Bulletin of TICMI
Vol. 22, No. 2, 2018, 113–124

On Initial-Boundary Value Problem with Mixed Boundary Conditions for One System of Nonlinear Partial Integro-Differential Equations with Source Terms

Maia Kratsashvili*

Sokhumi State University
12 Politkovskaia St., 0186 Tbilisi, Georgia

(*Received May 23, 2018; Revised August 8, 2018; Accepted July 22, 2018*)

Initial-boundary value problem with mixed boundary conditions for one system of nonlinear partial integro-differential equations with source terms are studied. Fully discrete finite difference scheme is constructed and its stability and convergence is proved. Compared to previous researches, in this note more general case for nonlinear coefficient of terms with high order derivatives is considered.

Keywords: System of nonlinear partial differential equations, Source terms, Finite difference scheme, Convergence.

AMS Subject Classification: 35K55, 45K05, 65N06.

1. Introduction

In the cylinder $Q = [0,1] \times [0,\infty)$ let us consider the following initial-boundary value problem:

$$\begin{aligned}
&\frac{\partial U}{\partial t} - \frac{\partial}{\partial x}\left[a\left(\int_0^t \left[\left(\frac{\partial U}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial x}\right)^2\right] d\tau\right)\frac{\partial U}{\partial x}\right] + g(U) = f_1(x,t),\\
&\frac{\partial V}{\partial t} - \frac{\partial}{\partial x}\left[a\left(\int_0^t \left[\left(\frac{\partial U}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial x}\right)^2\right] d\tau\right)\frac{\partial V}{\partial x}\right] + g(V) = f_2(x,t),
\end{aligned} \tag{1}$$

$$U(0,t) = V(0,t) = \left.\frac{\partial U(x,t)}{\partial x}\right|_{x=1} = \left.\frac{\partial V(x,t)}{\partial x}\right|_{x=1} = 0, \tag{2}$$

$$U(x,0) = U_0(x), \quad V(x,0) = V_0(x), \tag{3}$$

where $a = a(S)$, g, f_1, f_2, U_0 and V_0 are given functions of their arguments.

*Email: maiakratsashvili@gmail.com

ISSN: 1512-0082 print

System (1) is obtained by adding the source term to the following integro-differential vector equation [5]:

$$\frac{\partial H}{\partial t} = -rot\left[a\left(\int_0^t |rotH|^2 \, d\tau\right) rotH\right], \tag{4}$$

where $H = (H_1, H_2, H_3)$ is a vector of the magnetic field and function $a = a(S)$ is defined for $S \in [0, \infty)$. If the vector of the magnetic field has the form $H = (0, U, V)$, where $U = U(x,t)$, $V = V(x,t)$, then from (4) we get the system of nonlinear parabolic integro-differential equations (1) without the source terms. Vector equation (4) itself comes from the system of Maxwell equation [13]:

$$\begin{aligned} \frac{\partial H}{\partial t} &= -rot(\nu_m rotH), \\ c_\nu \frac{\partial \theta}{\partial t} &= \nu_m \left(rotH\right)^2, \end{aligned} \tag{5}$$

where again $H = (H_1, H_2, H_3)$ is a vector of the magnetic field, θ is temperature, c_ν and ν_m characterize the thermal heat capacity and electroconductivity of the substance. Note that reduction of system (5) to the integro-differential form (4) at first, as we already mentioned above, was given in [5]. Later a number of scientists studied proposed above integro-differential models for different cases of magnetic field and different kind of diffusion coefficient (see, for example, [1] - [12], [14], [15], [17] - [19] and references therein).

2. Uniqueness and large time behavior of solution

Here we give the main identity for proving the uniqueness of solution of problem (1) - (3). Let us assume that there exist two different solutions of problem (1) - (3): (U_1, V_1) and (U_2, V_2). To show that $U_2 - U_1 = V_2 - V_1 \equiv 0$ the methodology of proving the convergence theorem, which is given in the next section, monotone growth and positiveness features of function g and the following identity are mainly used:

$$\begin{aligned} &\left\{a\left(\int_0^t \left[\left(\frac{\partial U_2}{\partial x}\right)^2 + \left(\frac{\partial V_2}{\partial x}\right)^2\right] d\tau\right) \frac{\partial U_2}{\partial x}\right. \\ &\left. -a\left(\int_0^t \left[\left(\frac{\partial U_1}{\partial x}\right)^2 + \left(\frac{\partial V_1}{\partial x}\right)^2\right] d\tau\right) \frac{\partial U_1}{\partial x}\right\} \left(\frac{\partial U_2}{\partial x} - \frac{\partial U_1}{\partial x}\right) \\ &+\left\{a\left(\int_0^t \left[\left(\frac{\partial U_2}{\partial x}\right)^2 + \left(\frac{\partial V_2}{\partial x}\right)^2\right] d\tau\right) \frac{\partial V_2}{\partial x}\right. \\ &\left. -a\left(\int_0^t \left[\left(\frac{\partial U_1}{\partial x}\right)^2 + \left(\frac{\partial V_1}{\partial x}\right)^2\right] d\tau\right) \frac{\partial V_1}{\partial x}\right\} \left(\frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial x}\right) \end{aligned}$$

$$= \int\limits_0^1 \frac{d}{d\mu} a \left(\int\limits_0^t \left\{ \left[\frac{\partial U_1}{\partial x} + \mu \left(\frac{\partial U_2}{\partial x} - \frac{\partial U_1}{\partial x} \right) \right]^2 + \left[\frac{\partial V_1}{\partial x} + \mu \left(\frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial x} \right) \right]^2 \right\} d\tau \right)$$
$$\times \left[\frac{\partial U_1}{\partial x} + \mu \left(\frac{\partial U_2}{\partial x} - \frac{\partial U_1}{\partial x} \right) \right] d\mu \left(\frac{\partial U_2}{\partial x} - \frac{\partial U_1}{\partial x} \right)$$
$$+ \int\limits_0^1 \frac{d}{d\mu} a \left(\int\limits_0^t \left\{ \left[\frac{\partial U_1}{\partial x} + \mu \left(\frac{\partial U_2}{\partial x} - \frac{\partial U_1}{\partial x} \right) \right]^2 + \left[\frac{\partial V_1}{\partial x} + \mu \left(\frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial x} \right) \right]^2 \right\} d\tau \right)$$
$$\times \left[\frac{\partial V_1}{\partial x} + \mu \left(\frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial x} \right) \right] d\mu \left(\frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial x} \right).$$

For obtaining stabilization of solution the method of a-priori estimates based on analogical methodology given in [11] is used and large time behavior of solution is obtained.

Thus, combining the above said the following statement takes place.

Theorem 2.1: *If $a = a(S) \geq a_0 = Const > 0$, $a'(S) \geq 0$, $a''(S) \leq 0$, g is a monotonically increased and positively defined function, $U_0, V_0 \in H^1(0,1)$, $U_0(0) = V_0(0) = \left.\frac{dU_0(x)}{dx}\right|_{x=1} = \left.\frac{dV_0(x)}{dx}\right|_{x=1} = 0$, f_1, f_2, $\frac{\partial f_1}{\partial x}$, $\frac{\partial f_2}{\partial x} \in L_2(Q)$ and problem (1) - (3) has a solution then it is unique and exponential stabilization of solution as $t \to \infty$ takes place.*

Here we use usual L_2 and Sobolev H^1 spaces.

3. Finite difference scheme and its convergence

In the rectangle $Q_T = [0,1] \times [0,T]$, where T is a positive constant, let us consider again problem (1) - (3). On Q_T let us introduce a net with mesh points denoted by $(x_i, t_j) = (ih, j\tau)$, where $i = 0, 1, ..., M$; $j = 0, 1, ..., N$ with $h = 1/M$, $\tau = T/N$. Denote discrete approximation at (x_i, t_j) by $(u_i^j,\, v_i^j)$ and the exact solution to the problem (1) - (3) by $(U_i^j,\, V_i^j)$. Below, in this section, we will use the following known notations [20] of forward and backward derivatives:

$$r_{x,i}^j = \frac{r_{i+1}^j - r_i^j}{h}, \quad r_{\bar{x},i}^j = \frac{r_i^j - r_{i-1}^j}{h}, \quad r_{t,i}^j = \frac{r_i^{j+1} - r_i^j}{\tau}$$

and inner products and norms:

$$(r^j, y^j) = h \sum_{i=1}^{M-1} r_i^j y_i^j, \quad (r^j, y^j] = h \sum_{i=1}^{M} r_i^j y_i^j,$$
$$\|r^j\| = (r^j, r^j)^{1/2}, \quad \|r^j]| = (r^j, r^j]^{1/2}.$$

For problem (1) - (3) let us consider the following finite difference scheme:

$$
\begin{aligned}
&\frac{u_i^{j+1}-u_i^j}{\tau}-\left\{a\left(\tau\sum_{k=1}^{j+1}\left[(u_{\bar{x},i}^k)^2+(v_{\bar{x},i}^k)^2\right]\right)u_{\bar{x},i}^{j+1}\right\}_x+g(u_i^{j+1})=f_{1,i}^j,\\
&\frac{v_i^{j+1}-v_i^j}{\tau}-\left\{a\left(\tau\sum_{k=1}^{j+1}\left[(u_{\bar{x},i}^k)^2+(v_{\bar{x},i}^k)^2\right]\right)v_{\bar{x},i}^{j+1}\right\}_x+g(v_i^{j+1})=f_{2,i}^j,\\
&\qquad i=1,2,...,M-1;\ \ j=0,1,...,N-1,
\end{aligned}
\tag{6}
$$

$$
u_0^j=v_0^j=u_{\bar{x},M}^j=v_{\bar{x},M}^j=0,\quad j=0,1,...,N, \tag{7}
$$

$$
u_i^0=U_{0,i},\ v_i^0=V_{0,i},\ \ i=0,1,...,M. \tag{8}
$$

Multiplying equations in (6) scalarly by u_i^{j+1} and v_i^{j+1} respectively, it is not difficult to get the inequalities:

$$
\|u^n\|^2+\sum_{j=1}^{n}\|u_{\bar{x}}^j]|^2\tau<C,\quad \|v^n\|^2+\sum_{j=1}^{n}\|v_{\bar{x}}^j]|^2\tau<C,\quad n=1,2,...,N, \tag{9}
$$

where here and below C is a positive constant independent from τ and h.

On the basis of one variant of the well known Brouwer fixed point theorem (see, [16], pp. 53-54) using the a priori estimate (9) the solvability of the scheme (6) - (8) is easily derived. The technique used to prove convergence theorem below, can be applied to prove the stability and uniqueness of the solution of the scheme (6) - (8) too.

The main statement of the present section can be stated as follows.

Theorem 3.1: *If $a=a(S)\ge a_0=Const>0$, $a'(S)\ge 0$, $a''(S)\le 0$, g is monotonically increased and positively defined function and problem (1) - (3) has a sufficiently smooth solution $(U(x,t),V(x,t))$, then the solution $u^j=(u_1^j,u_2^j,\dots,u_{M-1}^j)$, $v^j=(v_1^j,v_2^j,\dots,v_{M-1}^j)$, $j=1,2,\dots,N$ of the difference scheme (6) - (8) tends to the solution of continuous problem (1) - (3) $U^j=(U_1^j,U_2^j,\dots,U_{M-1}^j)$, $V^j=(V_1^j,V_2^j,\dots,V_{M-1}^j)$, $j=1,2,\dots,N$ as $\tau\to 0$, $h\to 0$ and the following estimates are true:*

$$
\max_{j=1,2,\dots,N}\|u^j-U^j\|\le C(\tau+h),\qquad \max_{j=1,2,\dots,N}\|v^j-V^j\|\le C(\tau+h). \tag{10}
$$

Proof: Introducing the differences $z_i^j=u_i^j-U_i^j$ and $w_i^j=v_i^j-V_i^j$ we get the following relations:

$$
\begin{gathered}
z_{t,i}^{j+1} - \left\{ a\left(\tau \sum_{k=1}^{j+1} \left[(u_{\bar{x},i}^{k})^2 + (v_{\bar{x},i}^{k})^2 \right] \right) u_{\bar{x},i}^{j+1} \right. \\
\left. -a\left(\tau \sum_{k=1}^{j+1} \left[(U_{\bar{x},i}^{k})^2 + (V_{\bar{x},i}^{k})^2 \right] \right) U_{\bar{x},i}^{j+1} \right\}_x + g(u_i^{j+1}) - g(U_i^{j+1}) = -\psi_{1,i}^j, \\
w_{t,i} - \left\{ a\left(\tau \sum_{k=1}^{j+1} \left[(u_{\bar{x},i}^{k})^2 + (v_{\bar{x},i}^{k})^2 \right] \right) v_{\bar{x},i}^{j+1} \right. \\
\left. -a\left(\tau \sum_{k=1}^{j+1} \left[(U_{\bar{x},i}^{k})^2 + (V_{\bar{x},i}^{k})^2 \right] \right) V_{\bar{x},i}^{j+1} \right\}_x + g(v_i^{j+1}) - g(V_i^{j+1}) = -\psi_{2,i}^j,
\end{gathered} \tag{11}
$$

$$
z_0^j = w_0^j = z_{\bar{x},M}^j = w_{\bar{x},M}^j = 0, \tag{12}
$$

$$
z_i^0 = w_i^0 = 0, \tag{13}
$$

where $\psi_{1,i}^j$ and $\psi_{2,i}^j$ are approximation errors of scheme (6) and

$$
\psi_{k,i}^j = O(\tau + h), \quad k = 1,\, 2.
$$

Multiplying the first equation of system (11) scalarly by the grid function $\tau z^{j+1} = \tau(z_1^{j+1}, z_2^{j+1}, \ldots, z_{M-1}^{j+1})$ and using the boundary conditions (12) we get

$$
\begin{gathered}
\|z^{j+1}\|^2 - (z^{j+1}, z^j) + \tau h \sum_{i=1}^{M} \left\{ a\left(\tau \sum_{k=1}^{j+1} \left[(u_{\bar{x},i}^{k})^2 + (v_{\bar{x},i}^{k})^2 \right] \right) u_{\bar{x},i}^{j+1} \right. \\
\left. -a\left(\tau \sum_{k=1}^{j+1} \left[(U_{\bar{x},i}^{k})^2 + (V_{\bar{x},i}^{k})^2 \right] \right) U_{\bar{x},i}^{j+1} \right\} z_{\bar{x},i}^{j+1} \\
+ \left(g(u_i^{j+1}) - g(U_i^{j+1}),\, u^{j+1} - U^{j+1} \right) = -\tau(\psi_1^j, z^{j+1}).
\end{gathered}
$$

Analogously,

$$
\begin{gathered}
\|w^{j+1}\|^2 - (w^{j+1}, w^j) + \tau h \sum_{i=1}^{M} \left\{ a\left(\tau \sum_{k=1}^{j+1} \left[(u_{\bar{x},i}^{k})^2 + (v_{\bar{x},i}^{k})^2 \right] \right) v_{\bar{x},i}^{j+1} \right. \\
\left. -a\left(\tau \sum_{k=1}^{j+1} \left[(U_{\bar{x},i}^{k})^2 + (V_{\bar{x},i}^{k})^2 \right] \right) V_{\bar{x},i}^{j+1} \right\} w_{\bar{x},i}^{j+1} \\
+ \left(g(v_i^{j+1}) - g(V_i^{j+1}),\, v^{j+1} - V^{j+1} \right) = -\tau(\psi_2^j, w^{j+1}).
\end{gathered}
$$

Taking into account monotonicity and positiveness of the function g, from the last two equalities we have

$$
\begin{aligned}
&\|z^{j+1}\|^2-(z^{j+1},z^j)+\|w^{j+1}\|^2-(w^{j+1},w^j)\\
&+\tau h\sum_{i=1}^{M}\left\{a\left(\tau\sum_{k=1}^{j+1}\left[(u_{\bar{x},i}^k)^2+(v_{\bar{x},i}^k)^2\right]\right)u_{\bar{x},i}^{j+1}\right.\\
&\left.-a\left(\tau\sum_{k=1}^{j+1}\left[(U_{\bar{x},i}^k)^2+(V_{\bar{x},i}^k)^2\right]\right)U_{\bar{x},i}^{j+1}\right\}z_{\bar{x},i}^{j+1}\\
&+\tau h\sum_{i=1}^{M}\left\{a\left(\tau\sum_{k=1}^{j+1}\left[(u_{\bar{x},i}^k)^2+(v_{\bar{x},i}^k)^2\right]\right)v_{\bar{x},i}^{j+1}\right.\\
&\left.-a\left(\tau\sum_{k=1}^{j+1}\left[(U_{\bar{x},i}^k)^2+(V_{\bar{x},i}^k)^2\right]\right)V_{\bar{x},i}^{j+1}\right\}w_{\bar{x},i}^{j+1}=-\tau(\psi_1^j,z^{j+1})-\tau(\psi_2^j,w^{j+1}).
\end{aligned}
\tag{14}
$$

Note that, using the Hadamard formula

$$
\varphi(y)-\varphi(z)=\int_0^1\frac{d}{d\mu}\varphi[z+\mu(y-z)]d\mu,
$$

below we prove one of the main inequalities to estimate terms with the nonlinear diffusion coefficient $a(S)$

$$
\begin{aligned}
&\left\{a\left(\tau\sum_{k=1}^{j+1}\left[(u_{\bar{x},i}^k)^2+(v_{\bar{x},i}^k)^2\right]\right)u_{\bar{x},i}^{j+1}-a\left(\tau\sum_{k=1}^{j+1}\left[(U_{\bar{x},i}^k)^2+(V_{\bar{x},i}^k)^2\right]\right)U_{\bar{x},i}^{j+1}\right\}\\
&\times\left(u_{\bar{x},i}^{j+1}-U_{\bar{x},i}^{j+1}\right)+\left\{a\left(\tau\sum_{k=1}^{j+1}\left[(u_{\bar{x},i}^k)^2+(v_{\bar{x},i}^k)^2\right]\right)v_{\bar{x},i}^{j+1}\right.\\
&\left.-a\left(\tau\sum_{k=1}^{j+1}\left[(U_{\bar{x},i}^k)^2+(V_{\bar{x},i}^k)^2\right]\right)V_{\bar{x},i}^{j+1}\right\}\left(v_{\bar{x},i}^{j+1}-V_{\bar{x},i}^{j+1}\right)\\
&=\int_0^1\frac{d}{d\mu}a\left(\tau\sum_{k=1}^{j+1}\left\{\left[U_{\bar{x},i}^k+\mu(u_{\bar{x},i}^k-U_{\bar{x},i}^k)\right]^2+\left[V_{\bar{x},i}^k+\mu(v_{\bar{x},i}^k-V_{\bar{x},i}^k)\right]^2\right\}\right)\\
&\times\left[U_{\bar{x},i}^{j+1}+\mu(u_{\bar{x},i}^{j+1}-U_{\bar{x},i}^{j+1})\right]d\mu\left(u_{\bar{x},i}^{j+1}-U_{\bar{x},i}^{j+1}\right)\\
&+\int_0^1\frac{d}{d\mu}a\left(\tau\sum_{k=1}^{j+1}\left\{\left[U_{\bar{x},i}^k+\mu(u_{\bar{x},i}^k-U_{\bar{x},i}^k)\right]^2+\left[V_{\bar{x},i}^k+\mu(v_{\bar{x},i}^k-V_{\bar{x},i}^k)\right]^2\right\}\right)\\
&\times\left[V_{\bar{x},i}^{j+1}+\mu(v_{\bar{x},i}^{j+1}-V_{\bar{x},i}^{j+1})\right]d\mu\left(v_{\bar{x},i}^{j+1}-V_{\bar{x},i}^{j+1}\right)
\end{aligned}
$$

$$= 2\int_0^1 a'\left(\tau\sum_{k=1}^{j+1}\left\{\left[U^k_{\bar{x},i}+\mu(u^k_{\bar{x},i}-U^k_{\bar{x},i})\right]^2+\left[V^k_{\bar{x},i}+\mu(v^k_{\bar{x},i}-V^k_{\bar{x},i})\right]^2\right\}\right)$$

$$\times\tau\sum_{k=1}^{j+1}\left\{\left[U^k_{\bar{x},i}+\mu(u^k_{\bar{x},i}-U^k_{\bar{x},i})\right]\left(u^k_{\bar{x},i}-U^k_{\bar{x},i}\right)+\left[V^k_{\bar{x},i}+\mu(v^k_{\bar{x},i}-V^k_{\bar{x},i})\right]\left(v^k_{\bar{x},i}-V^k_{\bar{x},i}\right)\right\}$$

$$\times\left[U^{j+1}_{\bar{x},i}+\mu(u^{j+1}_{\bar{x},i}-U^{j+1}_{\bar{x},i})\right]d\mu\left(u^{j+1}_{\bar{x},i}-U^{j+1}_{\bar{x},i}\right)$$

$$+\int_0^1 a\left(\tau\sum_{k=1}^{j+1}\left\{\left[U^k_{\bar{x},i}+\mu(u^k_{\bar{x},i}-U^k_{\bar{x},i})\right]^2+\left[V^k_{\bar{x},i}+\mu(v^k_{\bar{x},i}-V^k_{\bar{x},i})\right]^2\right\}\right)$$

$$\times\left(u^{j+1}_{\bar{x},i}-U^{j+1}_{\bar{x},i}\right)d\mu\left(u^{j+1}_{\bar{x},i}-U^{j+1}_{\bar{x},i}\right)$$

$$+2\int_0^1 a'\left(\tau\sum_{k=1}^{j+1}\left\{\left[U^k_{\bar{x},i}+\mu(u^k_{\bar{x},i}-U^k_{\bar{x},i})\right]^2+\left[V^k_{\bar{x},i}+\mu(v^k_{\bar{x},i}-V^k_{\bar{x},i})\right]^2\right\}\right)$$

$$\times\tau\sum_{k=1}^{j+1}\left\{\left[U^k_{\bar{x},i}+\mu(u^k_{\bar{x},i}-U^k_{\bar{x},i})\right]\left(u^k_{\bar{x},i}-U^k_{\bar{x},i}\right)+\left[V^k_{\bar{x},i}+\mu(v^k_{\bar{x},i}-V^k_{\bar{x},i})\right]\left(v^k_{\bar{x},i}-V^k_{\bar{x},i}\right)\right\}$$

$$\times\left[V^{j+1}_{\bar{x},i}+\mu(v^{j+1}_{\bar{x},i}-V^{j+1}_{\bar{x},i})\right]d\mu\left(v^{j+1}_{\bar{x},i}-V^{j+1}_{\bar{x},i}\right)$$

$$+\int_0^1 a\left(\tau\sum_{k=1}^{j+1}\left\{\left[U^k_{\bar{x},i}+\mu(u^k_{\bar{x},i}-U^k_{\bar{x},i})\right]^2+\left[V^k_{\bar{x},i}+\mu(v^k_{\bar{x},i}-V^k_{\bar{x},i})\right]^2\right\}\right)$$

$$\times\left(v^{j+1}_{\bar{x},i}-V^{j+1}_{\bar{x},i}\right)d\mu\left(v^{j+1}_{\bar{x},i}-V^{j+1}_{\bar{x},i}\right)$$

$$= 2\int_0^1 a'\left(\tau\sum_{k=1}^{j+1}\left\{\left[U^k_{\bar{x},i}+\mu(u^k_{\bar{x},i}-U^k_{\bar{x},i})\right]^2+\left[V^k_{\bar{x},i}+\mu(v^k_{\bar{x},i}-V^k_{\bar{x},i})\right]^2\right\}\right)$$

$$\times\tau\sum_{k=1}^{j+1}\left\{\left[U^k_{\bar{x},i}+\mu(u^k_{\bar{x},i}-U^k_{\bar{x},i})\right]\left(u^k_{\bar{x},i}-U^k_{\bar{x},i}\right)+\left[V^k_{\bar{x},i}+\mu(v^k_{\bar{x},i}-V^k_{\bar{x},i})\right]\left(v^k_{\bar{x},i}-V^k_{\bar{x},i}\right)\right\}$$

$$\times\left\{\left[U^{j+1}_{\bar{x},i}+\mu(u^{j+1}_{\bar{x},i}-U^{j+1}_{\bar{x},i})\right]\left(u^{j+1}_{\bar{x},i}-U^{j+1}_{\bar{x},i}\right)\right.$$

$$\left.+\left[V^{j+1}_{\bar{x},i}+\mu(v^{j+1}_{\bar{x},i}-V^{j+1}_{\bar{x},i})\right]\left(v^{j+1}_{\bar{x},i}-V^{j+1}_{\bar{x},i}\right)\right\}d\mu$$

$$+\int_0^1 a\left(\tau\sum_{k=1}^{j+1}\left\{\left[U^k_{\bar{x},i}+\mu(u^k_{\bar{x},i}-U^k_{\bar{x},i})\right]^2+\left[V^k_{\bar{x},i}+\mu(v^k_{\bar{x},i}-V^k_{\bar{x},i})\right]^2\right\}\right)$$

$$\times\left[\left(u^{j+1}_{\bar{x},i}-U^{j+1}_{\bar{x},i}\right)^2+\left(v^{j+1}_{\bar{x},i}-V^{j+1}_{\bar{x},i}\right)^2\right]d\mu$$

$$= 2\int_0^1 a'\left(\tau\sum_{k=1}^{j+1}\left\{\left[U^k_{\bar{x},i}+\mu(u^k_{\bar{x},i}-U^k_{\bar{x},i})\right]^2+\left[V^k_{\bar{x},i}+\mu(v^k_{\bar{x},i}-V^k_{\bar{x},i})\right]^2\right\}\right)$$

$$\times\xi^{j+1}_i(\mu)\xi^j_{t,\,i}(\mu)\,d\mu$$

$$+\int_0^1 a\left(\tau\sum_{k=1}^{j+1}\left\{\left[U_{\bar{x},i}^k+\mu(u_{\bar{x},i}^k-U_{\bar{x},i}^k)\right]^2+\left[V_{\bar{x},i}^k+\mu(v_{\bar{x},i}^k-V_{\bar{x},i}^k)\right]^2\right\}\right)$$
$$\times\left[\left(u_{\bar{x},i}^{j+1}-U_{\bar{x},i}^{j+1}\right)^2+\left(v_{\bar{x},i}^{j+1}-V_{\bar{x},i}^{j+1}\right)^2\right]d\mu,$$

where

$$\xi_i^{j+1}(\mu)=\tau\sum_{k=1}^{j+1}\left\{\left[U_{\bar{x},i}^k+\mu(u_{\bar{x},i}^k-U_{\bar{x},i}^k)\right]\left(u_{\bar{x},i}^k-U_{\bar{x},i}^k\right)\right.$$
$$\left.+\left[V_{\bar{x},i}^k+\mu(v_{\bar{x},i}^k-V_{\bar{x},i}^k)\right]\left(v_{\bar{x},i}^k-V_{\bar{x},i}^k\right)\right\},\quad \xi_i^0(\mu)=0,$$

and therefore,

$$\xi_{t,\,i}^j(\mu)=\left[U_{\bar{x},i}^{j+1}+\mu(u_{\bar{x},i}^{j+1}-U_{\bar{x},i}^{j+1})\right]\left(u_{\bar{x},i}^{j+1}-U_{\bar{x},i}^{j+1}\right)$$
$$+\left[V_{\bar{x},i}^{j+1}+\mu(v_{\bar{x},i}^{j+1}-V_{\bar{x},i}^{j+1})\right]\left(v_{\bar{x},i}^{j+1}-V_{\bar{x},i}^{j+1}\right).$$

Introducing the following notation

$$s_i^{j+1}(\mu)=\tau\sum_{k=1}^{j+1}\left\{\left[U_{\bar{x},i}^k+\mu(u_{\bar{x},i}^k-U_{\bar{x},i}^k)\right]^2+\left[V_{\bar{x},i}^k+\mu(v_{\bar{x},i}^k-V_{\bar{x},i}^k)\right]^2\right\},$$

from the previous equality we have

$$\left\{a\left(\tau\sum_{k=1}^{j+1}\left[(u_{\bar{x},i}^k)^2+(v_{\bar{x},i}^k)^2\right]\right)u_{\bar{x},i}^{j+1}\right.$$
$$\left.-a\left(\tau\sum_{k=1}^{j+1}\left[(U_{\bar{x},i}^k)^2+(V_{\bar{x},i}^k)^2\right]\right)U_{\bar{x},i}^{j+1}\right\}\left(u_{\bar{x},i}^{j+1}-U_{\bar{x},i}^{j+1}\right)$$
$$+\left\{a\left(\tau\sum_{k=1}^{j+1}\left[(u_{\bar{x},i}^k)^2+(v_{\bar{x},i}^k)^2\right]\right)v_{\bar{x},i}^{j+1}\right.$$
$$\left.-a\left(\tau\sum_{k=1}^{j+1}\left[(U_{\bar{x},i}^k)^2+(V_{\bar{x},i}^k)^2\right]\right)V_{\bar{x},i}^{j+1}\right\}\left(v_{\bar{x},i}^{j+1}-V_{\bar{x},i}^{j+1}\right)$$
$$=2\int_0^1 a'\left(s_i^{j+1}(\mu)\right)\xi_i^{j+1}\xi_{t,\,i}^j\,d\mu$$
$$+\int_0^1 a\left(s_i^{j+1}(\mu)\right)\left[\left(u_{\bar{x},i}^{j+1}-U_{\bar{x},i}^{j+1}\right)^2+\left(v_{\bar{x},i}^{j+1}-V_{\bar{x},i}^{j+1}\right)^2\right]d\mu.$$

After substituting this equality in (14) we get

$$\|z^{j+1}\|^2-(z^{j+1},z^j)+\|w^{j+1}\|^2-(w^{j+1},w^j)$$
$$+2\tau h\sum_{i=1}^{M}\int_0^1 a'\left(s_i^{j+1}(\mu)\right)\xi_i^{j+1}\xi_{t,\,i}^{j}\,d\mu$$
$$+\tau h\sum_{i=1}^{M}\int_0^1 a\left(s_i^{j+1}(\mu)\right)\left[\left(u_{\bar{x},i}^{j+1}-U_{\bar{x},i}^{j+1}\right)^2+\left(v_{\bar{x},i}^{j+1}-V_{\bar{x},i}^{j+1}\right)^2\right]d\mu \tag{15}$$
$$=-\tau(\psi_1^j,z^{j+1})-\tau(\psi_2^j,w^{j+1}).$$

Taking into account that $a(S)\geq a_0=Const>0$ and relations $s^{j+1}(\mu)\geq 0$,

$$(r^{j+1},r^j)=\frac{1}{2}\|r^{j+1}\|^2+\frac{1}{2}\|r^j\|^2-\frac{1}{2}\|r^{j+1}-r^j\|^2,$$
$$\tau\,\xi_i^{j+1}\xi_{t,\,i}^{j}=\frac{1}{2}\left(\xi_i^{j+1}\right)^2-\frac{1}{2}\left(\xi_i^{j}\right)^2+\frac{\tau^2}{2}\left(\xi_{t,\,i}^{j}\right)^2$$

from (15) we have

$$\|z^{j+1}\|^2-\frac{1}{2}\|z^{j+1}\|^2-\frac{1}{2}\|z^j\|^2+\frac{1}{2}\|z^{j+1}-z^j\|^2+\|w^{j+1}\|^2-\frac{1}{2}\|w^{j+1}\|^2$$
$$-\frac{1}{2}\|w^j\|^2+\frac{1}{2}\|w^{j+1}-w^j\|^2+h\sum_{i=1}^{M}\int_0^1 a'\left(s_i^{j+1}(\mu)\right)\left[\left(\xi_i^{j+1}\right)^2-\left(\xi_i^{j}\right)^2\right]d\mu$$
$$+\tau^2 h\sum_{i=1}^{M}\int_0^1 a'\left(s_i^{j+1}(\mu)\right)\left(\xi_{t,\,i}^{j}\right)^2d\mu \tag{16}$$
$$+\tau h a_0\sum_{i=1}^{M}\left[\left(u_{\bar{x},i}^{j+1}-U_{\bar{x},i}^{j+1}\right)^2+\left(v_{\bar{x},i}^{j+1}-V_{\bar{x},i}^{j+1}\right)^2\right]$$
$$\leq-\tau(\psi_1^j,z^{j+1})-\tau(\psi_2^j,w^{j+1}).$$

From (16) we arrive at

$$\frac{1}{2}\|z^{j+1}\|^2-\frac{1}{2}\|z^j\|^2+\frac{\tau^2}{2}\|z_t^j\|^2+\frac{1}{2}\|w^{j+1}\|^2-\frac{1}{2}\|w^j\|^2+\frac{\tau^2}{2}\|w_t^j\|^2$$
$$+h\sum_{i=1}^{M}\int_0^1 a'\left(s_i^{j+1}(\mu)\right)\left[\left(\xi_i^{j+1}\right)^2-\left(\xi_i^{j}\right)^2\right]d\mu \tag{17}$$
$$+\tau a_0\left(\|z_{\bar{x}}^{j+1}]|^2+\|w_{\bar{x}}^{j+1}]|^2\right)$$
$$\leq\frac{\tau}{2a_0}\left(\|\psi_1^j\|^2+\|\psi_2^j\|^2\right)+\frac{\tau a_0}{2}\left(\|z^{j+1}\|^2+\|w^{j+1}\|^2\right).$$

Using the discrete analogue of the Poincaré inequality [20]

$$\|r^{j+1}\|^2 \le \|r_{\bar{x}}^{j+1}]|^2,$$

from (17) we get

$$\begin{aligned}
&\|z^{j+1}\|^2 - \|z^j\|^2 + \tau^2\|z_t^j\|^2 + \|w^{j+1}\|^2 - \|w^j\|^2 + \tau^2\|w_t^j\|^2 \\
&+2h\sum_{i=1}^{M}\int_0^1 a'\left(s_i^{j+1}(\mu)\right)\left[\left(\xi_i^{j+1}\right)^2 - \left(\xi_i^j\right)^2\right] d\mu \qquad (18) \\
&+\tau a_0\left(\|z_{\bar{x}}^{j+1}]|^2 + \|w_{\bar{x}}^{j+1}]|^2\right) \le \frac{\tau}{a_0}\left(\|\psi_1^j\|^2 + \|\psi_2^j\|^2\right).
\end{aligned}$$

Summing (18) from $j=0$ to $j=n-1$ we arrive at

$$\begin{aligned}
&\|z^n\|^2 + \tau^2\sum_{j=0}^{n-1}\|z_t^j\|^2 + \|w^n\|^2 + \tau^2\sum_{j=0}^{n-1}\|w_t^j\|^2 \\
&+2h\sum_{j=0}^{n-1}\sum_{i=1}^{M}\int_0^1 a'\left(s_i^{j+1}(\mu)\right)\left[\left(\xi_i^{j+1}\right)^2 - \left(\xi_i^j\right)^2\right] d\mu \qquad (19) \\
&+\tau a_0\sum_{j=0}^{n-1}\left(\|z_{\bar{x}}^{j+1}]|^2 + \|w_{\bar{x}}^{j+1}]|^2\right) \le \frac{\tau}{a_0}\sum_{j=0}^{n-1}\left(\|\psi_1^j\|^2 + \|\psi_2^j\|^2\right).
\end{aligned}$$

Note, that since $s_i^{j+1}(\mu) \ge s_i^j(\mu)$, $a'(S) \ge 0$ and $a''(S) \le 0$, for the second line of the last formula we have

$$\begin{aligned}
&\sum_{j=0}^{n-1} a'\left(s_i^{j+1}(\mu)\right)\left[\left(\xi_i^{j+1}\right)^2 - \left(\xi_i^j\right)^2\right] \\
&= a'\left(s_i^1(\mu)\right)\left(\xi_i^1\right)^2 - a'\left(s_i^1(\mu)\right)\left(\xi_i^0\right)^2 \\
&+a'\left(s_i^2(\mu)\right)\left(\xi_i^2\right)^2 - a'\left(s_i^2(\mu)\right)\left(\xi_i^1\right)^2 \\
&+\cdots + a'\left(s_i^n(\mu)\right)\left(\xi_i^n\right)^2 - a'\left(s_i^n(\mu)\right)\left(\xi_i^{n-1}\right)^2 \\
&= a'\left(s_i^n(\mu)\right)\left(\xi_i^n\right)^2 + \sum_{j=1}^{n-1}\left[a'\left(s_i^j(\mu)\right) - a'\left(s_i^{j+1}(\mu)\right)\right]\left(\xi_i^j\right)^2 \ge 0.
\end{aligned}$$

Taking into account the last relation and (19) one can deduce

$$
\begin{gathered}
\|z^n\|^2+\|w^n\|^2+\tau^2\sum_{j=0}^{n-1}\|z_t^j\|^2+\tau^2\sum_{j=0}^{n-1}\|w_t^j\|^2 \\
+\tau a_0\sum_{j=0}^{n-1}\left(\|z_{\bar{x}}^{j+1}]|^2+\|w_{\bar{x}}^{j+1}]|^2\right)\le\frac{\tau}{a_0}\sum_{j=0}^{n-1}\left(\|\psi_1^j\|^2+\|\psi_2^j\|^2\right).
\end{gathered}
\tag{20}
$$

From (20) we get (10), and thus Theorem 3.1 is proved. □

Acknowledgement

The author thanks Shota Rustaveli National Science Foundation (project PhDF2016_19) for the financial support.

References

[1] Y. Bai, P. Zhang, *On a class of volterra nonlinear equations of parabolic type*, Appl. Math. Comp., **216** (2010), 236-240

[2] T. Dzhangveladze, *A nonlinear integro-differential equations of parabolic type* (Russian), Differ. Uravn., **21** (1985), 41-46, English translation: Differ. Equ., **21** (1985), 32-36

[3] T.A. Dzhangveladze, *First boundary value problem for a nonlinear equation of parabolic type* (Russian), Dokl. Akad. Nauk SSSR, **269** (1983), 839-842, English translation: Soviet Phys. Dokl., **28** (1983), 323-324

[4] T.A. Dzhangveladze, Z.V. Kiguradze, *On the stabilization of solutions of an initial-boundary value problem for a nonlinear integro-differential equation* (Russian), Differ. Uravn., **43** (2007), 833-840, English translation: Differ. Equ., **43** (2007), 854-861

[5] D.G. Gordeziani, T.A. Dzhangveladze, T.K. Korshia, *Existence and uniqueness of a solution of certain nonlinear parabolic problems* (Russian), Differ. Uravn., **19** (1983), 1197-1207, English translation: Differ. Equ., **19** (1983), 887-895

[6] T.A. Jangveladze, *Convergence of a difference scheme for a nonlinear integro-differential equation*, Proc. I. Vekua Inst. Appl. Math., **48** (1998), 38-43

[7] T. Jangveladze, *On one class of nonlinear integro-differential equations*, Sem. I.Vekua Inst. Appl. Math., REPORTS, **23** (1997), 51-87

[8] T.A. Jangveladze, Z.V. Kiguradze, *Asymptotics for large time of solutions to nonlinear system associated with the penetration of a magnetic field into a substance*, Appl. Math., **55** (2010), 471-493

[9] T.A. Jangveladz, Z.V. Kiguradze, *Asymptotics of a solution of a nonlinear system of diffusion of a magnetic field into a substance* (Russian), Sibirsk. Mat. Zh., **47** (2006), 1058-1070, English translation: Siberian. Math. J., **47**, (2006), 867-878

[10] T.A. Jangveladze, Z.V. Kiguradze, *Semi-discrete scheme for one nonlinear integro-differential system describing diffusion process of electromagnetic field*, Advanc. Appl. Pure Math. Proc. 7th International Conference on Finite Differences, Finite Elements, Finite Volumes, Boundary Elements (2014), 118-122

[11] T. Jangveladze, Z. Kiguradze, B. Neta, *Numerical Solution of Three Classes of Nonlinear Parabolic Integro-Differential Equations*, Elsevier, Academic Press, 2016

[12] Z. Kiguradze, *Finite difference scheme for a nonlinear integro-differential system*, Proc. I.Vekua Inst. Appl. Math., **50-51** (2000-2001), 65-72

[13] L. Landau, E. Lifschitz, *Electrodynamics of Continuous Media, Course of Theoretical Physics,* Moscow, 1957

[14] G. Laptev, *Mathematical features of the problem of the penetration of a magnetic field into matter* (Russian), Zh. Vychisl. Mat. Mat. Fiz., **28** (1988), 1332-1345, English translation: U.S.S.R. Comput. Math. Math. Phys., **28** (1990), 35-45

[15] G. Laptev, *Quasilinear parabolic equations which contains in coefficients volterra's operator* (Russian), Math. Sbornik, **136** (1988), 530-545, English translation: Sbornik Math., **64** (1989), 527-542

[16] J.-L. Lions, *Quelques Methodes de Resolution des Problemes aux Limites Non-lineaires*, Dunod Gauthier-Villars, Paris, 1969

[17] H. Liao, Y. Zhao, *Linearly localized difference schemes for the nonlinear Maxwell model of a magnetic field into a substance*, Appl. Math. Comput., **233** (2014), 608-622

[18] Y. Lin, H.M.Yin, *Nonlinear parabolic equations with nonlinear functionals*, J. Math. Anal. Appl., **168** (1992), 28-41

[19] N. Long, A. Dinh, *Nonlinear parabolic problem associated with the penetration of a magnetic field into a substance*, Math. Meth. Appl. Sci., **16** (1993), 81-295

[20] A.A. Samarskii, *The Theory of Difference Schemes* (Russian), Nauka, Moscow, 1977