Bulletin of TICMI
Vol. 19, No. 2, 2015, 3–9

The Wavelet Characterization of the Variable Exponent Lebesgue Space

Nino Samashvili*

Faculty of Exact and Natural Sciences of I. Javakhishvili Tbilisi State University
2 University St., 0186, Tbilisi, Georgia

(*Received March 11, 2015; Revised July 17, 2015; Accepted October 16, 2015*)

The paper deals with unconditional wavelet bases in variable exponent Lebesgue spaces. Inhomogeneous wavelets of Daubechies type are considered. Some conditions for exponents are found for which the Daubechies wavelet system is an unconditional basis in $L^{p(\cdot)}(\mathbb{R}^n)$ space.

Keywords: Variable Lebesgue spaces, Hardy-Littlewood maximal operator, Wavelet, Unconditional bases.

AMS Subject Classification: 42B25, 46E30, 35L05.

1. Introduction

The Lebesgue spaces $L^{p(\cdot)}(\mathbb{R}^n)$ with variable exponent and the corresponding variable Sobolev spaces $W^{k,p(\cdot)}(\mathbb{R}^n)$ are of interest for their applications to modelling problems in physics, and to the study of variational integrals and partial differential equations with non-standard growth condition (see [1],[3]).

Let $p : \mathbb{R} \longrightarrow [1, \infty)$ be a measurable function. Denote by $L^{p(\cdot)}(\mathbb{R})$ the space of all measurable functions f on $\mathbb{R}$ such that for some $\lambda > 0$

$$\int_{\mathbb{R}} \left| \frac{f(x)}{\lambda} \right|^{p(x)} dx < \infty,$$

with the norm

$$\|f\|_{p(\cdot)} = \inf \left\{ \lambda > 0 : \int_{\mathbb{R}} \left| \frac{f(x)}{\lambda} \right|^{p(x)} dx \le 1 \right\}.$$

Methods of wavelet analysis are an important tool in investigating properties of function spaces. Due to wavelet bases we can define isomorphisms between function spaces of Hardy-Sobolev-Triebel type and corresponding sequence spaces. These isomorphisms reduce many problems from the function spaces level to the sequence spaces level. The main advantage of that approach is that interesting issues often simplify in sequence spaces. So the question about existence of an unconditional

*Corresponding author. E-mail: n.samashvili@gmail.com

ISSN: 1512-0082 print

basis in function spaces or wavelet characterization is very important to investigate their properties (see [8],[11]).

Definition 1.1: We call a scaling function (father wavelet) a function $\varphi \in L^2(\mathbb{R})$ and a wavelet (mother wavelet) a function $\psi \in L^2(\mathbb{R})$ such that the system

$$(\Phi, \Psi) = \{\varphi_m; \ \ m \in \mathbb{Z}\} \cup \{\psi_{jk}, \ \ j \in \mathbb{Z}, \ j > 0, \ k \in \mathbb{Z}\},$$

where $\varphi_m(x) = \varphi(x - m)$ and $\psi_{jk}(x) = 2^{j/2}\psi(2^j x - k)$, is an unconditional basis in the space $L^2(\mathbb{R})$.

Below for the simplicity we mean that $\varphi_m = \psi_{0,m}, m \in \mathbb{Z}$ and we have

$$(\Phi, \Psi) = \{\psi_{jm} \ \ j \in \mathbb{N}_0, m \in \mathbb{Z}\}.$$

There are no wavelets belonging to the class C^∞ with compact support. However I. Daubechies constructed systems of compactly supported wavelets with any finite smoothness [7]. Such a system of wavelets will be called the Daubechies systems. That properties of Daubechies wavelets make easy to prove theorems about isomorphisms between function spaces and sequence spaces easier.

Theorem 1.2: *([7]) There exists a constant $c > 0$ such that for every $k = 1, 2, ...$ there are scaling function φ and wavelet ψ such that (i) φ, $\psi \in C^k(\mathbb{R})$, (ii) φ and ψ have compact support and supp φ, and supp ψ are subsets of $[-kc, \, kc]$.*

Given a locally integrable function f on $\mathbb{R}$, the Hardy-Littlewood maximal operator M is defined by the equality

$$Mf(x) = \sup \frac{1}{|Q|} \int_Q |f(y)| dy,$$

where the supremum is taken over all intervals Q containing x.

Let f be a locally integrable function f on $\mathbb{R}$. We consider the local variant of the Hardy-Littlewood maximal operator given by

$$M^{loc} f(x) = \sup_{Q \ni x, |Q| \leq 1} \frac{1}{|Q|} \int_Q |f(y)| dy.$$

Denote by $\mathcal{B}(\mathbb{R})$ ($\mathcal{B}^{loc}(\mathbb{R})$) the class of all measurable functions $p : \mathbb{R} \longrightarrow [1, \infty)$ for which the operator M (operator M^{loc}) is bounded on $L^{p(\cdot)}(\mathbb{R})$. Given any measurable function $p(\cdot)$, let $p_- = \inf_{x \in \mathbb{R}} p(x)$ and $p_+ = \sup_{x \in \mathbb{R}} p(x)$. Below we assume that $1 < p_- \leq p_+ < \infty$.

In harmonic analysis a fundamental operator is the Hardy-Littlewood maximal operator M. In many applications a crucial step has been to show that operator M is bounded on a variable L^p space. Note that many classical operators in harmonic analysis such as singular integrals, commutators and fractional integrals are bounded on the variable Lebesgue space $L^{p(\cdot)}$ whenever the Hardy-Littlewood maximal operator is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$ (see monographes [1],[3]).

The conditions of boundednes of the local maximal function M^{loc} are given in [4].

Kopaliani [6] and Idzuki [5] obtained characterization of variable exponent Lebesgue spaces $L^{p(\cdot)}(\mathbb{R})$ by wavelets in case when $p(\cdot) \in \mathcal{B}(\mathbb{R})$. Our aim is to investigate the same question in case when $p(\cdot) \in \mathcal{B}^{loc}(\mathbb{R})$.

For Daubechies scaling function φ and wavelet ψ we define the following square function:

$$W(f)(x) = \left(\sum_{j=0}^{+\infty} \sum_{k\in\mathbb{Z}} 2^j |a_{j,k}\chi_{j,k}(x)|^2 \right)^{1/2}, \quad f \in L^{p(\cdot)}(\mathbb{R});$$

where

$$a_{j0} = \int_{\mathbb{R}} f(x)\varphi_j(x)dx \text{ and } a_{jk} = \int_{\mathbb{R}} f(x)\psi_{jk}(x)dx, \;\; k > 0,$$

and χ_{jk} denotes characteristic functions of dyadic intervals Q_{jk}. (A dyadic interval is the interval with sides 2^{-j} and center $2^{-j}(m+1/2)$ is denoted by Q_{jk} for $j, k \in \mathbb{Z}$.)

We prove the following theorem

Theorem 1.3: *Let $p(\cdot) \in B^{loc}(\mathbb{R})$ and let $\{\Phi, \Psi\}$ be a system associated with Daubechies scaling function φ and wavelet ψ with smoothness $k \geq 1$. Then*

(i) the system $\{\Phi, \Psi\}$ forms an unconditional bases in space $L^{p(\cdot)}(\mathbb{R})$;

(ii) there exist constants $c, C > 0$ such that for all $f \in L^{p(\cdot)}(\mathbb{R})$

$$c||f||_{p(\cdot)} \leq ||W(f)||_{p(\cdot)} \leq C||f||_{p(\cdot)}. \tag{1.1}$$

2. Proof of the main result

Let's introduce some necessary definitions and theorems.

Let us recall the definition of the local Muckenhoupt weight.We define the class of weights $\mathcal{A}_p^{loc}(\mathbb{R}),\ (1 < p < \infty)$, which consist of all nonnegative locally integrable functions w defined on $\mathbb{R}$

$$\mathcal{A}_p^{loc}(w) := \sup \frac{1}{|Q|^p} \int_Q w(x)dx \left(\int_Q w(x)^{-\frac{1}{p-1}} dx \right)^{p-1} < \infty,$$

where the supremum is taken over all intervals $Q,\ |Q| \leq 1$.

We say that $w \in \mathcal{A}_\infty^{loc}(\mathbb{R})$ if for any $0 < \alpha < 1$

$$\sup_{|Q|\leq 1} \left(\sup_{F\subset Q, |F|>\alpha|Q|} \frac{\int_Q w(x)dx}{\int_E w(x)dx} \right) < \infty.$$

Note that if $w \in \mathcal{A}_\infty^{loc}(\mathbb{R})$, then $w \in \mathcal{A}_p^{loc}(\mathbb{R})$, for some $p < \infty$. In consequence we can define for $w \in \mathcal{A}_\infty^{loc}(\mathbb{R})$ a positive number

$$r_w := \inf\{1 \leq p < \infty : \; w \in \mathcal{A}_p^{loc}(\mathbb{R})\}.$$

Below we formulate the analog of Rubio de Francia theorem in case of variable exponent space with local weights. Hereafter, $\mathcal{F}$ will denote a family of ordered pairs of non-negative, measurable functions (f, g). If we say that for some p, $1 < p < \infty$, and $w \in A_p^{loc}$

$$\int_{\mathbb{R}^n} f(x)^p w(x)dx \leq C \int_{\mathbb{R}^n} g(x)^p w(x)dx, \quad (f, g) \in \mathcal{F}, \quad (f, g) \in \mathcal{F}, \tag{2.1}$$

we mean that this inequality holds for any $(f, g) \in \mathcal{F}$ such that the left-hand side is finite and that the constant C depends only on p and the constant $A_p^{loc}(w)$ of w.

Theorem 2.1 : *[2] Given a family $\mathcal{F}$, assume that (2.1) holds for some $1 < p_0 < \infty$, for every weight $\omega \in \mathcal{A}_{p_0}^{loc}(\mathbb{R})$. Let $p(\cdot)$ be such that there exists $1 < p_1 < p_-$, with $(p(\cdot)/p_1)' \in \mathcal{B}^{loc}(\mathbb{R})$. Then*

$$\|f\|_{p(\cdot)} \leq C\|g\|_{p(\cdot)} \textit{ for all } (f, g) \in \mathcal{F}$$

such that $f \in L^{p(\cdot)}(\mathbb{R})$.

By $\mathcal{S}_e(\mathbb{R})$ we denote the set of all $f \in C^\infty(\mathbb{R})$ such that

$$q_N(f) := \sup_{x\in\mathbb{R}} e^{N|x|} \sum_{0\leq k\leq N} |f^k(x)| < \infty, \text{ for all } N \in \mathbb{N}_0.$$

We equip $\mathcal{S}_e(\mathbb{R})$ with the locally convex topology which is defined by the system of the semi norms q_N. Let $\mathcal{S}'_e(\mathbb{R})$ is the collection of all continuous linear functionals on $\mathcal{S}_e(\mathbb{R})$. We equip $\mathcal{S}'_e(\mathbb{R})$ with the strong topology.

Following Rychkov ([9]) we define Triebel-Lizorkin spaces with local Muckenhoupt weights.

Definition 2.2: Let $0 < p < \infty$, $0 < q \leq \infty$, $s \in \mathbb{R}$ and $w \in \mathcal{A}_\infty^{loc}(\mathbb{R})$. Let the function $\varphi_0 \in C_0^\infty(\mathbb{R})$ satisfy

$$\int_{\mathbb{R}} \varphi_0(x)dx \neq 0,$$

and

$$\int_{\mathbb{R}} x^k \varphi(x)dx = 0, \ 0 \leq k \leq B,$$

where $\varphi(x) = \varphi_0(x) - \frac{1}{2}\varphi_0(\frac{x}{2})$ and $B > [s]$. We define weighted Triebel-Lizorkin space $F_{p,q}^{s,w}(\mathbb{R})$ to be a set of all $f \in \mathcal{S}'_e(\mathbb{R})$ for which the following quasi-norm

$$||f|F_{pq}^{s,\omega}(\mathbb{R})||\varphi_0 = \left\| \left(\sum_{j=0}^{\infty} 2^{jsq} |\varphi_j * f|^q \right)^{1/q} |L_p^w(\mathbb{R}) \right\| < \infty.$$

The definition of the above spaces is independent of a choice of the function φ_0 up to the equivalence of quasi-norms. The spaces are quasi-Banach and Banach spaces if $p > 1$ and $q \geq 1$. The definition covers the earlier definitions of Triebel-Lizorkin spaces for Muckenhoupt weights, admissible and locally regular weight. All the above properties can be found in [9].

Definition 2.3: Let $0 < p < \infty,\ 0 < q \leq \infty,\ s \in \mathbb{R}$ and $w \in \mathcal{A}_\infty^{loc}(\mathbb{R})$. Then $f_{p,q}^{s,w}$ is a a collection of all sequences

$$\lambda = \{\lambda_{jm} \in \mathbb{R},\ j \in \mathbb{N}_0,\ m \in \mathbb{Z}\}$$

such that

$$||\lambda|f_{pq}^{s,\omega}|| = \left\| \left(\sum_{j=0}^{\infty} \sum_{m \in \mathbb{Z}} 2^{jsq} |\lambda_{jk}\chi_{jm}|^q \right)^{1/q} |L_w^p(\mathbb{R}) \right\| < \infty.$$

For $w \in \mathcal{A}_\infty^{loc}$ let us define

$$\sigma_p(w) = \left(\frac{r_w}{\min(p, r_w)} - 1 \right) + (r_w - 1)$$

$$\sigma_q = \frac{1}{\min(1,q)} - 1,\ \sigma_{pq}(w) = \max(\sigma_p(w),\, \sigma_q).$$

Theorem 2.4: *Let $0 < p < \infty,\ 0 < q \leq \infty,\ s \in \mathbb{R}$ and $w \in \mathcal{A}_\infty^{loc}(\mathbb{R})$. Let for Daubechies wavelet system $\{\Phi, \Psi\}$ with smoothness k we have*

$$k > \max\left(0,\, [s]+1,\, [r_w/p - 1/p - s] + 1,\, [\sigma_{pq}(w) - s]\right).$$

Let $f \in \mathcal{S}_e'(\mathbb{R})$. Then $f \in F_{p,q}^{s,w}(\mathbb{R})$ if and only if it can be represented as

$$f = \sum_{j=0}^{\infty} \sum_{m \in \mathbb{Z}} \lambda_{jm} 2^{-j/2} \psi_{jm},$$

where $\{\lambda_{jm}\} \in f_{p,q}^{s,w}$ and the series converges in $\mathcal{S}_e'(\mathbb{R})$. The representation is unique with

$$\lambda_{0m} = \int_{\mathbb{R}} f(x)\varphi(x-m)dx,\ m \in \mathbb{Z},\ \ \lambda_{jm} = \int_{\mathbb{R}} f(x)\psi_{jm}dx,\ j \in \mathbb{N},\, m \in \mathbb{Z}$$

and

$$I : f \to \{2^{j/2}\lambda_{jm}\}$$

is a linear isomorphism of $F_{p,q}^{s,w}(\mathbb{R})$ onto $f_{p,q}^{s,w}$.

Let now prove the main result. It is known see [9] that, if $1 < p < \infty$ and $\omega \in \mathcal{A}_p^{loc}(\mathbb{R})$ then $L_p^{\omega}(\mathbb{R}) = F_{p,2}^{0,\omega}(\mathbb{R})$.

It follows from Theorem 2.4 that if $1 < p < \infty$, $\omega \in \mathcal{A}_p^{loc}(\mathbb{R})$ and $k \geq max\{1, p-1\}$, then Daubehies wavelet system $\{\Phi, \Psi\}$ with smoothness k is an unconditional basis in $L_p^{\omega}(\mathbb{R})$ and the followiwing equivalence is true for some positive constants $c, C > 0$

$$c\|f\|_{L^p_\omega(\mathbb{R})} \leq \left\| \left(\sum_{j=0}^{+\infty} \sum_{k\in\mathbb{Z}} 2^j |a_{j,k}\chi_{j,k}(x)|^2 \right)^{1/2} \right\|_{L^p_\omega(\mathbb{R})} \leq C\|f\|_{L^p_\omega(\mathbb{R})}. \tag{2.2}$$

Fix $1 < p_0 < 2$. Then $\max\{1, p_0 - 1\} = 1$. If $p(\cdot) \in \mathcal{B}^{loc}(\mathbb{R})$ then there exists $1 < p_1 < p_-$ with $(p(\cdot)/p_1)' \in \mathcal{B}^{loc}(\mathbb{R})$ (see [4]). Note that (2.2) inequalities are valid when $k \geq 1, w \in \mathcal{A}_p^{loc}(\mathbb{R})$. From this fact and Theorem 2.1 with the pair $(Wf, |f|)$, we get the right side of inequality (1.1) when $f \in C_0^\infty(\mathbb{R})$. Note that the set $C_0^\infty(\mathbb{R})$ is dense in $L^{p(\cdot)}(\mathbb{R})$ and consequently right side of inequality (1.1) is also valid for all $f \in L^{p(\cdot)}(\mathbb{R})$. Analogously we obtain the left side of (1.1).

Let $f \in L^{p(\cdot)}(\mathbb{R})$, then we have

$$c\|f\|_{L^{p(\cdot)}(\mathbb{R})} \leq \left\| \left(\sum_{j=0}^{+\infty} \sum_{k\in\mathbb{Z}} 2^j |a_{j,k}\chi_{j,k}(x)|^2 \right)^{1/2} \right\|_{L^{p(\cdot)}(\mathbb{R})} \leq C\|f\|_{L^{p(\cdot)}(\mathbb{R})}. \tag{2.3}$$

For any $N \in \mathbb{N}$ we have

$$0 \leq \left\| f - \sum_{j=0}^{N} \sum_{m\in\mathbb{Z}} \lambda_{jm}\psi_{jm} \right\|_{p(\cdot)} = \left\| \sum_{j=N+1}^{\infty} \sum_{m\in\mathbb{Z}} \lambda_{jm}\psi_{jm} \right\|_{p(\cdot)}$$

$$= \left\| \left(\sum_{j=N+1}^{+\infty} \sum_{k\in\mathbb{Z}} 2^j a_{j,k}^2 \chi_{j,k}(x) \right)^{1/2} \right\|_{p(\cdot)} < \infty.$$

The last converges to 0 if $N \to \infty$.

Finally we prove that $\{\Phi, \Psi\}$ system is an unconditional basis for $L^{p(\cdot)}(\mathbb{R})$. Let $\Lambda_n \subset \mathbb{N}_0 \times \mathbb{Z}$ be an increasing sequence of finite sets such that $\Lambda_n \uparrow \mathbb{N}_0 \times \mathbb{Z}$. Since

$$\Big(\sum_{(j,k)\in\Lambda_n} 2^j |a_{jk}\chi_{jk}|^2 \Big)^{1/2} \in L^{p(\cdot)}(\mathbb{R}),$$

the series converges a.e. so that

$$\Big(\sum_{\mathbb{N}_0\times\mathbb{Z}\setminus\Lambda_n} 2^j |a_{jk}\chi_{jk}|^2 \Big)^{1/2} \to 0 \text{ a.e as } n \to \infty.$$

Then (2.3) implies

$$\|f - \sum_{(j,k)\in\Lambda_n} a_{jk}\psi_{jk})\|_{p(\cdot)} \leq C\|(\sum_{\mathbb{N}_0\times\mathbb{Z}\setminus\Lambda_n} 2^j|a_{jk}\chi_{jk}|^2)^{1/2}\|_{p(\cdot)}$$

and the last norm tends to 0 as $n \to \infty$ by the dominated convergence theorem for Banach function spaces with absolutely continuous norm. □

Acknowledgements

The research was supported by Shota Rustaveli National Science Foundation grant no. 13/06 (Geometry of function spaces, interpolation and embedding theorems); Georgian Research and Development Foundation, award agreement no. A60796 (Some Approximation Problems in Nonstandard Function Spaces).

References

[1] D. Cruz-Uribe, A. Fiorenza, *Variable Lebesgue Spaces: Foundations and Harmonic Analysis*, Birkhauser, Basel, 2013

[2] D. Cruz-Uribe, SFO, J.M. Martell and C. Pérez, Weights, *Extrapolation and the Theory of Rubio de Francia*, Series: Operator Theory Advances and Applications, vol. 215, Birkhäuser, 2011

[3] L. Diening, P. Hasto, P. *Harjuleto and Ruzicka, Lebesgue and Sobolev spaces with variable exponent*, Springer Lecture Notes, vol. 2017, Springer-Verlag, Berlin 2011

[4] A. Gogatishvili, A. Danelia, T. Kopaliani, *Local Hardy-Littlewood maximal operator variable Lebesgue spaces*, Banach Journal of Mathematical Analysis, **8**, 2 (2014), 229-244

[5] M. Izuki, *Wavelets and modular inequalities in variable L^p space*, Georgian Math. J., **15** (2008), 281-293

[6] T.S. Kopaliani, *Greediness of the wavelet system in $L^{p(t)}(R)$ space*, East J. Approx, **14** (2008), 59-67

[7] I. Daubechies, *Ten Lectures on Wavelets*, SIAM, Philadelphia, 1992

[8] E. Hernandez and G. Weiss, *A first course on wavelets*, CRC Press, Boca Raton, Fl., 1996

[9] V.S. Rychkov, *Littlewood-Paley theory and function spaces vith A_p^{loc} weights*, Math. Nach., **224** (2001), 145-180

[10] H. Triebel. *local means and wavelets in function spaces*, in: Function spaces VIII, Banach Center Publication, **79** (2007), 215-234

[11] A. Wojciechowska, *A Remark on Wavelet Bases in Weighted L^p Spaces*, Journal of Function Spaces and Applications, 2012

[12] P. Wojtaszczyk, *A Mathematical Introduction to Wavelets*, London Math. Soc. Student Texts, vol. 37, Cambridge Univ. Press, Cambridge, 1997