{i}^{j+1}(\mu)) \left[ (\xi_{i}^{j+1})^2 - (\xi_{i}^{j})^2 \right] d\mu
\]
\[+ \tau^2 h \sum_{i=1}^{M} \int_{0}^{1} a'(s_{i}^{j+1}(\mu)) (\xi_{i}^{j})^2 d\mu
\]
\[+ \tau h a_{0} \sum_{i=1}^{M} \left[ (w_{x,i}^{j+1} - U_{x,i}^{j+1})^2 + (v_{x,i}^{j+1} - V_{x,i}^{j+1})^2 \right]
\[\leq -\tau(\psi_{1}, z^{j+1}) - \tau(\psi_{2}, w^{j+1}).\]

From (16) we arrive at
\[
\frac{1}{2} \|z^{j+1}\|^{2} - \frac{1}{2} \|z^{j}\|^{2} + \frac{\tau^2}{2} \|z^{j}\|^{2} + \frac{1}{2} \|w^{j+1}\|^{2} - \frac{1}{2} \|w^{j}\|^{2} + \frac{\tau^2}{2} \|w^{j}\|^{2}
\]
\[+ h \sum_{i=1}^{M} \int_{0}^{1} a'(s_{i}^{j+1}(\mu)) \left[ (\xi_{i}^{j+1})^2 - (\xi_{i}^{j})^2 \right] d\mu
\]
\[+ \tau a_{0} \left( \|z^{j+1}\|^{2} + \|w^{j+1}\|^{2} \right)
\[\leq \frac{\tau}{2 a_{0}} \left( \|\psi_{1}\|^{2} + \|\psi_{2}\|^{2} \right) + \frac{\tau a_{0}}{2} \left( \|z^{j+1}\|^{2} + \|w^{j+1}\|^{2} \right).\]
Using the discrete analogue of the Poincaré inequality [20]

$$
\|r_{j+1}\|^2 \leq \|r_{j}\|^2,
$$

from (17) we get

$$
\|z^j\|_2^2 - \|z^j\|_2^2 + \tau^2 \|z^j\|_2^2 + \|w^{j+1}\|_2^2 - \|w^j\|_2^2 + \tau^2 \|w^j\|_2^2
+ 2h \sum_{i=1}^{M} \int_0^1 a' \left(s^j_{i+1}(\mu)\right) \left(\xi^j_{i+1}\right)^2 - \left(\xi^j_{i}\right)^2 \, d\mu
+ \tau a_0 \left(\|z^j\|_2^2 + \|w^j\|_2^2\right) \leq \frac{\tau}{a_0} \left(\|z_0\|_2^2 + \|w_0\|_2^2\right).
$$

Summing (18) from $j = 0$ to $j = n - 1$ we arrive at

$$
\|z^n\|_2^2 + \tau^2 \sum_{j=0}^{n-1} \|z^j\|_2^2 + \|w^n\|_2^2 + \tau^2 \sum_{j=0}^{n-1} \|w^j\|_2^2
+ 2h \sum_{j=0}^{n-1} \sum_{i=1}^{M} \int_0^1 a' \left(s^j_{i+1}(\mu)\right) \left(\xi^j_{i+1}\right)^2 - \left(\xi^j_{i}\right)^2 \, d\mu
+ \tau a_0 \sum_{j=0}^{n-1} \left(\|z^j\|_2^2 + \|w^j\|_2^2\right) \leq \frac{\tau}{a_0} \sum_{j=0}^{n-1} \left(\|z^j\|_2^2 + \|w^j\|_2^2\right).
$$

Note, that since $s^j_{i+1}(\mu) \geq s^j_{i}(\mu)$, $a'(S) \geq 0$ and $a''(S) \leq 0$, for the second line of the last formula we have

$$
\sum_{j=0}^{n-1} a' \left(s^j_{i+1}(\mu)\right) \left(\xi^j_{i+1}\right)^2 - \left(\xi^j_{i}\right)^2
= a' \left(s^j_{i}(\mu)\right) (\xi^j_{i})^2 - a' \left(s^j_{i}(\mu)\right) (\xi^j_{i})^2
+ a' \left(s^j_{i}(\mu)\right) (\xi^j_{i})^2 - a' \left(s^j_{i}(\mu)\right) (\xi^j_{i})^2
+ \cdots + a' \left(s^j_{i}(\mu)\right) (\xi^j_{i})^2 - a' \left(s^j_{i}(\mu)\right) (\xi^j_{i})^2
= a' \left(s^j_{i}(\mu)\right) (\xi^j_{i})^2 + \sum_{j=1}^{n-1} \left[a' \left(s^j_{i}(\mu)\right) - a' \left(s^j_{i+1}(\mu)\right)\right] (\xi^j_{i})^2 \geq 0.
$$
Taking into account the last relation and (19) one can deduce
\[
\|z^n\|^2 + \|w^n\|^2 + \tau^2 \sum_{j=0}^{n-1} \|z_j^2\|^2 + \tau^2 \sum_{j=0}^{n-1} \|w_j^2\|^2 +
\]
\[+ \tau a_0 \sum_{j=0}^{n-1} \left( \|z_{j+1}^2\|^2 + \|w_{j+1}^2\|^2 \right) \leq \frac{\tau}{a_0} \sum_{j=0}^{n-1} \left( \|\psi_j^2\|^2 + \|\psi_j^2\|^2 \right). \quad (20)
\]

From (20) we get (10), and thus Theorem 3.1 is proved.

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