Bulletin of TICMI
Vol. 19, No. 1, 2015, 26–36

Solutions of BVPs in the Fully Coupled Theory of Elasticity for a Sphere with Double Porosity

Ivane Tsagareli[a*] and Lamara Bitsadze[a]

[a] *I. Vekua Institute of Applied Mathematics of I. Javakhishvili Tbilisi State University 2 University St., 0186, Tbilisi, Georgia*

(*Received December 22, 2014; Revised June 10, 2015; Accepted June 20, 2015*)

The purpose of this paper is to consider the basic boundary value problems of the fully coupled equilibrium theory of elasticity for solids with double porosity and explicitly solve the BVPs of statics in the fully coupled theory for a sphere. The explicit solutions of these BVPs are represented by means of absolutely and uniformly convergent series.

Keywords: Double porosity, Explicit solution, Sphere.

AMS Subject Classification: 74F10, 74G05.

1. Introduction

The theory of consolidation with double porosity was first proposed by Aifantis and co-authors in the papers [1,3]. This theory unifies a model proposed by Biot for the consolidation of deformable single porosity media with a model proposed by Barenblatt for seepage in undeformable media with two degrees of porosity. In a material with two degrees of porosity, there are two pore systems, the primary and the secondary. For example in a fissured rock (i.e.a mass of porous blocks separated from each other by an interconnected and continuously distributed system of fissures) most of the porosity is provided by the pores of the blocks or primary porosity, while most of permeability is provided by the fissures or the secondary porosity. In part I of a series of paper on the subject, Wilson and Aifantis [1] gave detailed physical interpretations of the phenomenological coefficients appearing in the double porosity theory. In part II of this series, uniqueness and variational principles were established by Beskos and Aifantis [2] for the equations of double porosity, while in part III Khaled, Beskos and Aifantis [3] provided a related finite element to consider the numerical solution of Aifantis' equations of double porosity (see [1],[2],[3] and the references cited therein.) The basic results and the historical information on the theory of porous media may be found in [4]. However, Aifantis' quasi-static theory ignored the cross-coupling effect between the volume change of the pores and fissures in the system. The cross-coupled terms were included in the equations of conservation of mass for the pore and fissure fluid and in Darcy's law for solids with double porosity by Khalili and coauthors in [5,8]. The phenomeno-

*Corresponding author. E-mail: i.tsagareli@yahoo.com

logical equations of the quasi-static theory for double porous media are established in [9,10], where a method to calculate the relevant coefficients is also presented.

For the past years many authors have investigated the BVPs of the theory of elasticity for materials with double porosity, publishing a large number of papers(for details see [11-16] and references therein).

In [17-20] the fully coupled linear theory of elasticity is considered for solids with double porosity. Four special cases of the dynamical equations are considered. The fundamental solutions are constructed by means of elementary functions and the basic properties of the fundamental solutions are established. In [21,22] for Aifantis' equations, explicit solution of the problems of elastostatics for an elastic circle with double porosity, are considered. In [23-25], for Aifantis' equations, the explicit solutions of some BVPs of elasticity for an elastic sphere, for the space with a spherical cavity and for the half-space are constructed.

The purpose of this paper is to consider the basic boundary value problems of the fully coupled equilibrium theory of elasticity for solids with double porosity and explicitly solve the BVPs of statics in the fully coupled theory for a sphere. The explicit solutions of these BVPs are represented by means of absolutely and uniformly convergent series.

2. Basic equations and boundary value problems

Let $\mathbf{x} = (x_1, x_2, x_3)$ be a point of the Euclidean three-dimensional space E^3. Let us assume that D is a ball of radius R, centered at point $O(0,0,0)$ in space E^3 and S is a spherical surface of radius R. Let us assume that the domain D is filled with an isotropic material with double porosity.

The system of homogeneous equations in the fully coupled linear equilibrium theory of elasticity for materials with double porosity can be written as follows [6,17]

$$\mu\Delta\mathbf{u} + (\lambda+\mu) graddiv\mathbf{u} - grad(\beta_1 p_1 + \beta_2 p_2) = 0, \tag{1}$$

$$\begin{aligned} &(k_1\Delta - \gamma)p_1 + (k_{12}\Delta + \gamma)p_2 = 0, \\ &(k_{21}\Delta + \gamma)p_1 + (k_2\Delta - \gamma)p_2 = 0, \end{aligned} \tag{2}$$

where $\mathbf{u}(\mathbf{x}) = \mathbf{u}(u_1, u_2, u_3)$ is the displacement vector in a solid, $p_1(\mathbf{x})$ and $p_2(\mathbf{x})$ are the pore and fissure fluid pressures respectively. β_1 and β_2 are the effective stress parameters, $\gamma > 0$ is the internal transport coefficient and corresponds to fluid transfer rate with respect to the intensity of flow between the pore and fissures, λ, μ, are constitutive coefficients, $k_j = \dfrac{\kappa_j}{\mu'}$, $k_{12} = \dfrac{\kappa_{12}}{\mu'}$, $k_{21} = \dfrac{\kappa_{21}}{\mu'}$. μ' is the fluid viscosity, κ_1 and κ_2 are the macroscopic intrinsic permeabilities associated with matrix and fissure porosity, respectively, κ_{12} and κ_{21} are the cross-coupling permeabilities for fluid flow at the interface between the matrix and fissure phases, Δ is the Laplace operator. If needed, we consider vectors as column matrices.

Introduce the definition of a regular vector-function.

Definition 2.1: A vector-function $\mathbf{U}(\mathbf{x}) = (u_1, u_2, u_3, p_1, p_2)$ defined in the domain D is called regular if it has integrable continuous second derivatives in D and $\mathbf{U(x)}$ itself and its first order derivatives are continuously extendable at every point of the boundary of D, i.e., $\mathbf{U(x)} \in C^2(D) \bigcap C^1(\overline{D})$.

For system (1),(2) we pose the following BVPs.

Problem 1: Find in the domain D a regular solution $\mathbf{U}(\mathbf{x}) = (\mathbf{u}, p_1, p_2)$, of equations (1),(2) by the boundary conditions

$$\mathbf{u}^+(\mathbf{z}) = \mathbf{\Psi}(\mathbf{z}), \quad p_1^+(\mathbf{z}) = f_4(\mathbf{z}), \quad p_2^+(\mathbf{z}) = f_5(\mathbf{z}), \quad \mathbf{z} \in S. \tag{3}$$

Problem 2: Find in the domain D a regular solution $\mathbf{U}(\mathbf{x}) = (\mathbf{u}, p_1, p_2)$, of equations (1),(2) by the boundary conditions

$$[\mathbf{P}(\partial\mathbf{x}, \mathbf{n})\mathbf{U}]^+ = \mathbf{G}(\mathbf{z}), \quad \left(\mathbf{P}^{(1)}(\partial\mathbf{x}, \mathbf{n})\mathbf{p}\right)^+ = \mathbf{g}(\mathbf{z}), \quad \mathbf{z} \in S, \tag{4}$$

where $[\cdot]^+$ denotes the limiting value from D, the vector-functions $\mathbf{\Psi}(\mathbf{z}) = (\Psi_1, \Psi_2, \Psi_3)$, $\mathbf{G}(\mathbf{z}) = (G_1, G_2, G_3)$, $\mathbf{g}(\mathbf{z}) = (g_1, g_2)$ and the functions $f_4(\mathbf{z}), f_5(\mathbf{z})$ are given functions on S, $\mathbf{n}(\mathbf{z}) = (n_1, n_2, n_3)$ is the external unit normal vector on S at $\mathbf{z}$ and $\mathbf{P}(\partial\mathbf{x}, \mathbf{n})\mathbf{U}$ is the stress vector in the considered theory, which acts on the elements of S with the normal $\mathbf{n}$,

$$\mathbf{P}(\partial\mathbf{x}, \mathbf{n})\mathbf{U} = \mathbf{T}(\partial\mathbf{x}, \mathbf{n})\mathbf{u} - \mathbf{n}(\beta_1 p_1 + \beta_2 p_2),$$

$\mathbf{T}(\partial\mathbf{x}, \mathbf{n})\mathbf{u}$ is the stress vector in the classical theory of elasticity

$$\mathbf{T}(\partial\mathbf{x}, \mathbf{n})\mathbf{u}(\mathbf{x}) = 2\mu\frac{\partial\mathbf{u(x)}}{\partial n} + \lambda\mathbf{n}div\mathbf{u}(\mathbf{x}) + \mu[\mathbf{n} \times rot\mathbf{u}(\mathbf{x})],$$

and

$$\mathbf{P}^{(1)}(\partial\mathbf{x}, \mathbf{n})\mathbf{p} = \begin{pmatrix} k_1 & k_{12} \\ k_{21} & k_2 \end{pmatrix} \frac{\partial\mathbf{p}}{\partial\mathbf{n}}, \quad \mathbf{p} = (p_1, p_2),$$

$$\frac{\partial}{\partial\mathbf{n}} = n_1\frac{\partial}{\partial x_1} + n_2\frac{\partial}{\partial x_2} + n_3\frac{\partial}{\partial x_3}.$$

Note that BVPs for the system (2) which contain only p_1 and p_2 can be investigated separately. Then suppose p_j as known we can study BVPs for system (1) with respect to $\mathbf{u}$. Combining the results obtained we arrive at explicit solutions of BVPs for system (1)-(2).

On the basis of equations (2) we can write

$$\Delta(\Delta + \lambda_1^2)p_j = 0, \qquad j = 1, 2.$$

We can easily see that the solution of system (2) can be represented in the form

$$p_1(\mathbf{x}) = \varphi(\mathbf{x}) + A\varphi_1(\mathbf{x}), \quad p_2(\mathbf{x}) = \varphi(\mathbf{x}) + \varphi_1(\mathbf{x}), \tag{5}$$

where the functions φ and φ_1 are the solutions of the following equations

$$\Delta\varphi = 0, \quad (\Delta + \lambda_1^2)\varphi_1 = 0, \tag{6}$$

respectively,

$$A = \frac{\gamma - k_{12}\lambda_1^2}{\gamma + k_1\lambda_1^2} = -\frac{k_2 + k_{12}}{k_1 + k_{21}},$$

$$\lambda_1 = i\sqrt{\frac{\gamma k_0}{k_1k_2 - k_{12}k_{21}}} = i\lambda_0, \quad i = \sqrt{-1}, \quad k_0 = k_1 + k_2 + k_{12} + k_{21};$$

$$k_1 > 0, \quad k_2 > 0, \quad \gamma > 0, \quad k_1k_2 - k_{12}k_{21} > 0, \quad k_0 > 0.$$

Let us substitute the expression (5) into (1) and let us search the particular solution of the following nonhomogeneous equation

$$\mu\Delta\mathbf{u} + (\lambda + \mu) graddiv\mathbf{u} = grad[(\beta_1 + \beta_2)\varphi + (A\beta_1 + \beta_2)\varphi_1].$$

It is well-known that a general solution of the last equation can be presented in the form

$$\mathbf{u}(\mathbf{x}) = \mathbf{v}(\mathbf{x}) + \mathbf{v}_0(\mathbf{x}), \tag{7}$$

where $\mathbf{v}(\mathbf{x})$ is a general solution of equation

$$\mu\Delta\mathbf{v} + (\lambda + \mu) graddiv\mathbf{v} = 0, \tag{8}$$

and $\mathbf{v}_0(\mathbf{x})$ is a particular solution of the nonhomogeneous equation. It is easy to see, that the vector $\mathbf{v}_0(\mathbf{x})$ has the form

$$\mathbf{v}_0(\mathbf{x}) = \frac{1}{\lambda + 2\mu} grad\left[(\beta_1 + \beta_2)\varphi_0(\mathbf{x}) - \frac{\beta_1 A + \beta_2}{\lambda_1^2}\varphi_1(\mathbf{x})\right], \quad \mathbf{x} \in D, \tag{9}$$

where the function φ_0 must satisfy the condition $\Delta\varphi_0 = \varphi$. Thus, $\Delta\Delta\varphi_0 = 0$.

We will study separately the following BVPs:

Problem $\mathbf{B}_1$. Find in the domain D the regular solutions of system (6) satisfying the following boundary conditions

$$\varphi^+(\mathbf{z}) = h(\mathbf{z}), \quad \varphi_1^+(\mathbf{z}) = h_1(\mathbf{z}), \quad \mathbf{z} \in S, \tag{10}$$

respectively, where $h(\mathbf{z})$ and $h_1(\mathbf{z})$ are known functions defined by formulas

$$h(\mathbf{z}) = \frac{1}{k_0}[(k_1 + k_{21})f_4(\mathbf{z}) + (k_2 + k_{12})f_5(\mathbf{z})],$$

$$h_1(\mathbf{z}) = \frac{1}{k_0}(k_1 + k_{21})[f_5(\mathbf{z}) - f_4(\mathbf{z})].$$

Problem B$_2$. Find in the domain D the solutions of system (6) satisfying the following boundary conditions

$$\left(\frac{\partial \varphi}{\partial R}\right)^+ = h_2(\mathbf{z}), \quad \left(\frac{\partial \varphi_1}{\partial R}\right)^+ = h_3(\mathbf{z}), \quad \mathbf{z} \in S, \tag{11}$$

respectively, where

$$h_2(\mathbf{z}) = \frac{g_1(\mathbf{z}) + g_2(\mathbf{z})}{k_0},$$

$$h_3(\mathbf{z}) = \frac{(k_1 + k_{21})[(k_1 + k_{12})g_2(\mathbf{z}) - (k_2 + k_{21})g_1(\mathbf{z})]}{k_0(k_1k_2 - k_{12}k_{21})}.$$

Problem A$_1$. Find in the domain D a regular solution $\mathbf{v}(\mathbf{x})$ of equation (8), satisfying the following boundary condition

$$\mathbf{v}^+(\mathbf{z}) = \mathbf{\Psi}(\mathbf{z}) - \mathbf{v}_0(\mathbf{z}) = \boldsymbol{\omega}(\mathbf{z}), \quad \mathbf{z} \in S. \tag{12}$$

Problem A$_2$. Find in the domain D a solution $\mathbf{v}(\mathbf{x})$ of equation (8), satisfying the following boundary condition

$$[\mathbf{T}(\partial \mathbf{z}, \mathbf{n})\mathbf{v}(\mathbf{z})]^+ = \boldsymbol{G}(\mathbf{z}) - \mathbf{T}(\partial \mathbf{z}, \mathbf{n})\mathbf{v}_0(\mathbf{z}) + \mathbf{n}[\beta_1 p_1(\mathbf{z}) + \beta_2 p_2(\mathbf{z})] = \mathbf{\Omega}(\mathbf{z}), \quad \mathbf{z} \in S. \tag{13}$$

In problems A_1 and A_2 the functions $p_1(\mathbf{x})$ and $p_2(\mathbf{x})$ are solutions of problems B_1 and B_2 respectively.

3. Explicit solutions of the boundary value problems

3.1. *Problem B_1.*

Let us introduce the equalities of spherical coordinates

$$x_1 = \rho \sin\vartheta \cos\eta, \quad x_2 = \rho \sin\vartheta \sin\eta, \quad x_3 = \rho\cos\vartheta, \quad x \in D^+,$$

$$y_1 = R\sin\vartheta_0 \cos\eta_0, \quad y_2 = R\sin\vartheta_0 \sin\eta_0, \quad y_3 = R\cos\vartheta_0, \quad y \in S,$$

$$|x|^2 = \rho^2 = x_1^2 + x_2^2 + x_3^2, \quad 0 \le \vartheta \le \pi, \quad 0 \le \eta \le 2\pi \quad 0 \le \rho \le R.$$

Let us expand the functions h and h_1 in spherical harmonics

$$h(\mathbf{z}) = \sum_{n=0}^{\infty} h_n(\vartheta, \eta), \quad h_1(\mathbf{z}) = \sum_{n=0}^{\infty} h_{1n}(\vartheta, \eta),$$

where h_n and h_{1n} are the spherical harmonics of order n :

$$h_n = \frac{2n+1}{4\pi R^2} \int\limits_S P_n(\cos\gamma) h(\mathbf{y}) d_y S,$$

$$h_{1n} = \frac{2n+1}{4\pi R^2} \int\limits_S P_n(\cos\gamma) h_1(\mathbf{y}) d_y S,$$

P_n is Legender polynomial of the n-th order, γ is an angle formed by the radius-vectors Ox and Oy,

$$\cos\gamma = \frac{1}{|\mathbf{x}||\mathbf{y}|} \sum_{k=1}^{3} x_k y_k.$$

For the unknowns harmonic function $\varphi(\mathbf{x})$ and metaharmonic function $\varphi_1(\mathbf{x})$ we obtain the Dirichlet BVPs for system (6) with boundary conditions (10). The solutions of Problem B_1 in D have the form [26,27]:

$$\begin{aligned} \varphi(\mathbf{x}) &= \sum_{n=0}^{\infty} \frac{\rho^n}{R^n} h_n(\vartheta,\eta), \quad \rho < R, \\ \varphi_1(\mathbf{x}) &= \sum_{n=0}^{\infty} \phi_n^{(1)}(\lambda_1 \rho) h_{1n}(\vartheta,\eta), \quad \rho < R, \end{aligned} \tag{14}$$

respectively, where

$$\phi_n^{(1)}(\lambda_1\rho) = \frac{\sqrt{R} J_{n+\frac{1}{2}}(\lambda_1\rho)}{\sqrt{\rho} J_{n+\frac{1}{2}}(\lambda_1 R)},$$

$J_{n+\frac{1}{2}}(\lambda_1\rho)$ is the Bessel function.

If we substitute the values of $\varphi(\mathbf{x})$ and $\varphi_1(\mathbf{x})$ from (14) into (5), we find the functions $p_1(\mathbf{x})$ and $p_2(\mathbf{x})$ in D.

For the solution of equation $\Delta\varphi_0 = \varphi$, where φ is given by (14), we have

$$\varphi_0(\mathbf{x}) = \frac{1}{2} \sum_{n=0}^{\infty} \frac{\rho^{n+2} h_n(\vartheta,\eta)}{(3+2n)R^n}, \quad \mathbf{x} \in D. \tag{15}$$

3.2. *Problem B_2.*

For the unknowns harmonic function $\varphi(\mathbf{x})$ and metaharmonic function $\varphi_1(\mathbf{x})$ we obtain the Neumann BVPs for system (6) with boundary conditions (11). The solutions of Problem B_2 in D have the form [27,28]:

$$\varphi(\mathbf{x}) = C + \sum_{n=1}^{\infty} \frac{\rho^n}{nR^{n-1}} h_{2n}(\vartheta, \eta), \quad \rho < R,$$
$$\varphi_1(\mathbf{x}) = \sum_{n=0}^{\infty} \frac{\phi_n^{(1)}(\lambda_1\rho) h_{3n}(\vartheta, \eta)}{H_n(R)}, \quad \rho < R. \tag{16}$$

respectively, where

$$H_n(\rho) = \frac{\partial}{\partial\rho}\phi_n^{(1)}(\lambda_1\rho), \quad \phi_n^{(1)}(\lambda_1\rho) = \frac{J_{n+\frac{1}{2}}(\lambda_1\rho)}{\sqrt{\rho}},$$

$$h_{2n} = \frac{2n+1}{4\pi R^2}\int\limits_S P_n(\cos\gamma) h_2(\mathbf{y}) d_y S,$$

$$h_{3n} = \frac{2n+1}{4\pi R^2}\int\limits_S P_n(\cos\gamma) h_3(\mathbf{y}) d_y S,$$

C is an arbitrary constant.

For the solution to exist it is necessary that the condition

$$h_{20} = \int\limits_S h_2(\mathbf{y}) ds = 0$$

be fulfilled.

If we substitute the values of $\varphi(\mathbf{x})$ and $\varphi_1(\mathbf{x})$ from (16) into (5), we find the functions $p_1(\mathbf{x})$ and $p_2(\mathbf{x})$ in D.

For the solution of equation $\Delta\varphi_0 = \varphi$, where φ is given by (16), we have

$$\varphi_0(\mathbf{x}) = \frac{\rho^2}{6}C + \frac{1}{2}\sum_{n=1}^{\infty} \frac{\rho^{n+2} h_{2n}(\vartheta, \eta)}{n(3+2n)R^{n-1}}, \quad \mathbf{x} \in D. \tag{17}$$

3.3. *Problem A_1.*

For a ball the solution $\mathbf{v}(\mathbf{x})$ of equation (8), with boundary condition (12) is given in [29,30] in the following form

$$\rho\mathbf{v}(\mathbf{x}) = \mathbf{x}\psi_1(\mathbf{x}) + \left[\frac{\partial\psi_2(\mathbf{x})}{\partial s} \cdot \mathbf{x}\right] + \rho\frac{\partial\psi_3(\mathbf{x})}{\partial s}, \quad \mathbf{x} \in D, \tag{18}$$

where

$$\frac{\partial}{\partial s} = \left(\frac{\partial}{\partial s_1}, \frac{\partial}{\partial s_2}, \frac{\partial}{\partial s_3}\right); \quad \frac{\partial}{\partial s_k(\mathbf{x})} = [\mathbf{x} \cdot \nabla]_k, \quad k = 1, 2, 3;$$

∇ is the Hamiltonian operator and the functions $\psi_j, \;\; j=1,2,3$ are represented in the following form

$$\psi_1(\mathbf{x}) = \frac{\rho}{R} f_0(\theta,\eta) + \frac{a-1}{2(a+1)} \sum_{n=1}^{\infty} \left\{ \left[\frac{(n+b)(n+1)}{n+\alpha} \left(\frac{\rho}{R}\right)^{n+1} \right.\right.$$
$$\left.\left. -\frac{(n+c)n}{n+\alpha} \left(\frac{\rho}{R}\right)^{n-1} \right] f_n(\theta,\eta) + \frac{n+b}{n+\alpha} \left[\left(\frac{\rho}{R}\right)^{n+1} - \left(\frac{\rho}{R}\right)^{n-1} \right] F_n(\theta,\eta) \right\},$$
$$\psi_2(\mathbf{x}) = \frac{a-1}{2(a+1)} \sum_{n=1}^{\infty} \left\{ \left[\frac{(n+c)}{(n+1)(n+\alpha)} \left(\frac{\rho}{R}\right)^{n+1} \right.\right. \tag{19}$$
$$\left.\left. -\frac{(n+b)}{n(n+\alpha)} \left(\frac{\rho}{R}\right)^{n-1} \right] F_n(\theta,\eta) + \frac{n+c}{n+\alpha} \left[\left(\frac{\rho}{R}\right)^{n+1} - \left(\frac{\rho}{R}\right)^{n-1} \right] f_n(\theta,\eta) \right\},$$
$$\psi_3(\mathbf{x}) = -\sum_{n=1}^{\infty} \frac{1}{n(n+1)} \left(\frac{\rho}{R}\right)^{n} \Phi_n(\theta,\eta).$$

Here

$$a = \frac{\mu}{\lambda+2\mu}, \quad \alpha = \frac{a}{a+1} < 1, \;\; b = \frac{2a}{a-1}, \;\; c = \frac{a-3}{a-1},$$

$$f_n = \frac{2n+1}{4\pi R^2} \int_S P_n(\cos\gamma) f(\mathbf{y}) d_y S, \quad F_n = \frac{2n+1}{4\pi R^2} \int_S P_n(\cos\gamma) F(\mathbf{y}) d_y S,$$
$$\Phi_n = \frac{2n+1}{4\pi R^2} \int_S P_n(\cos\gamma) \Phi(\mathbf{y}) d_y S.$$

The functions $f, \;\; F$, and Φ are given by the formulas

$$f(\mathbf{z}) = \frac{1}{R} \sum_{k=1}^{3} z_k \omega_k(\mathbf{z}), \quad F(\mathbf{z}) = \sum_{k=1}^{3} \left[\frac{\partial}{\partial s_k(\mathbf{x})} \frac{[\mathbf{x}\cdot\boldsymbol{\omega}(\mathbf{x})]_k}{\rho} \right]_{\rho=R},$$
$$\Phi(\mathbf{z}) = \sum_{k=1}^{3} \left[\frac{\partial \omega_k(\mathbf{x})}{\partial s_k} \right]_{\rho=R}, \quad \mathbf{z}\in S.$$

Note that $F_0 = 0, \;\; \Phi_0 = 0$.

Thus, the solutions of the Problem 1 for the sphere is represented by formulas (5),(7),(14),(15),(9),(18), (19).

For absolutely and uniformly convergence of obtained series together with their first derivatives it is sufficient to assume that

$$\omega(\mathbf{z}) \in C^5(S), \quad h(\mathbf{z}) \in C^5(S), \;\; h_1(\mathbf{z}) \in C^5(S).$$

Solutions obtained under such conditions are regular in D.

3.4. *Problem A_2.*

For a ball the solution of equation (8), with boundary condition (13) was constructed in [29,30] and it is represented as (18), where

$$\psi_1(\mathbf{x}) = \frac{l(x)}{\rho} + \frac{f_0}{3\mu+2\mu} + c_1\frac{\rho^2-R^2}{R}f_1(\theta,\eta)$$

$$+\sum_{n=2}^{\infty}\frac{1}{2\Delta_n}\left(\frac{\rho}{R}\right)^{n-1}\left\{\left[A_{1n}\left(\frac{\rho}{R}\right)^2+A_{2n}\right]R[A_{3n}f_n(\theta,\eta)+A_{4n}F_n(\theta,\eta)]\right.$$

$$\left.+\frac{\rho^2-R^2}{R}A_{5n}(A_{6n}F_n(\theta,\eta)+n(n+1)A_{4n}f_n(\theta,\eta))\right\},$$

$$\psi_2(\mathbf{x}) = \frac{l(x)}{\rho} - \frac{R}{c_2}\left[(2-a)\left(\frac{\rho}{R}\right)^2+3a-1\right]f_1(\theta,\eta) \tag{20}$$

$$-\sum_{n=2}^{\infty}\frac{1}{2n(n+1)\Delta_n}\left(\frac{\rho}{R}\right)^{n-1}\left\{(n+1)A_{2n}\frac{\rho^2-R^2}{R}[A_{3n}f_n(\theta,\eta)+A_{4n}F_n(\theta,\eta)]\right.$$

$$\left.+R\left[A_{2n}\left(\frac{\rho}{R}\right)^2+(n+1)A_{5n}\right][A_{6n}F_n(\theta,\eta)+n(n+1)A_{4n}f_n(\theta,\eta)]\right\},$$

$$\psi_3(\mathbf{x}) = q(x) - \frac{R}{\mu}\sum_{n=2}^{\infty}\frac{1}{n(n^2-1)}\left(\frac{\rho}{R}\right)^n\Phi_n(\theta,\eta).$$

Here

$$f_1 = F_1, \quad A_{1n} = [(a-1)n+2a](n+1), \quad A_{2n} = [(1-a)n+3-a]n,$$

$$A_{3n} = 2n^2+(1-2a)n-2a, \quad A_{4n} = 2an+2a-3, \quad A_{5n} = (a-1)n+2a,$$

$$A_{6n} = 2n^2+(1-4a)n+3-4a, c_1 = (3a-1)c_2^{-1}, \quad c_2 = 2\mu(3-4a),$$

$$\Delta_n = 2\mu(n-1)[2(1-a^2)n^3+4(1-2a^2)n^2+(3+3a-10a^2)n+a(3-4a)],$$

$$l = (\mathbf{x}\cdot\mathbf{b}), \quad q = (\mathbf{x}\cdot\mathbf{q}), \quad \Delta_n \neq 0, \quad n = 2,3,...;$$

$\mathbf{q}(q_1,q_2,q_3)$ and $\mathbf{b}(b_1,b_2,b_3)$ are arbitrary constant vectors,

$$f_n = \frac{2n+1}{4\pi R^2}\int_S P_n(\cos\gamma)f(\mathbf{y})d_yS, \quad F_n = \frac{2n+1}{4\pi R^2}\int_S P_n(\cos\gamma)F(\mathbf{y})d_yS,$$

$$\Phi_n = \frac{2n+1}{4\pi R^2}\int_S P_n(\cos\gamma)\Phi(\mathbf{y})d_yS.$$

The functions $f,\ \ F,\ $ and Φ are given in the form

$$f(\mathbf{z}) = \frac{1}{R}\sum_{k=1}^{3} z_k\Omega_k(\mathbf{z}), \quad F(\mathbf{z}) = \sum_{k=1}^{3}\left[\frac{\partial}{\partial s_k(\mathbf{x})}\frac{[\mathbf{x}\cdot\mathbf{\Omega}(\mathbf{x})]_{\mathbf{k}}}{\rho}\right]_{\rho=R},$$

$$\Phi(\mathbf{z}) = \sum_{k=1}^{3}\left[\frac{\partial\Omega_k(\mathbf{x})}{\partial s_k(\mathbf{x})}\right]_{\rho=R}, \quad \mathbf{z}\in S.$$

For determining the stress vector we obtain

$$\mathbf{T}(\partial\mathbf{x},\mathbf{n})\mathbf{v}(\mathbf{x}) = 2\mu\frac{\partial\mathbf{v}(\mathbf{x})}{\partial n} + \lambda\mathbf{n}div\mathbf{v}(\mathbf{x}) + \mu[\mathbf{n}\times rot\mathbf{v}(\mathbf{x})],$$

where the vector $\mathbf{v}$ is determined by (18).

Thus, the solution of Problem A_2 for the sphere is represented by formulas (5),(7), (16), (17),(9),(18),(20).

For absolutely and uniformly convergence of series in (20), together with their first derivatives, it is sufficient to assume that $\mathbf{\Omega}\in C^5(S),\ \ |\mathbf{\Omega}|\leq\frac{1}{n^4}$. Solutions of Problem A_2 obtained under such conditions are regular in D. For the solution to exist it is necessary that the conditions $F_0=\Phi_0=\Phi_1=0, \quad f_1=F_1$ be fulfilled.

Note that Problem 2 is solvable if the principal vector and the principal moment of external stresses are equal to zero

$$\int_S \mathbf{\Omega}(\mathbf{y})dS = 0, \quad \int_S [\mathbf{y}\times\mathbf{\Omega}(\mathbf{y})]dS = 0.$$

References

[1] Wilson R.K. and Aifantis E.C., *On the theory of consolidation with double porosity-I*, International Journal of Engineering Science, **20** (1982), 1009-1035

[2] Beskos D.E. and Aifantis E.C., *On the theory of consolidation with double porosity-II*, International Journal of Engineering Science, **24** (1986), 1697-1716

[3] Khaled M.Y., Beskos D.E. and Aifantis E.C., *On the theory of consolidation with double porosity-III*, International Journal for Numerical and Analytical Methods in Geomechanics, **8**, 2 (1984), 101-123

[4] R. De Boer., *Theory of Porous Media: Highlights in the historical development and current state*, Springer, Berlin-Heidelberg- New York, 2000

[5] Khalili N., Valliappan S., *Unified theory of flow and deformation in double porous media*, European Journal of Mechanics, A/Solids, **15** (1996), 321-336

[6] Khalili N., *Coupling effect in double porosity media with deformable matrix*, Geophysical Research Letters, **30** (2003), 21-53

[7] Khalili N., Selvadurai A.P.S., *A fully coupled constitutive model for thermo-hydro-mechanical analysis in elastic media with double porosity*, Geophysical Research Letters, **30** (2003), 2268

[8] Khalili N. and Selvadurai P.S., *On the constitutive modelling of thermo-hydro-mechanical coupling in elastic media with double porosity*, Elsevier Geo-Engineering Book Series, **2** (2004), 559-564

[9] Berryman J.G., Wang H.F., *The elastic coefficients of double porosity models for fluid transport in jointed rock*, Journal of Geophysical Research, **100** (1995), 34611-34627

[10] Berryman J.G. Wang H.F., *Elastic wave propagation and attenuation in a double porosity dual-permiability medium*, International Journal of Rock Mechanics and Mining Sciences, **37** (2000), 63-78

[11] Cowin S.C., *Bone poroelasticity*, Journal of Biomechanics, **32** (1999), 217-238

[12] Biot M.A., *Theory of finite deformations of porous solids*, Indiana University Mathematics journal, **21** (1972), 597-620

[13] Masters I., Pao W.K.S. and Lewis R.W., *Coupled temperature to a double porosity model of deformable porous media*, Int. J. for Numerical Methods in Engineering, **49** (2000), 421-438

[14] Straughan B., *Stability and uniqueness in double porosity elasticity*, Int. J. of Engineering Science, **65** (2013), 1-8

[15] Ciarletta M., Passarela F. and Svanadze M., *Plane waves and uniqueness theorems in the coupled linear theory of elasticity for solids with double porosity*, J. of elasticity, **114** (2014), 55-68.

[16] Bai M., Roegiers J.C., *Fluid flow and heat flow in deformable fractured porous media*, International Journal of Engineering Science, **32** (1994), 1615-1633

[17] Svanadze M., De Cicco S., *Fundamental solutions in the full coupled theory of elasticity for solids with double porosity*, Arch. Mech., **65**, 5 (2013), 367-390

[18] Svanadze M., *Fundamental solution in the theory of consolidation with double porosity*, J. Mech. Behavior of Materials, **16** (2005), 123-130

[19] Svanadze M.and Scalia A., *Mathematical problems in the theory of bone poroelasticity*, Int. J. on Mathem. Methods and Models in Biosciences, **1**, 2, 1211225 (2012), 1-4

[20] Svanadze M., *The boundary value problems of the full coupled theory of poroelasticity for materials with double porosity*, PAMM. Proc. Appl. Math., **12**, 279-282, DOI 10.1002/pamm.201210130

[21] Tsagareli I., Svanadze M.M., *Explicit solution of the BVPs of the theory of elasticity for solids with double porosity*, PAMM. Proc. Appl. Math. Mech., **10** (2010), 337-338

[22] Tsagareli I., Svanadze M.M., *Explicit solution of the problems of elastostatics for an elastic circle with double porosity*, Mechanics Research Communications, **46** (2012), 76-80

[23] Basheleishvili M., Bitsadze L., *The basic BVPs of the theory of consolidation with double porosity for the sphere*, Bulletin of TICMI, **16**, 1 (2012), 15-26

[24] Basheleishvili M., Bitsadze L., *Solution of the third and fourth BVPs of the theory of consolidation with double Porosity for the sphere*, Seminar of I. Vekua Institute of Applied Mathematics, reports, **36-37** (2010-2011), 5-15

[25] Basheleishvili M., Bitsadze L., *Explicit solutions of the BVPs of the theory of consolidation with double porosity for the half-space*, Bulletin of TICMI, **14** (2010), 9-15

[26] Smirnov V.I., *Course of Higher Mathematics* (Russian) Moscow, **3**, 2 (1969)

[27] Tikhonov A.N.,Samarski A.A., *Equation of Mathematical physics* (Russian), Moscow, 1966

[28] Natroshvili D.G., Svanadze M.G., *Some dinamical problems of the theory of coupled thermoelasticity for the piecewise homogeneous bodies* (Russian), Proceedings of I. Vekua Institute of Applied Mathematics, **10** (1981), 99-190

[29] Basheleishvili M.O., *Some representation of the solutions of equations of the elasticity theory*, Bull. of the Academy of Sciences of the Georgian SSR,**102**, 3 (1981), 569-571

[30] Basheleishvili M., Tsagareli I., *Effective solution of the basic BVPs of the elasticity theory for a sphere*, Bull. of the Academy of Sciences of the Georgian SSR, **108**, 1 (1982), 41-44