Hierarchical Models for Viscoelastic Kelvin-Voigt Prismatic Shells with Voids

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The present paper is devoted to construction of hierarchical models for porous elastic and viscoelastic Kelvin-Voigt prismatic shells on the basis of linear theories. Using I. Vekua's [1], [2] dimension reduction method, governing systems are derived and in the Nth approximation boundary value problems are set.

1. Introduction

The present paper is devoted to construction of hierarchical models for porous elastic and viscoelastic Kelvin-Voigt prismatic shells on the basis of linear theories. Using I. Vekua's [1], [2] dimension reduction method, governing systems are derived and in the Nth approximation boundary value problems are set. In the $N = 0$ approximation, considering plates of a constant thickness, governing systems mathematically coincide with the governing systems of the plane strain corresponding to the basic three-dimensional (3D) linear theories [3]-[6] up to a separate equation for the out of plane component of the displacement vector in our cases. The ways of investigation of boundary value problems (BVPs) and initial boundary value problems (IBVPs), including the case of cusped prismatic shells [2], are indicated and some preliminary results are presented.

2. Field equations for Kelvin-Voigt materials

The field equations have the following form [4], [5]:
  Motion Equations

$$X_{ji,j} + \Phi_i = \rho \ddot{u}_i(x_1, x_2, x_3, t), \quad (x_1, x_2, x_3) \in \Omega \subset \mathbb{R}^3, \quad t > t_0, \quad i, j = 1, 2, 3; \quad (2.1)$$

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$$H_{j,j} + H_0 = \rho_0 \ddot{\varphi} - \mathcal{F}, \tag{2.2}$$

where $X_{ij} \in C^1(\Omega)$ is the stress tensor; Φ_i are the volume force components; $\rho_0 := \rho k'$ (k' is equilibrated inertia), ρ is the reference mass density; $u_i \in C^2(\Omega)$ are the displacements; $H_j \in C^1(\Omega)$ is the component of the equilibrated stress vector, H_0 and $\mathcal{F}$ are the intrinsic and extrinsic equilibrated volume forces; the points as superscripts mean differentiation with respect to the time, and Einstein's summation convention is used; indices after comma mean differentiation with respect to the corresponding variables of the Cartesian frame $Ox_1x_2x_3$ (throughout the paper we assume existence of the indicated (continuous) derivatives); dots as superscripts of the symbols mean derivatives with respect to time t;

Constitutive Equations (isotropic case)

$$X_{ij} = \lambda e_{kk}\delta_{ij} + 2\mu e_{ij} + \lambda^* \dot{e}_{kk}\delta_{ij} + 2\mu^* \dot{e}_{ij} + b\varphi\delta_{ij} + b^*\dot{\varphi}\delta_{ij}, \quad i,j = 1,2,3, \tag{2.3}$$

$$H_j = \tilde{\alpha}\varphi_{,j} + \alpha^*\dot{\varphi}_{,j}, \quad j = 1,2,3, \tag{2.4}$$

$$H_0 = -be_{kk} - \xi\varphi - \nu^*\dot{e}_{kk} - \xi^*\dot{\varphi}, \tag{2.5}$$

where $e_{ij} \in C^1(\Omega)$ is the strain tensor; $\varphi := \nu_0 - \nu \in C^2(\Omega)$ is the change in the volume fraction from the matrix reference volume fraction ν (clearly, the bulk reference density $\rho = \nu\gamma$, $0 < \nu \leq 1$, here γ is the matrix reference density); λ, λ^*, μ, μ^*, b, b^*, $\tilde{\alpha}$, α^*, ν^*, ξ, ξ^* are the constitutive coefficients;

Kinematic Relations

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}), \quad i,j = 1,2,3. \tag{2.6}$$

3. Construction of hierarchical models. Nth approximation.

Let us consider prismatic shells (see, e.g., Fig. 1 and [2], [7]) occupying 3D domain Ω with the projection ω (on the plane $x_3 = 0$) and the face surfaces

$$x_3 = \overset{(+)}{h}(x_1, x_2) \in C^2(\omega) \text{ and } x_3 = \overset{(-)}{h}(x_1, x_2) \in C^2(\omega), \quad (x_1, x_2) \in \omega.$$
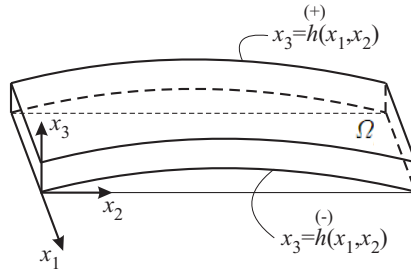

Figure 1. Prismatic shell of a constant thickness. $\partial\Omega$ is a Lipschitz boundary

rth order moments of the following quantities are defined as

$$\left(u_{ir}, X_{ijr}, e_{ijr}, \Phi_{jr}, H_{ir}, H_{0r}, \varphi_r, \mathcal{F}_r \right)(x_1, x_2, t)$$

$$:= \int\limits_{\overset{(-)}{h}(x_1,x_2)}^{\overset{(+)}{h}(x_1,x_2)} \left(u_i, X_{ij}, e_{ij}, \Phi_j, H_i, H_0, \varphi, \mathcal{F} \right)(x_1, x_2, x_3, t) P_r(ax_3 - b) \, dx_3,$$

$$i, j = 1, 2, 3, \quad (3.1)$$

where

$$P_r(ax_3 - b) \quad \left(a(x_1, x_2) := \frac{2}{\overset{(+)}{h} - \overset{(-)}{h}}, \quad b(x_1, x_2) := \frac{\overset{(+)}{h} + \overset{(-)}{h}}{\overset{(+)}{h} - \overset{(-)}{h}} \right), \quad r = 0, 1, \cdots,$$

are the rth order Legendre polynomials.

$$2h(x_1, x_2) := \overset{(+)}{h}(x_1, x_2) - \overset{(-)}{h}(x_1, x_2), \quad (x_1, x_2) \in \omega,$$

is the thickness of the prismatic shell.

Under the well-know restrictions (see, e.g., [1]) the following Fourier-Legendre series

$$\left(u_i, X_{ij}, e_{ij}, H_i, H_0, \varphi, \mathcal{F} \right)(x_1, x_2, x_3, t)$$

$$= \sum_{r=0}^{\infty} a\left(r + \frac{1}{2}\right)\left(u_{ir}, X_{ijr}, e_{ijr}, H_{ir}, H_{0r}, \varphi_r, \mathcal{F}_r \right)(x_1, x_2, t) P_r(ax_3 - b), (3.2)$$

are convergent.

Therefore on the upper and lower face surfaces of the prismatic shell under consideration

$$\overset{(\pm)}{u}_i := u_i(x_1, x_2, \overset{(\pm)}{h}(x_1, x_2), t) = \sum_{s=0}^{\infty} a\left(s + \frac{1}{2}\right) u_{is}(\pm 1)^s = \sum_{s=0}^{\infty} \frac{(\pm 1)^s (2s+1)}{2h} u_{is},$$

$$i = 1, 2, 3,$$

whence

$$\overset{(+)}{u}_i - (-1)^r \overset{(-)}{u}_i = -\sum_{s=0}^{\infty} \overset{r}{a}_{3s} u_{is}, \quad i = 1, 2, 3,$$

$$\overset{(+)}{u}_i \overset{(+)}{h}_{,\alpha} - (-1)^r \overset{(-)}{u}_i \overset{(-)}{h}_{,\alpha} = \sum_{s=0}^{\infty} \overset{r}{a}^*_{\alpha s} u_{is}, \quad \alpha = 1, 2,$$

where

$$\overset{r}{a}{}^{*}_{\alpha s} = \overset{r}{a}_{\alpha s}, \quad s \neq r, \quad \overset{r}{a}{}^{*}_{\alpha r} = (2r+1)\frac{h_{,\alpha}}{h},$$

$$\overset{r}{a}_{\alpha \underline{r}} := r\frac{h_{,\alpha}}{h}, \quad \overset{r}{a}_{\alpha s} := (2s+1)\frac{\overset{(+)}{h}_{,\alpha} -(-1)^{r+s}\overset{(-)}{h}_{,\alpha}}{2h}, \quad s \neq r.$$

$$\overset{r}{a}_{3s} := -(2s+1)\frac{1-(-1)^{s+r}}{2h}.$$

Using

$$\int\limits_{\overset{(-)}{h}(x_1,x_2)}^{\overset{(+)}{h}(x_1,x_2)} P_r(ax_3-b)f_{,\alpha}\,dx_3 = f_{r,\alpha} + \sum_{s=0}^{r}\overset{r}{a}_{\alpha s}f_s - \overset{(+)(+)}{f\ h}_{,\alpha} + (-1)^r\overset{(-)(-)}{f\ h}_{,\alpha}, \quad \alpha = 1,2,$$

$$\int\limits_{\overset{(-)}{h}(x_1,x_2)}^{\overset{(+)}{h}(x_1,x_2)} P_r(ax_3-b)f_{,3}\,dx_3 = \sum_{s=0}^{r}\overset{r}{a}_{3s}f_s + \overset{(+)}{f} - (-1)^r\overset{(-)}{f},$$

from (2.1)-(2.6), after multiplying them by $P_r(ax_3-b)$ and then integrating within the limits $\overset{(-)}{h}(x_1,x_2)$ and $\overset{(+)}{h}(x_1,x_2)$ with respect to the thickness variable x_3, we obtain the following formulas in ω:

$$X_{\alpha ir,\alpha} + \sum_{s=0}^{r}\overset{r}{a}_{js}X_{jis} + \overset{r}{X}_i = \rho\frac{\partial^2 u_{ir}}{\partial t^2}, \quad i = \overline{1,3}, \;\; r = 0,1,\cdots, \qquad (3.3)$$

$$H_{\alpha r,\alpha} + \sum_{s=0}^{r}\overset{r}{a}_{is}H_{is} + H_{0r} + \overset{r}{H} = \rho_0\ddot{\varphi}_r - \mathcal{F}_r, \quad r = 0,1,\cdots, \qquad (3.4)$$

$$\overset{r}{H} := \overset{(+)}{H_3} - \overset{(+)}{H_\alpha}\overset{(+)}{h}_{,\alpha} + (-1)^r\left[-\overset{(-)}{H_3} + \overset{(-)}{H_\alpha}\overset{(-)}{h}_{,\alpha}\right] + H_{0r}$$

$$= \overset{(+)}{H}\sqrt{1 + \left(\overset{(+)}{h}_{,1}\right)^2 + \left(\overset{(+)}{h}_{,2}\right)^2} + (-1)^r\overset{(-)}{H}\sqrt{1 + \left(\overset{(-)}{h}_{,1}\right)^2 + \left(\overset{(-)}{h}_{,2}\right)^2}, \quad r = 0,1,\cdots,$$

$$X_{ijr} = \lambda e_{kkr}\delta_{ij} + 2\mu e_{ijr} + \lambda^* \dot{e}_{kkr}\delta_{ij} + 2\mu^* \dot{e}_{ijr} + b\varphi_r\delta_{ij} + b^*\dot{\varphi}_r\delta_{ij},$$
$$i,j = \overline{1,3}, \quad r = 0,1,\cdots, \qquad (3.5)$$

$$H_{\alpha r} = \tilde{\alpha}\Big[\varphi_{r,\alpha} + \sum_{s=0}^{r} a_{\alpha s}^r \varphi_s - \overset{(+)}{\varphi}\overset{(+)}{h}_{,\alpha} + (-1)^r \overset{(-)}{\varphi}\overset{(-)}{h}_{,\alpha}\Big]$$

$$+\alpha^*\Big[\dot{\varphi}_{r,\alpha} + \sum_{s=0}^{r} a_{\alpha s}^r \dot{\varphi}_s - \overset{(+)}{\dot{\varphi}}\overset{(+)}{h}_{,\alpha} - (-1)^r \overset{(-)}{\dot{\varphi}}\overset{(-)}{h}_{,\alpha}\Big], \quad \alpha = 1,2, \qquad (3.6)$$

$$H_{3r} = \tilde{\alpha}\Big[\sum_{s=0}^{r} a_{3s}^r \varphi_s + \overset{(+)}{\varphi} - (-1)^r \overset{(-)}{\varphi}\Big] + \alpha^*\Big[\sum_{s=0}^{r} a_{3s}^r \dot{\varphi}_s + \overset{(+)}{\dot{\varphi}} - (-1)^r \overset{(-)}{\dot{\varphi}}\Big], \qquad (3.7)$$

i.e.,

$$H_{jr} = \tilde{\alpha}\Big(\varphi_{r,j} + \sum_{s=r}^{\infty} b_{js}^r \varphi_s\Big) + \alpha^*\Big(\dot{\varphi}_{r,j} + \sum_{s=r}^{\infty} b_{js}^r \dot{\varphi}_s\Big), \quad j = \overline{1,3}, \ r = 0,1,\cdots, \quad (3.8)$$

evidently,

$$H_{jr} = \tilde{\alpha}\Big(h^{r+1}\psi_{r,j} + \sum_{s=r+1}^{\infty} h^{s+1} b_{js}^r \psi_s\Big) + \alpha^*\Big(h^{r+1}\dot{\psi}_{r,j} + \sum_{s=r+1}^{\infty} h^{s+1} b_{js}^r \dot{\psi}_s\Big),$$
$$j = \overline{1,3}, \ r = 0,1,\cdots,$$

$$\psi_r := \frac{\varphi_r}{h^{r+1}},$$

$$H_{0r} = -b e_{kkr} - \xi\varphi_r - \nu^*\dot{e}_{kkr} - \xi^*\dot{\varphi}_r, \quad r = 0,1,\cdots. \qquad (3.9)$$

$$H_{0r} = -b\Big(h^{r+1} v_{\gamma r,\gamma} + \sum_{s=r+1}^{\infty} h^{s+1} b_{is}^r v_{is}\Big) - \xi h^{r+1}\psi_r$$

$$-\nu^*\Big(h^{r+1}\dot{v}_{\gamma r,\gamma} + \sum_{s=r+1}^{\infty} h^{s+1} b_{is}^r \dot{v}_{is}\Big) - \xi^* h^{r+1}\dot{\psi}_r, \quad r = 0,1,\cdots.$$

$$e_{ijr} = \frac{1}{2}\left(u_{ir,j} + u_{jr,i}\right) + \frac{1}{2}\sum_{s=r}^{\infty} \overset{r}{b}_{is}u_{js} + \frac{1}{2}\sum_{s=r}^{\infty} \overset{r}{b}_{js}u_{is},$$

$$i, j = \overline{1,3}, \quad r = 0, 1, \cdots, \tag{3.10}$$

where

$$\theta_r := e_{iir} = u_{\gamma r,\gamma} + \sum_{s=r}^{\infty} \overset{r}{b}_{ks}u_{ks}, \quad \overset{r}{b}_{\alpha\underline{r}} := -(r+1)\frac{h_{,\alpha}}{h}, \quad \overset{r}{b}_{3\underline{r}} = 0,$$

$$\overset{r}{b}_{js} := \begin{cases} 0, & s < r, \\ -\overset{r}{a}_{\alpha s} = -(2s+1)\dfrac{\overset{(+)}{h}_{,\alpha} - (-1)^{r+\underline{s}}\overset{(-)}{h}_{,\alpha}}{2h}, & j = \alpha, \ s > r, \\ -\overset{r}{a}_{3s} = (2s+1)\dfrac{1 - (-1)^{s+r}}{2h}, & j = 3, \ s > r, \end{cases}$$

$$\alpha = 1, 2, \ j = \overline{1,3}, \quad r, s = 0, 1, 2, \cdots;$$

$$\overset{r}{X}_j := \overset{(+)}{X}_{3j} - \overset{(+)}{X}_{\alpha j}\overset{(+)}{h}_{,\alpha} + (-1)^r\left[-\overset{(-)}{X}_{3j} + \overset{(-)}{X}_{\alpha j}\overset{(-)}{h}_{,\alpha}\right] + \Phi_{jr}$$

$$= Q_{\overset{(+)}{n}j}\sqrt{1 + \left(\overset{(+)}{h}_{,1}\right)^2 + \left(\overset{(+)}{h}_{,2}\right)^2}$$

$$+ (-1)^r Q_{\overset{(-)}{n}j}\sqrt{1 + \left(\overset{(-)}{h}_{,1}\right)^2 + \left(\overset{(-)}{h}_{,2}\right)^2} + \Phi_{jr},$$

$$j = \overline{1,3}, \ r = 0, 1, 2, \cdots;$$

$Q_{\overset{(+)}{n}j}$ and $Q_{\overset{(-)}{n}j}$ are components of the stress vectors acting on the upper and lower face surfaces with normals $\overset{(+)}{n}$ and $\overset{(-)}{n}$, respectively. So, we get the equivalent[1] to (2.2)-(3.6), infinite system (3.3)-(3.5), (3.8)-(3.10) with respect to the so called r-th order moments X_{ijr}, e_{ijr}, u_{ir}, H_{jr}, H_{0r}, φ_r. Then, substituting (3.10) into (3.5) and the obtained into (3.3), and (3.8) into (3.4) we construct an equivalent infinite system with respect to the r-th order moments u_{ir}, φ_r. After this, if we suppose that the moments whose subscripts, indicating moments' order, are greater than N equal zero and consider only the first $N+1$ equations ($r = \overline{0,N}$) in the obtained infinite system of equations with respect to the r-th order moments u_{ir}, we obtain the $N-$th order approximation (hierarchical model) governing system consisting

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[1] in the following sense: if X_{ij}, e_{ij}, u_i, H_i, H_0, and φ satisfy the relations (3.2)-(3.6), then constructed by (3.1) functions X_{ijr}, e_{ijr}, u_{ir}, H_{ir}, H_{0r}, φ_r will satisfy the infinite relations (3.3)-(3.5), (3.8)-(3.10) and, vice versa, if X_{ijr}, e_{ijr}, u_{ir}, H_{ir}, H_{0r}, φ_r satisfy the infinite relations (3.3)-(3.5), (3.8)-(3.10), then constructed by means of (3.2) functions X_{ij}, e_{ij}, u_i, H_i, H_0, φ will satisfy the relations (3.2)-(3.5).

of $4N + 4$ equations with respect to $4N + 4$ unknown functions $\overset{N}{u}_{ir}$, $\overset{N}{\varphi}_r$ (roughly speaking $\overset{N}{u}_{ir}$, $\overset{N}{\varphi}_r$ is an "approximate value" of u_{ir}, φ_r, since $\overset{N}{u}_{ir}$, $\overset{N}{\varphi}_r$ are solutions of the derived finite system), $i = \overline{1,3}$, $r = \overline{0,N}$. Because of

$$\overset{r}{b}_{3r} = 0, \quad h^{r+1}(h^{-r-1})_{,\alpha} = \overset{r}{b}_{\alpha \underline{r}}, \quad \alpha = 1, 2,$$

we can rewrite (3.3) for

$$v_{ir} := h^{-r-1} u_{ir}$$

as follows

$$e_{ijr} = \frac{1}{2} h^{r+1} \left(v_{ir,j} + v_{jr,i} \right) + \frac{1}{2} \sum_{s=r+1}^{\infty} h^{s+1} \left(\overset{r}{b}_{is} v_{js} + \overset{r}{b}_{js} v_{is} \right),$$

$$i, j = \overline{1,3}, \ r = 0, 1, \cdots. \qquad (3.11)$$

$$\theta_r := e_{iir} = h^{r+1} v_{\gamma r,\gamma} + \sum_{s=r+1}^{\infty} h^{s+1} \overset{r}{b}_{ks} v_{ks}, \quad r = 0, 1, \cdots.$$

Multiplying equality (3.3) by h^r and, taking into account that $\overset{r}{a}_{i\underline{r}} = rh^{-1} h_{,\alpha}$, we get

$$(h^r X_{\alpha jr})_{,\alpha} + h^r \sum_{s=0}^{r-1} \overset{r}{a}_{is} X_{ijs} + h^r \overset{r}{X}_j = \rho h^r \frac{\partial^2 h^{r+1} v_{jr}}{\partial t^2},$$

$$j = \overline{1,3}, \ r = 0, 1, \cdots. \qquad (3.12)$$

Substituting (3.11) into (3.5) we have

$$X_{ijr} = \lambda \delta_{ij} h^{r+1} v_{\gamma r,\gamma} + \mu h^{r+1} (v_{ir,j} + v_{jr,i}) + \sum_{s=r+1}^{\infty} \overset{r}{B}_{ijks} h^{s+1} v_{ks}$$

$$+ \lambda^* \delta_{ij} h^{r+1} \dot{v}_{\gamma r,\gamma} + \mu^* h^{r+1} (\dot{v}_{ir,j} + \dot{v}_{jr,i}) + \sum_{s=r+1}^{\infty} \overset{r}{B}^*_{ijks} h^{s+1} \dot{v}_{ks}$$

$$+ b h^{r+1} \psi_r \delta_{ij} + b^* h^{r+1} \dot{\psi}_r \delta_{ij}, \quad i, j = \overline{1,3}, \quad r = 0, 1, \cdots,$$

where

$$\overset{r}{B}_{ijks} := \lambda \delta_{ij} \overset{r}{b}_{ks} + \mu \delta_{kj} \overset{r}{b}_{is} + \mu \delta_{ik} \overset{r}{b}_{js},$$

$$\overset{r}{B}{}^{*}_{ijks} := \lambda^{*}\delta_{ij}\overset{r}{b}_{ks} + \mu^{*}\delta_{kj}\overset{r}{b}_{is} + \mu^{*}\delta_{ik}\overset{r}{b}_{js},$$

Multiplying (3.4) by h^{r} we obtain

$$(h^{r}H_{\alpha r})_{,\alpha} + h^{r}\sum_{s=0}^{r-1}\overset{r}{a}_{is}H_{is} + h^{r}H_{0r} + h^{r}\overset{r}{H} = \rho_{0}h^{r}\ddot{\varphi}_{r} - h^{r}\mathcal{F}_{r}. \qquad (3.13)$$

In the N-th Approximation the governing system has the following form

$$\tilde{\alpha}\Big[\Big(h^{2r+1}\psi_{r,\alpha}\Big)_{,\alpha} + \sum_{s=r+1}^{N}\Big(h^{r+s+1}\overset{r}{b}_{\alpha s}\psi_{s}\Big)_{,\alpha}\Big]$$

$$+\alpha^{*}\Big[\Big(h^{2r+1}\dot{\psi}_{r,\alpha}\Big)_{,\alpha} + \sum_{s=r+1}^{N}\Big(h^{r+s+1}\overset{r}{b}_{\alpha s}\dot{\psi}_{s}\Big)_{,\alpha}\Big]$$

$$+\tilde{\alpha}\sum_{s=0}^{r-1}\overset{r}{a}_{is}\Big(h^{r+s+1}\psi_{s,i} + \sum_{l=s+1}^{N}h^{r+l+1}\overset{s}{b}_{il}\psi_{l}\Big)$$

$$+\alpha^{*}\sum_{s=0}^{r-1}\overset{r}{a}_{is}\Big(h^{r+s+1}\dot{\psi}_{s,i} + \sum_{l=s+1}^{N}\Big(h^{r+l+1}\overset{s}{b}_{il}\dot{\psi}_{l}\Big)\Big)$$

$$-b\Big(h^{2r+1}v_{\gamma r,\gamma} + \sum_{s=r+1}^{N}h^{r+s+1}\overset{r}{b}_{is}v_{is}\Big) - \xi h^{2r+1}\psi_{r}$$

$$-\nu^{*}\Big(h^{2r+1}\dot{v}_{\gamma r,\gamma} + \sum_{s=r+1}^{N}h^{r+s+1}\overset{r}{b}_{is}\dot{v}_{is}\Big)$$

$$-\xi^{*}h^{2r+1}\dot{\psi}_{r} + h^{r}\overset{r}{H} = \rho_{0}h^{r}\ddot{\varphi}_{r} - h^{r}\mathcal{F}_{r}, \quad r = \overline{0, N}. \qquad (3.14)$$

$$\mu\Big[\Big(h^{2r+1}\overset{N}{v}_{\alpha r,i}\Big)_{,\alpha} + \Big(h^{2r+1}\overset{N}{v}_{ir,\alpha}\Big)_{,\alpha}\Big] + \lambda\delta_{\alpha i}\Big(h^{2r+1}\overset{N}{v}_{\gamma r,\gamma}\Big)_{,\alpha}$$

$$+\sum_{s=r+1}^{N}\Big(\overset{r}{B}_{\alpha iks}h^{r+s+1}\overset{N}{v}_{ks}\Big)_{,\alpha} + \sum_{l=0}^{r-1}\overset{r}{a}_{jl}\Big[\lambda\delta_{ji}h^{r+l+1}\overset{N}{v}_{\gamma l,\gamma} + \mu h^{r+l+1}\Big(\overset{N}{v}_{jl,i} + \overset{N}{v}_{il,j}\Big)$$

$$+\sum_{s=l+1}^{N}\overset{l}{B}_{jiks}h^{r+s+1}\overset{N}{v}_{ks}\Big]$$

$$+\mu^{*}\Big[\Big(h^{2r+1}\overset{N}{\dot{v}}_{\alpha r,i}\Big)_{,\alpha} + \Big(h^{2r+1}\overset{N}{\dot{v}}_{ir,\alpha}\Big)_{,\alpha}\Big] + \lambda^{*}\delta_{\alpha i}\Big(h^{2r+1}\overset{N}{\dot{v}}_{\gamma r,\gamma}\Big)_{,\alpha}$$

$$+ \sum_{s=r+1}^{N} \left(\overset{r}{B}^{*}_{\alpha iks}\, h^{r+s+1}\, \overset{N}{\dot{v}}_{ks} \right)_{,\alpha} + \sum_{l=0}^{r-1} \overset{r}{a}_{jl} \left[\lambda^{*}\delta_{ji} h^{r+l+1}\, \overset{N}{\dot{v}}_{\gamma\, l,\gamma} + \mu h^{r+l+1} \left(\overset{N}{\dot{v}}_{jl,i} + \overset{N}{\dot{v}}_{il,j} \right) \right.$$

$$\left. + \sum_{s=l+1}^{N} \overset{l}{B}^{*}_{jiks}\, h^{r+s+1}\, \overset{N}{\dot{v}}_{ks} \right]$$

$$+ b \left[\left(h^{2r+1}\overset{N}{\psi}_{r} \right)_{,i} + \sum_{s=0}^{r-1} \overset{r}{a}_{is} h^{r+s+1} \overset{N}{\psi}_{s} \right] + b^{*} \left[\left(h^{2r+1}\overset{\dot{N}}{\psi}_{r} \right)_{,i} + \sum_{s=0}^{r-1} \overset{r}{a}_{is} h^{r+s+1} \overset{\dot{N}}{\psi}_{s} \right]$$

$$+ h^{r}\overset{r}{X}_{i} = \rho h^{r} \frac{\partial^{2} h^{r+1}\overset{N}{v}_{ir}}{\partial t^{2}}, \quad r = \overline{0,N}, \quad i = \overline{1,3}, \quad \sum_{q}^{q-1}(\cdots) \equiv 0, \tag{3.15}$$

where

$$\overset{N}{v}_{kr} := \frac{\overset{N}{u}_{kr}}{h^{r+1}}, \quad k = \overline{1,3}, \quad r = \overline{0,N}. \tag{3.16}$$

Now, we consider the following two mixed 3D BCs:
- on the face surfaces of the prismatic shell under consideration the stress vectors $Q_{\overset{(+)}{n}\, j}$, $j = 1,2,3$, $Q_{\overset{(-)}{n}\, j}$, $j = 1,2,3$, and equilibrated stress vectors $\overset{(+)}{H}_{j}$, $j = 1,2,3$, $\overset{(-)}{H}_{j}$, $j = 1,2,3$, are prescribed;
- on the lateral boundary

$$\Gamma := \{(x_{1},x_{2},x_{3}) \in \mathbb{R}^{3} : (x_{1},x_{2}) \in \partial\omega,\ \overset{(-)}{h}_{j}(x_{1},x_{2}) < x_{3} < \overset{(+)}{h}(x_{1},x_{2})\}$$

of the prismatic shell
either BCs:

$$u_{i} = \tilde{u}_{i}, \quad \varphi = \tilde{\varphi},$$

or BCs:

$$X_{ji}n_{j} = f_{i}, \ H_{j}n_{j} = g,$$

where $\tilde{u}_{i}, f_{i}, \tilde{\varphi}, g$ are given continuous functions, are prescribed.
To the last BCs on the lateral boundary Γ correspond the following BCs in the Nth approximation:

$$v_{ir} = \frac{\tilde{u}_{ir}}{h^{r+1}}, \quad i = 1,2,3, \quad \psi_{r} = \frac{\tilde{\varphi}_{r}}{h^{r+1}}, \quad r = \overline{0,N}, \ \text{on} \ \partial\omega,$$

and

$$X_{jir}n_j = f_{ir}, \quad i = 1, 2, 3, \quad H_{jr}n_j = g_r, \quad r = \overline{0, N}, \quad \text{on} \ \partial\omega,$$

respectively, where $\tilde{u}_{ir}, \tilde{\varphi}_r, f_{ir}, g_r, \ i = 1, 2, 3, \ r = \overline{0, N}$ are rth order moments of $\tilde{u}_i, \tilde{\varphi}, f_i, g, \ i = 1, 2, 3$, correspondiugly.

Note that, if we take

$$\lambda^* = 0, \ \mu^* = 0, \ b^* = 0, \ \alpha^* = 0, \ \nu^* = 0, \ \xi^* = 0, \tag{3.17}$$

from the above obtained governing system, we get hierarchical models for porous elastic prismatic shells.

4. N=0 approximation for viscoelastic Kelvin-Voigt prismatic shells

The governing system has the following form (see (3.14), (3.15)):

$$\mu\Big[(hv_{\alpha0,\beta}),_\alpha + (hv_{\beta0,\alpha}),_\alpha\Big] + \lambda(hv_{\gamma0,\gamma}),_\beta + b(h\psi_0),_\beta + \mu^*\Big[(h\dot{v}_{\alpha0,\beta}),_\alpha + (h\dot{v}_{\beta0,\alpha}),_\alpha\Big]$$

$$+\lambda^*(h\dot{v}_{\gamma0,\gamma}),_\beta + b^*(h\dot{\psi}_0),_\beta + \overset{0}{X}_\beta = \rho h\ddot{v}_{\beta0}, \quad \beta = 1, 2; \tag{4.1}$$

$$\mu(hv_{30,\alpha}),_\alpha + \mu^*(h\dot{v}_{30,\alpha}),_\alpha + \overset{0}{X}_3 = \rho h\ddot{v}_{30}; \tag{4.2}$$

$$\tilde{\alpha}(h\psi_{0,\alpha}),_\alpha - bhv_{\gamma0,\gamma} - \xi h\psi_0 + \alpha^*(h\dot{\psi}_{0,\alpha}),_\alpha - \nu^* h\dot{v}_{\gamma0,\gamma} - \xi^* h\dot{\psi}_0 + \overset{0}{H} = \rho\ddot{\psi}_0 - \mathcal{F}_0. \tag{4.3}$$

5. N=0 approximation for porous elastic prismatic shells

From (4.1)-(4.3), taking into account (3.17), we get the following governing system

$$\mu\Big[(hv_{\alpha0,\beta}),_\alpha + (hv_{\beta0,\alpha}),_\alpha\Big] + \lambda(hv_{\gamma0,\gamma}),_\beta + b(h\psi_0),_\beta + \overset{0}{X}_\beta = \rho h\ddot{v}_{\beta0}, \quad \beta = 1, 2; \tag{5.1}$$

$$\mu(hv_{30,\alpha}),_\alpha + \overset{0}{X}_3 = \rho h\ddot{v}_{30}; \tag{5.2}$$

$$\tilde{\alpha}(h\psi_{0,\alpha}),_\alpha - bhv_{\gamma0,\gamma} - \xi h\psi_0 + \overset{0}{H} = \rho\ddot{\varphi}_0 - \mathcal{F}_0. \tag{5.3}$$

BCs for the weighted displacements and the weighted volume fraction are non-classical in the case of cusped prismatic shells (see Figures 2, 3). Namely, we are not always able to prescribe them at cusped edges.

Let ω be a domain bounded by a sufficiently smooth arc $(\partial\omega \setminus \overline{\gamma}_0)$ lying in the half -plane $x_2 > 0$ and a segment $\overline{\gamma}_0$ of the x_1−axis ($x_2 = 0$).
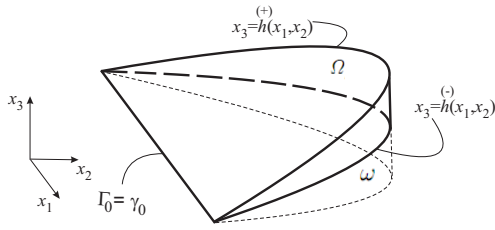
Figure 2. A sharp cusped prismatic shell with a semicircle projection. $\partial\Omega$ is a Lipschitz boundary
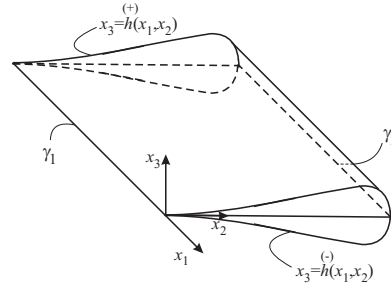
Figure 3. A cusped plate with sharp γ_1 and blunt γ_2 edges, $\gamma_0 = \gamma_1 \cup \gamma_2$. $\partial\Omega$ is a non-Lipschitz boundary

If the thickness looks like

$$2h(x_1, x_2) = h_0 x_2^\kappa, \qquad h_0, \ \kappa = const > 0, \tag{5.4}$$

then the displacements and volume fraction we can prescribe at cusped edge $\overline{\gamma}_0$ if $\kappa < 1$, while we cannot do it if $\kappa \geq 1$.

Let us show it for the particular case of deformation when

$$v_{\alpha 0} \equiv 0, \ \ \alpha = 1, 2; \ \ v_{30} \not\equiv 0.$$

Then in the static case, taking into account (5.4), from (5.2), (5.3) we get

$$x_2 v_{30,\alpha\alpha} + \kappa v_{30,2} = 2(\mu h_0)^{-1} x_2^{1-\kappa} \overset{\circ}{X}_3, \tag{5.5}$$

$$x_2 \psi_{0,\alpha\alpha} + \kappa \psi_{0,2} - \xi \alpha^{-1} x_2 \psi_0 = -2(\tilde{\alpha} h_0)^{-1} x_2^{1-\kappa} \left(\overset{\circ}{H} + \mathcal{F}_0 \right), \tag{5.6}$$

respectively.

Problem D (Find $v_{30}, \Psi_0 \in C^2(\omega) \cap C(\overline{\omega})$ by their values prescribed on $\partial\omega$) and Problem E (Find bounded $v_{30}, \Psi_0 \in C^2(\omega) \cap C(\omega \cup (\partial\omega \setminus \overline{\gamma}_0))$ by their values prescribed only on the arc $\partial\omega \setminus \overline{\gamma}_0$) are uniquely solvable for equations (5.5), (5.6) by $\kappa_2 < 1$ and $\kappa_2 \geq 1$, correspondingly. It follows from the theorem (see [8]).

Theorem 5.1: *If the coefficients a_α, $\alpha = 1, 2$, and c of the equation*

$$x_2^{\kappa_\alpha} u_{,\alpha\alpha} + a_\alpha(x_1, x_2) u_{,\alpha} + c(x_1, x_2) u = 0, \ \ c \leq 0, \ \ \kappa_\alpha = const \geq 0, \ \ \alpha = 1, 2,$$

are analytic in $\overline{\omega}$, then

(i) if either $\kappa_2 < 1$, or $\kappa_2 \geq 1$,

$$a_2(x_1, x_2) < x_2^{\kappa_2 - 1} \tag{5.7}$$

in $\overline{\omega}_\delta$ for some $\delta = const > 0$, where

$$\omega_\delta := \{(x_1, x_2) \in \omega \ : \ 0 < x_2 < \delta\},$$

the Dirichlet problem (Problem D) is well-posed;

(ii) if $\kappa_2 \geq 1$,

$$a_2(x_1, x_2) \geq x_2^{\kappa_2 - 1} \tag{5.8}$$

in ω_δ and $a_1(x_1, x_2) = O(x_2^{\kappa_1})$, $x_2 \to 0_+$ (O is the Landau symbol), the Keldysh problem (Problem E) is well-posed.

Indeed, from (5.7) and (5.8), it follows $a_2(x_1, x_2) = \kappa < 1$ for Problem D and $a_2(x_1, x_2) = \kappa \geq 1$ for Problem E, respectively, since $\kappa_1 = \kappa_2 = 1$.

To the general system (5.1)-(5.3) in the static case we apply results obtained for the more general system (see [9]).

Remark 1: In the similar way we construct hierarchical models for piezoelectric thermoviscoelastic Kelvin-Voigt prismatic shells with voids as well. It turned out that considering cusped prismatic shells by setting BCs at cusped edges the electric and magnetic potentials expose the same peculiarities which characterize the displacements and volume fraction by setting BCs for the displacements, volume fraction, electric and magnetic potentials on the lateral boundary of the prismatic shells. These results will be published in the forthcoming paper.

Acknowledgements

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