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Some Applications of the Theory of Self-Conjugate Differential Forms

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The concept of self-conjugate differential forms provides a generalization in $\mathbb{R}^n$ of the notion of holomorphic functions of one complex variable. In this framework several extensions of classical results hold. In this paper we present a survey of this theory and different applications concerning Laplace series, the Brother Riesz Theorem, Boundary Value Problems and Cimmino system.

Key words: Self-conjugate differential forms, Laplace series, Brother Riesz theorem, Boundary value problem, Cimmino system.

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1. Introduction

This paper deals with non homogeneous differential forms $U = \sum_{k=1}^{n} u_k$, u_k being a differential form of degree k (briefly a k-form). Following [6], we say that U is self-conjugate if $dU = \delta U$, where d and δ are the differential and codifferential operator respectively. The equation $dU = \delta U$ provides a generalization in $\mathbb{R}^n$ of the Cauchy-Riemann system. The concept of self-conjugate differential forms includes classical "real" generalizations of holomorphic functions of one complex variable, like solutions of Moisil-Teodorescu, Fueter, Cimmino systems, harmonic vectors and harmonic forms.

The aim of the present work is to show some applications of the theory of self-conjugate differential forms. The paper is organized as follows.

Section 2 presents a review of the basic notions concerning k-forms and k-measures. Section 3 provides an overall treatment of the theory of self-conjugate differential forms. In Section 4, conditions for the existence of conjugate harmonic forms in multiply connected domains of $\mathbb{R}^n$ are given. Generalizations of the Brother Riesz Theorem are presented first for Laplace series in Section 5, considering the concept of conjugate Laplace series introduced in [4], and then for conjugate differential forms in Section 6. Section 7 is devoted to some BVPs for non homogeneous differential forms. In Section 8 necessary and sufficient conditions for the resolubility of the Dirichlet problem for the Cimmino system are given.

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2. *k*-forms and *k*-measures

Let $V^m \subset \mathbb{R}^n$ be an oriented m-dimensional differential simple manifold of class p. A *differential form of degree* k, or briefly a *k-form*, on V^m is represented in an admissible coordinate system $(t_1, \dots, t_m)$, as

$$u = \frac{1}{k!} u_{s_1,\dots,s_k} dt_{s_1} \dots dt_{s_k},$$

where $u_{s_1,\dots,s_k}$ are the components of a k-covector, i.e. the components of a skew-symmetric covariant tensor.
By $C_k^q(V^m)$ we denote the space of the k-forms whose coefficients are continuously up to the order q in a coordinate system of class C^{q+1} (and then in every coordinate system of class C^{q+1}). Moreover the symbol L_k^p stands for the space of all k-forms whose coefficients are L^p real valued functions.
Let $\Omega \subset \mathbb{R}^n$ be a domain. The representation of $u \in C_k^0(\Omega)$ in the natural coordinate system is

$$u = \frac{1}{k!} u_{s_1,\dots,s_k} dx_{s_1} \dots dx_{s_k}.$$

The *adjoint* of u is the $(n-k)$-form

$$*u = \frac{1}{(n-k)!} \frac{1}{k!} \delta^{1,\dots,n}_{s_1,\dots,s_k,i_1,\dots,i_{n-k}} u_{s_1,\dots,s_k} dx_{i_1} \dots dx_{i_{n-k}}.$$

We remark that $** u = (-1)^{k(n-k)} u$.

If $u \in C_k^1(\Omega)$, the *differential* of u is the $(k+1)$-form

$$du = \frac{1}{k!} \frac{\partial}{\partial x_j} u_{s_1,\dots,s_k} dx_j dx_{s_1} \dots dx_{s_k},$$

while the *codifferential* of u is the $(k-1)$-form

$$\delta u = (-1)^{n(k+1)+1} * d * u.$$

These operators are strictly related to the Laplacian; indeed if $u \in C_k^2$

$$-(d\delta + \delta d)u = \Delta u = \frac{1}{k!} \Delta u_{s_1,\dots,s_k} dx_{s_1} \dots dx_{s_k},$$

where $\Delta u_{s_1,\dots,s_k} = \sum_{h=1}^{n} \frac{\partial^2}{\partial x_h^2} u_{s_1,\dots,s_k}$.
Fore more details about differential forms see, e.g., [14, 15].
On a C^∞ manifold the concept of k-form has been generalized to the notion of current (see [12]). A k-measure can be considered as a current of order 0. Fichera [14] showed how a k-measure can be introduced in a direct and natural way on a differentiable manifold. We recall his definition.

A *k-measure* is the object determined (in a fixed coordinate system $(t^1, \dots, t^m)$) by the coefficients $\mu_{s_1,\dots,s_k}(B)$, where $\mu_{s_1,\dots,s_k}$ is a measure defined on the family $\{B\}$ of all the Borel sets of V^m, and $\mu_{s_1,\dots,s_k}(B)$ depends skew-symmetrically on the indices $s_1, \dots, s_k$.
Moreover if $\widetilde{\mu}_{j_1,\dots,j_k}(B)$ are the components of the k-measure μ in the coordinate system $(\tau_1, \dots, \tau_m)$ we have

$$\widetilde{\mu}_{j_1,\dots,j_k}(B) = \int_B \frac{\partial t^{s_1}}{\partial \tau^{j_1}} \dots \frac{\partial t^{s_k}}{\partial \tau^{j_k}} \left| \det \frac{\partial \tau}{\partial t} \right| \mu_{s_1,\dots,s_k}(B)$$

for every Borel set $B \in \{B\}$. $\mathcal{M}_k(V^m)$ denotes the space of all k-measure on V^m. If $\mu \in \mathcal{M}_k(V^m)$, the following Lebesgue-Radon-Nycodim decomposition holds

$$\mu_{s_1,\dots,s_k}(B) = \mu^*_{s_1,\dots,s_k}(B) + \int_B f_{s_1,\dots,s_k} d\sigma,$$

where $f \in L^1_k(V^m)$ and μ^* is a singular measure in $\mathcal{M}_k(V^m)$. If $\mu^* = 0$, we say that μ is *absolutely continuous*.
If $u \in C_k(V^m)$ and $\mu \in \mathcal{M}_h(V^m)$ the exterior product $\beta = u \wedge \mu$ is the $(k+h)$-measure whose components are

$$\beta_{i_1,\dots,i_{k+h}}(B) = \frac{1}{k!h!} \delta^{s_1 \dots s_k j_1 \dots j_h}_{i_1 \dots i_{k+h}} \int_B u_{s_1,\dots s_k} d\mu_{j_1,\dots,j_h},$$

for any Borel set B; $\mu \wedge u$ is by definition $(-1)^{kh}(u \wedge \mu)$.

3. Self-conjugate differential forms

We begin with some definitions.

Definition 3.1 ([5]): We say that $u \in C^1_k(\Omega)$ and $v \in C^1_{k+2}(\Omega)$ are conjugate if

$$\begin{aligned} du &= \delta v, \\ \delta u = 0, \quad dv &= 0. \end{aligned}$$

This definition can be extended to a non homogeneous differential form $U = \sum_{k=0}^{n} u_k$, u_k being a k-form. We set

$$dU = \sum_{k=0}^{n-1} du_k, \quad \delta U = \sum_{k=1}^{n} \delta u_k, \quad \Delta U = \sum_{k=0}^{n} \Delta u_k.$$

Since $d^2 = 0$ and $\delta^2 = 0$, we can write

$$\Delta = (d - \delta)^2. \tag{1}$$

We denote by $C^k(\Omega)$ the space $C_0^k(\Omega) \oplus \ldots \oplus C_n^k(\Omega)$; similarly $L^p(\Omega) = L_0^p(\Omega) \oplus \ldots \oplus L_n^p(\Omega)$ is the space composed by k-forms whose coefficients are L^p real valued functions in Ω.

Definition 3.2 ([5, 6]) **:** We say that $U \in C^1(\Omega)$ is *self-conjugate* if

$$dU = \delta U, \qquad \text{in } \Omega,$$

i.e.

$$\delta u_1 = 0, \quad du_{k-1} = \delta u_{k+1} \ (k = 1, \ldots, n-1), \quad du_{n-1} = 0.$$

From (1) it follows that if U is self-conjugate then U is harmonic, i.e. all the coefficients of u_k are harmonic functions.
Let us consider now some examples of self-conjugate differential forms.

If $n = 2$,

$$U = u_0 + u_2,$$

where $u_0 = u$ is a scalar function and $u_2 = v dx dy$ is a 2-form, we have

$$dU = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy, \qquad \delta U = \frac{\partial v}{\partial y} dx - \frac{\partial v}{\partial x} dy.$$

Then U is self-conjugate if and only if $u + iv$ is holomorphic.

If $n = 3$,

$$U = u_0 + u_2,$$

where $u_0 = u$ and $u_2 = v_1 dx_2 dx_3 + v_2 dx_3 dx_1 + v_3 dx_1 dx_2$, U is self-conjugate if and only if (u, v_1, v_2, v_3) is solution of the Moisil-Theodorescu system:

$$\begin{cases} \operatorname{div}(v_1, v_2, v_3) = 0, \\ \operatorname{grad} u = \operatorname{curl}(v_1, v_2, v_3). \end{cases}$$

If $n = 4$,

$$U = u_0 + u_2 + u_4, \tag{2}$$

where

$$u_0 = f_0, \quad u_2 = f_1(dx_1 dx_2 - dx_3 dx_4) + f_2(dx_1 dx_3 - dx_4 dx_2) + f_3(dx_1 dx_4 - dx_2 dx_3), \quad u_4 = f_0 dx_1 dx_2 dx_3 dx_4,$$

it follows that U is self-conjugate if and only if (f_0, f_1, f_2, f_3) is solution of the

Fueter system

$$\begin{cases} f_{0_{x_1}} - f_{1_{x_2}} - f_{2_{x_3}} - f_{3_{x_4}} = 0, \\ f_{0_{x_2}} + f_{1_{x_1}} - f_{2_{x_4}} + f_{3_{x_3}} = 0, \\ f_{0_{x_3}} + f_{1_{x_4}} + f_{2_{x_1}} - f_{3_{x_2}} = 0, \\ f_{0_{x_4}} - f_{1_{x_3}} + f_{2_{x_2}} + f_{3_{x_1}} = 0. \end{cases}$$

It is known that Fueter system characterizes quaternionic hyperholomorphic functions. Consequently, the form (2) is self-conjugate if and only if (f_0, f_1, f_2, f_3) are the components of a quaternionic hyperholomorphic function.

In a similar way it is possible to find a relation between self-conjugate differential forms and solutions of the Cimmino system. Namely, if $n = 4$, let be

$$U = u_0 + u_2 + u_4,$$

where

$$u_0 = f_0, \quad u_2 = f_1(dx_1dx_2 + dx_3dx_4) - f_2(dx_1dx_3 + dx_4dx_2) \\ + f_3(dx_1dx_4 + dx_2dx_3), \quad u_4 = -f_0dx_1dx_2dx_3dx_4;$$

U is self-conjugate if and only if (f_0, f_1, f_2, f_3) is a solution of the Cimmino system (see [9])

$$\begin{cases} f_{0_{x_1}} - f_{1_{x_2}} + f_{2_{x_3}} - f_{3_{x_4}} = 0, \\ f_{0_{x_2}} + f_{1_{x_1}} - f_{2_{x_4}} - f_{3_{x_3}} = 0, \\ f_{0_{x_3}} - f_{1_{x_4}} - f_{2_{x_1}} + f_{3_{x_2}} = 0, \\ f_{0_{x_4}} + f_{1_{x_3}} + f_{2_{x_2}} + f_{3_{x_1}} = 0. \end{cases} \tag{3}$$

For any $n \geq 2$ if

$$U = u_1, \quad \text{where} \quad u_1 = w_h dx_h,$$

U is self-conjugate if and only if $(w_1, \ldots, w_n)$ is a *harmonic vector*, i.e. it is a solution of the following system

$$\begin{cases} \operatorname{div}(w_1, \ldots, w_n) = 0, \\ \operatorname{curl}(w_1, \ldots, w_n) = 0. \end{cases}$$

More generally, if

$$U = u_k,$$

U is self-conjugate if and only if u_k is a *harmonic form*, i.e.

$$\begin{cases} du_k = 0, \\ \delta u_k = 0. \end{cases}$$

The use of differential forms points out the connection between Analysis and Geometry, as we can remark by the ensuing result. It is well known that in a simply connected domain of $\mathbb{C}$ any harmonic function is the real part of a holomorphic function. The next theorem provides an extension of this result to $\mathbb{R}^n$. Here $H_k(\Omega)$ denotes the k-dimensional singular homology group of the domain $\Omega \subset \mathbb{R}^n$.

Theorem 3.3 ([6]) **:** *Let us fix $1 \leq k \leq n$. Suppose*

$$\begin{cases} H_{n-1}(\Omega) = 0, & \text{if } k = 0 \text{ or } k = n, \\ H_{k-1}(\Omega) = 0 \text{ and } H_{n-k-1}(\Omega) = 0, & \text{if } 2 \leq k \leq n-2, \\ H_{n-2}(\Omega) = 0, & \text{if } k = 1 \text{ or } k = n-1. \end{cases}$$

If u_k is a harmonic form defined in Ω such that

$$\begin{cases} \delta u_0 = 0, & & \text{if } k = 0, \\ \delta u_1 = 0, & \delta du_1 = 0, & \text{if } k = 1, \\ \delta du_k = 0, & d\delta u_k = 0, & \text{if } 2 \leq k \leq n-2, \\ du_{n-1} = 0, & d\delta u_{n-1} = 0, & \text{if } k = n-1, \\ d\delta u_n = 0, & & \text{if } k = n, \end{cases}$$

then there exists in Ω a self-conjugate form U such that u_k is the k-th component of U.

Let us consider now the double k-form introduced by Hodge

$$s_k(x,y) = \sum_{j_1 < \ldots < j_k} s(x-y) dx_{j_1} \ldots dx_{j_k} dy_{j_1} \ldots dy_{j_k},$$

where

$$s(x-y) = \begin{cases} \dfrac{1}{2\pi} \log |x-y|, & \text{if } n = 2, \\ -\dfrac{1}{(n-2)\omega_n} |x-y|^{2-n}, & \text{if } n > 2 \end{cases}$$

(ω_n being the hypersurface measure of the unit sphere of $\mathbb{R}^n$) is the fundamental solution of the Laplace equation. It satisfies the following identities for $x \neq y$ (see [10]):

$$d_y s_k(x,y) = \delta_x s_{k+1}(x,y), \qquad k = 0, \ldots, n-1,$$

from which one can prove that

$$\begin{cases} \delta_x \underset{y}{*} d_y s_k(x,y) = 0, & \delta_x d_y s_k(x,y) = 0, \\ d_x \delta_y s_k(x,y) = 0, & d_x \underset{y}{*} \delta_y s_k(x,y) = 0, \end{cases}$$

$$\begin{cases} d_x \underset{y}{*} d_y s_k(x,y) = -\delta_x \underset{y}{*} \delta_y s_{k+2}(x,y), \\ d_x d_y s_k(x,y) = -\delta_x \delta_y s_{k+2}(x,y). \end{cases}$$

Moreover,

$$\underset{x}{*} d_x s_k(x,y) = (-1)^{nk+1} \underset{y}{*} d_y s_{n-1-k}(x,y),$$

$$\underset{x}{*} s_k(x,y) = (-1)^{(n-k)k} \underset{y}{*} s_{n-k}(x,y).$$

Let now Ω be a *regular domain;* this means that Ω is a bounded domain, its boundary Σ is an orientable $(n-1)$-dimensional C^1 differentiable manifold and for any $u \in C^0_{n-1}(\overline{\Omega}) \cap C^1_{n-1}(\Omega)$ such that $du \in C^0_n(\overline{\Omega})$ the Stokes formula holds

$$\int_\Omega du = \int_{+\Sigma} u.$$

This implies

$$\int_\Omega du \wedge *v = \int_{+\Sigma} u \wedge *v + \int_\Omega \delta v \wedge *u, \quad \forall u \in C^1_k(\overline{\Omega}),\ v \in C^1_{k+1}(\overline{\Omega}).$$

If $U = \sum_{k=0}^{n} u_k \in C^0(\overline{\Omega}) \cap C^1(\Omega)$ is self-conjugate, we may write

$$\begin{cases} \displaystyle\int_\Omega dv \wedge *u_1 = \int_{+\Sigma} v \wedge *u_1, & \forall v \in C^1_0(\overline{\Omega}), \\ \displaystyle\int_\Omega [dv \wedge *u_{k+2} - \delta v \wedge *u_k] = \int_{+\Sigma} [u_k \wedge *v + v \wedge *u_{k+2}], & \\ \qquad\qquad \forall v \in C^1_{k+1}(\overline{\Omega}) \quad k = 0, \dots, n-2, & \\ \displaystyle -\int_\Omega \delta v \wedge *u_{n-1} = \int_{+\Sigma} u_{n-1} \wedge *v, & \forall v \in C^1_n(\overline{\Omega}). \end{cases}$$

Theorem 3.4 ([3, 6]) **:** *If Ω is a regular domain and $U \in C^0(\overline{\Omega}) \cap C^1(\Omega)$ is such that $dU - \delta U = F \in C^0(\overline{\Omega})$, then the following Cauchy integral formula holds*

$$\begin{aligned} -\int_\Omega [d_y s_k(x,y) \wedge *F_{k+1}(y) - \delta_y s_k(x,y) \wedge *F_{k-1}(y)] & \\ + \int_{+\Sigma} \Big[u_k(y) \wedge \underset{y}{*} d_y s_k(x,y) - \delta_y s_k(x,y) \wedge *u_k(y) & \\ + d_y s_k(x,y) \wedge *u_{k+2}(y) - u_{k-2}(y) \wedge \underset{y}{*} \delta_y s_k(x,y) \Big] & \\ = \begin{cases} u_k(x), & x \in \Omega, \\ 0, & x \notin \overline{\Omega}, \end{cases} \quad k = 0, \dots, n, & \quad (4) \end{aligned}$$

where $U = \sum_{k=0}^{n} u_k$, $F = \sum_{k=0}^{n} F_k$, $u_k \equiv 0$, $k = -2, -1, n+1, n+2$; $F_k \equiv 0$, $k = -1, n+1$.

We remark that in the case $n = 2$ formula (4) gives

$$-\frac{1}{2\pi}\int_\Omega d_\zeta \log|z-\zeta| \wedge *F_1(\zeta) - \frac{1}{2\pi}\int_\Sigma \Big[u(\zeta)\frac{\partial}{\partial n_\zeta}\log|z-\zeta| - v(\zeta)\frac{\partial}{\partial s_\zeta}\log|z-\zeta|\Big] ds_\zeta = \begin{cases} u(z), & z \in \Omega, \\ 0, & z \notin \overline{\Omega}, \end{cases}$$

$$\frac{1}{2\pi}\int_\Omega - * d_\zeta \log|z-\zeta| \wedge *F_1(\zeta) - \frac{1}{2\pi}\int_\Sigma \Big[v(\zeta)\frac{\partial}{\partial n_\zeta}\log|z-\zeta| + u(\zeta)\frac{\partial}{\partial s_\zeta}\log|z-\zeta|\Big] ds_\zeta = \begin{cases} v(z), & z \in \Omega, \\ 0, & z \notin \overline{\Omega}. \end{cases}$$

Putting $f(z) = u(z) + iv(z)$ we have the well known formula

$$\frac{1}{2\pi i}\int_{+\Sigma} \frac{f(\zeta)}{\zeta - z} d\zeta - \frac{1}{\pi}\int_\Omega \frac{f_{\overline{\zeta}}(\zeta)}{\zeta - z} d\zeta = \begin{cases} f(z), & z \in \Omega, \\ 0, & z \notin \overline{\Omega}. \end{cases}$$

This shows that (4) generalizes this classical result to $\mathbb{R}^n$.

4. Conjugate harmonic forms in multiply connected domains

Let us consider a domain Ω of the form

$$\Omega = \Omega_0 \setminus \bigcup_{j=1}^{m} \overline{\Omega}_j,$$

where Ω_j $(j = 0, \ldots, m)$ are bounded connected domains of $\mathbb{R}^n$, whose boundaries Σ_j are connected Lyapunov surfaces, such that

$$\overline{\Omega}_j \subset \Omega_0 \ \text{ and } \ \overline{\Omega}_j \cap \overline{\Omega}_k = \emptyset, \qquad j, k = 1, \ldots, m, \quad j \neq k.$$

For brevity, we shall call such a domain an $(m+1)$-*connected domain.*
If $\Omega \subset \mathbb{R}^2$ is a simply connected domain, it is well known that if u is a harmonic function, there exists a conjugate function v (i.e. $u + iv$ is holomorphic).
One can prove that if $\Omega \subset \mathbb{R}^2$ is a $(m+1)$-connected domain and $u \in C^1(\overline{\Omega}) \cap C^2(\Omega)$ is a harmonic function, there exists a conjugate harmonic function v, if and only if,

$$\int_{\Sigma_j} \frac{\partial u}{\partial n} ds = 0, \qquad j = 0, 1, \ldots, m.$$

The next theorem shows that a similar result holds in $\mathbb{R}^n$ ([8, Th 6.1]).

Theorem 4.1 : *Let $u \in C^1(\overline{\Omega})$ be a harmonic function, where $\Omega \subset \mathbb{R}^n$ is a $(m+1)$-connected domain. There exists a 2-form v conjugate to u in Ω (see Definition 3.1), if and only if,*

$$\int_{\Sigma_j} \frac{\partial u}{\partial n} d\sigma = 0, \qquad j = 0, 1, \ldots, m.$$

Moreover the 2-form v is given by

$$v(x) = - * \int_{\Sigma} \psi(y) d_y s_{n-2}(x, y) + \omega(x),$$

where ψ is the density of the double layer potential representing u and ω is an arbitrary closed and co-closed 2-form.

5. Conjugate Laplace series

Let us consider a trigonometric series

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos k\theta + b_k \sin k\theta).$$

It can be seen (at least formally) as the trace on the unit circle of the function harmonic in the unit disk

$$u(\rho, \theta) = \frac{a_0}{2} + \sum_{k=1}^{\infty} \rho^k (a_k \cos k\theta + b_k \sin k\theta).$$

The following series

$$v(\rho, \theta) = - \sum_{k=1}^{\infty} \rho^k (b_k \cos k\theta - a_k \sin k\theta) \tag{5}$$

is a harmonic function which is conjugate to u, i.e. (u, v) is a solution of the Cauchy-Riemann system $u_x = v_y$, $u_y = -v_x$. The conjugate trigonometric series is the series which we obtain taking $\rho = 1$ in (5)

$$- \sum_{k=1}^{\infty} (b_k \cos k\theta - a_k \sin k\theta).$$

In [18] F. and M. Riesz proved the important result

Theorem 5.1 : *If a trigonometric series and its conjugate series*

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos k\theta + b_k \sin k\theta), \qquad \frac{b_0}{2} - \sum_{k=1}^{\infty} (b_k \cos k\theta - a_k \sin k\theta)$$

are both Fourier-Stieltjes, then they are ordinary Fourier series.

In other words, if we have two real measures α, β defined on the Borel sets of $[0, 2\pi]$ such that

$$\int_0^{2\pi} \cos k\theta d\alpha = \int_0^{2\pi} \sin k\theta d\beta, \quad \int_0^{2\pi} \sin k\theta d\alpha = -\int_0^{2\pi} \cos k\theta d\beta \quad (k = 1, 2, \ldots),$$

then these measures have to be absolutely continuous.
The notion of conjugate Fourier series and the Brother Riesz Theorem have been generalized in $\mathbb{R}^n$ in different ways (see for instance [2, 11, 16, 17]). We describe here an extension of Theorem 5.1 hinging on a new concept of conjugate Laplace series given in [4, 7].
It is well known that if u is a harmonic function in the unit ball $B = \{x \in \mathbb{R}^n : |x| < 1\}$, it can be expanded by means of harmonic polynomials

$$u(x) = \sum_{h=0}^{\infty} |x|^h \sum_{k=1}^{p_{nk}} a_{hk} Y_{hk}\left(\frac{x}{|x|}\right),$$

where $p_{nh} = (2h + n - 2)\frac{(h+n-3)}{(n-2)!h!}$ and $\{Y_{hk}\}$ is a complete system of spherical harmonic functions. We suppose $\{Y_{hk}\}$ is orthonormal, i.e.

$$\int_\Sigma Y_{hk} Y_{rs} d\sigma = \begin{cases} 1, & \text{if } h = r \text{ and } k = s, \\ 0, & \text{otherwise.} \end{cases}$$

The trace of u on $\Sigma = \{x \in \mathbb{R}^n : |x| = 1\}$ is given by the expansion

$$\sum_{h=0}^{\infty} \sum_{k=1}^{p_{nk}} a_{hk} Y_{hk}(x) \qquad (|x| = 1). \tag{6}$$

If the coefficients a_{hk} are

$$a_{hk} = \int_\Sigma f Y_{hk} d\sigma \qquad \left(\text{resp.} \quad a_{hk} = \int_\Sigma Y_{hk} d\mu\right)$$

we say that (6) is the *Laplace series* of the function f (resp. of the measure μ).
Let us consider the 2-form

$$v = \sum_{h=0}^{\infty} \sum_{k=1}^{p_{nh}} \frac{a_{hk}}{(h+2)(n+h-2)} dY_{hk}\left(\frac{x}{|x|}\right) \wedge d\left(|x|^{h+2}\right) \tag{7}$$

and its adjoint

$$*v = \sum_{h=0}^{\infty} \sum_{k=1}^{p_{nh}} \frac{a_{hk}}{(h+2)(n+h-2)} * \left(dY_{hk}\left(\frac{x}{|x|}\right) \wedge d\left(|x|^{h+2}\right)\right). \tag{8}$$

It is possible to show that $dv = 0$, $\delta v = du$ in B, i.e. $u + v$ is self-conjugate.
We say that (8), with $|x| = 1$, is the *Laplace conjugate series* of (6). It represents

the restriction of $*v$ on Σ, while the restriction of v, provided it does exist, is equal to 0, because of (7).
In [4] the following expansion is introduced as the series conjugate to (6):

$$\sum_{h=0}^{\infty}\sum_{k=1}^{p_{nh}}\sqrt{\frac{h}{n+h-2}}a_{hk}\psi_{hk}, \tag{9}$$

with

$$\psi_{hk}=\frac{1}{(h+2)\sqrt{h(n+h-2)}}*(dY_{hk}\wedge d\rho^{h+2})=$$

$$=\frac{1}{\sqrt{h(n+h-2)}}\rho^{h+1}\sum_{j=1}^{n-1}(-1)^{n-1-j}\sqrt{g}g^{jj}\frac{\partial Y_{hk}}{\partial\phi_j}d\phi_1\ldots\widehat{j}\ldots d\phi_{n-1},$$

where $\phi_1,\ldots,\phi_{n-1}$ are the usual polar coordinates and g^{ij} is the inverse matrix of the relevant metric tensor g_{ij} on Σ.
The next result provides a Brother Riesz theorem for Laplace series.

Theorem 5.2 ([4]) **:** *Let* (6) *be a Laplace series of a measure* $\mu\in\mathcal{M}_0(\Sigma)$. *If its conjugate series* (9) *is a Fourier series of a* $(n-2)$*-measure, i.e. there exists* $\beta\in\mathcal{M}_{n-2}(\Sigma)$ *such that*

$$a_{hk}=\sqrt{\frac{n+h-2}{h}}\int_{+\Sigma}\beta\wedge\underset{\Sigma}{*}\psi_{hk}\qquad(h=1,2,\ldots;\,k=1,\ldots p_{nh})$$

($\underset{\Sigma}{*}\psi$ *denotes the adjoint of* ψ *on* Σ *with respect to the usual metric on* Σ*) and*

$$\int_{+\Sigma}\beta\wedge\underset{\Sigma}{*}\gamma=0,\qquad\forall\gamma\in C^{\infty}_{n-2}(\mathbb{R}^n):d\gamma=0\ \text{on}\ \Sigma,$$

then μ *and* β *are absolutely continuous with respect to the Lebesgue* $(n-1)$*-dimensional measure on* Σ.

6. The Brother Riesz theorem for conjugate differential forms

In this section we lay out the results obtained in [5].
The Brother Riesz theorem 5.1 can be reformulated in the following way.

Theorem 6.1 : *If* $u(x,y)$ *and* $v(x,y)$ *are two conjugate real functions in a planar domain* Ω *and both of them have traces on* $\partial\Omega$ *in the sense of measures, then these measures have to be absolutely continuous with respect to the arc-length measure on* $\partial\Omega$.

In order to obtain a generalization of this statement in $\mathbb{R}^n$ for conjugate differential forms, we introduce the concept of the trace in the sense of k-measures (see [5]). To this aim let us construct a family of approximating domains Ω_ρ. Let $\lambda(x)$ be a unit vector defined and continuously differentiable on Σ such that

$\lambda(x)\cdot\nu(x)\geq p_0>0$, $\forall x\in\Sigma$. By Σ_ρ we denote the hypersurface $x_\rho=x+\rho\lambda(x)$, $x\in\Sigma$, where $|\rho|\leq\rho_0$ (ρ_0 small enough). Ω_ρ is the bounded domain whose boundary is Σ_ρ. The domains Ω_ρ with $0<\rho\leq\rho_0$ are contained in Ω, while Ω is contained in Ω_ρ with $-\rho_0\leq\rho<0$. We call $\{\Sigma_\rho\}$ a *family of approximating hypersurfaces.*

Definition 6.2: We say that the k-form $u\in C_k(\Omega)$ admits a trace $\alpha\in\mathcal{M}_k(\Sigma)$ in the sense of k-measures with respect to the approximating family $\{\Sigma_\rho\}$ if

$$\lim_{\rho\to0^+}\int_{+\Sigma_\rho}p\wedge u=\int_{+\Sigma}p\wedge\alpha,\qquad\forall p\in C^\infty_{n-1-k}(\overline{\Omega}).$$

Concerning conjugate differential forms admitting traces in this sense, we have:

Theorem 6.3: *Let $u\in C^1_k(\Omega)$ and $v\in C^1_{k+2}(\Omega)$ be two conjugate forms. Let us suppose that they and their adjoint forms admit traces in the sense of h-measures with respect to the approximating family $\{\Sigma_\rho\}$. Namely let $\alpha\in\mathcal{M}_k(\Sigma)$, $\widetilde{\alpha}\in\mathcal{M}_{n-k}(\Sigma)$, $\beta\in\mathcal{M}_{k+2}(\Sigma)$, $\widetilde{\beta}\in\mathcal{M}_{n-k-2}(\Sigma)$, the traces of $u,*u,v,*v$, respectively. Then the following formulas hold*

$$\int_{+\Sigma}\left[\alpha_y\wedge\underset{y}{*}d_ys_k(y,x)-\delta_ys_k(y,x)\wedge\widetilde{\alpha}_y+d_ys_k(y,x)\wedge\widetilde{\beta}_y\right]=\begin{cases}u(x), & x\in\Omega,\\ 0, & x\notin\overline{\Omega},\end{cases}$$

$$\int_{+\Sigma}\left[\beta_y\wedge\underset{y}{*}s_{k+2}(y,x)-\delta_ys_{k+2}(y,x)\wedge\widetilde{\beta}_y-\alpha_y\wedge\underset{y}{*}\delta_ys_{k+2}(y,x)\right]=\begin{cases}v(x), & x\in\Omega,\\ 0, & x\notin\overline{\Omega}.\end{cases}$$

Moreover u, $*u$, v, $*v$ admit the same traces α, $\widetilde{\alpha}$, β, $\widetilde{\beta}$ with respect to any approximating family $\{\Sigma_\rho\}$.

We can now consider the generalization of the Brother Riesz Theorem for conjugate differential forms.

Theorem 6.4: *Let $u\in C^1_k(\Omega)$ and $v\in C^1_{k+2}(\Omega)$ be two conjugate forms. Let us suppose that u, $*u$, v, $*v$ admit traces in the sense of h-measures with respect to one approximating family (and therefore to any approximating family). Namely let $\alpha\in\mathcal{M}_k(\Sigma)$, $\widetilde{\alpha}\in\mathcal{M}_{n-k}(\Sigma)$, $\beta\in\mathcal{M}_{k+2}(\Sigma)$, $\widetilde{\beta}\in\mathcal{M}_{n-k-2}(\Sigma)$, be the traces of $u,*u,v,*v$, respectively. Then all these h-measures are absolutely continuous with respect to the Lebesgue $(n-1)$-dimensional measure on Σ.*

It is possible to extend this theorem to non homogeneous differential forms. We denote by $\mathcal{M}_0\oplus\cdots\oplus\mathcal{M}_{n-1}$ the space of non homogeneous k-measures $\Theta=\sum_{h=0}^{n-1}\theta_h$, where $\theta_h\in\mathcal{M}_h$. We say that a non homogeneous differential form $U=\sum_{h=0}^{n}u_h$ admits trace Θ in the sense of k-measures if $u_h(h=0,\ldots,n-1)$ admits a trace $\theta_h\in\mathcal{M}_h$ in the sense introduced before.

Theorem 6.5: *If the self-conjugate form $U\in C^1_0(\Omega)\oplus\ldots\oplus C^1_n(\Omega)$ is such that U and $*U$ admit traces on Σ in the sense of k-measures with respect to one approximating family (and therefore to any approximating family), then these traces are absolutely continuous with respect to the Lebesgue $(n-1)$-dimensional measure on Σ.*

7. The Dirichlet problem for the equation $dU - \delta U = F$

We say that $U \in L^1_{\rm loc}(\Omega)$ is a weak solution of $dU - \delta U = F \in L^1_{\rm loc}(\Omega)$ if

$$\int_\Omega (d\phi - \delta\phi) \wedge *U = -\int_\Omega F \wedge *\phi, \qquad \forall \phi \in \mathring{C}^\infty(\Omega). \tag{10}$$

We shall be interested in seeing whether the Dirichlet problem for the equation $dU - \delta U = F$ admits a solution. Next results give necessary and sufficient conditions for the existence of this solution. For the proofs we refer to [3].
In what follows, the symbol $L^1(\Sigma)$ stands for $L^1_0(\Sigma) \oplus \ldots \oplus L^1_{n-1}(\Sigma)$.
Let us denote by $w_h^{i_1,\ldots,i_k}$ the k-form $w_h dx_{i_1} \ldots dx_{i_k}$, where $\{w_h\}$ is a complete system of homogeneous harmonic polynomials. Such a system can be obtained by ordering in one sequence the polynomials:

$$|x|^k Y_s^k\left(\frac{x}{|x|}\right), \qquad k = 0, 1, 2, \ldots;$$

$$s = 1, \ldots, p_{nk}; \qquad p_{nk} = (2k + n - 2)\frac{(k + n - 3)!}{(n - 2)!k!},$$

where $Y_1^k(\omega), \ldots, Y_{p_{nk}}^k(\omega)$ is a complete system of (surface) spherical harmonics of degree k.

Theorem 7.1: *Let Ω be a regular domain such that $\mathbb{R}^n \setminus \overline{\Omega}$ is connected. Let $\phi = \sum_{k=0}^{n-1} \phi_k$, $\widetilde{\phi} = \sum_{k=0}^{n-1} \widetilde{\phi}_{n-k} \in L^1(\Sigma)$ and $F = \sum_{k=0}^{n} F_k \in L^1(\Omega)$ be given forms. There exists a non homogeneous differential form $U \in L^1(\Omega)$ solution[1] of*

$$\begin{cases} dU - \delta U = F, & \text{in } \Omega, \\ U = \phi, \quad *U = \widetilde{\phi}, & \text{on } \Sigma, \end{cases} \tag{11}$$

if and only if

$$\begin{aligned} (-1)^{(n-1)(k-1)+1} \int_\Omega \left[*F_{k+1} \wedge dw_h^{i_1,\ldots,i_k} - *F_{k-1} \wedge \delta w_h^{i_1,\ldots,i_k} \right] \\ - \frac{1}{2} \left\{ \int_{+\Sigma} \Big[\phi_k \wedge *dw_h^{i_1,\ldots,i_k} - \delta w_h^{i_1,\ldots,i_k} \wedge \widetilde{\phi}_k \right. \\ + dw_h^{i_1,\ldots,i_k} \wedge \widetilde{\phi}_{k+2} \\ \left. - \phi_{k-2} \wedge *\delta w_h^{i_1,\ldots,i_k} \Big] \right\} = 0 \end{aligned} \tag{12}$$

for any $1 \le i_1 < \ldots < i_k \le n, h = 1, 2, \ldots,\ k = 0, 1, \ldots, n$ ($\phi_k \equiv 0, k = -2, -1; \widetilde{\phi}_k \equiv 0, k = n + 1, n + 2; F_k \equiv 0, k = -1, n + 1$).

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[1] In the problem (11) the equation $dU - \delta U = F$ is considered in the weak sense (10).

It is possible to generalize the previous result to multiply connected domains. In this case, we have to add supplementary conditions to (12) for each hole.

Theorem 7.2: *Let $\Omega = \Omega_0 \setminus \bigcup_{j=1}^{m} \overline{\Omega}_j$ be an $(m+1)$-connected domain. Let $\phi = \sum_{k=0}^{n-1} \phi_k,\ \widetilde{\phi} = \sum_{k=0}^{n-1} \widetilde{\phi}_{n-k} \in L^1(\Sigma)$ and $F = \sum_{k=0}^{n} F_k \in L^1(\Omega)$ be given forms. There exists a non homogeneous differential form $U \in L^1(\Omega)$ solution of*

$$\begin{cases} dU - \delta U = F, & in\ \Omega, \\ U = \phi, \quad *U = \widetilde{\phi}, & on\ \Sigma, \end{cases} \tag{13}$$

if and only if

$$\begin{aligned} (-1)^{(n-1)(k-1)+1} \int_\Omega \left[*F_{k+1} \wedge dw_h^{i_1,\dots,i_k} - *F_{k-1} \wedge \delta w_h^{i_1,\dots,i_k} \right] \\ - \frac{1}{2} \left\{ \int_{+\Sigma} \Big[\phi_k \wedge *dw_h^{i_1,\dots,i_k} - \delta w_h^{i_1,\dots,i_k} \wedge \widetilde{\phi}_k \right. \\ \left. + dw_h^{i_1,\dots,i_k} \wedge \widetilde{\phi}_{k+2} - \phi_{k-2} \wedge *\delta w_h^{i_1,\dots,i_k} \Big] \right\} = 0, \end{aligned}$$

$$\begin{aligned} &(-1)^{(n-1)(k-1)+1} \int_\Omega \Big[*F_{k+1}(y) \wedge d_y[|y-x^j|^{2-n-2k} w_h^{i_1,\dots,i_k}(y-x^j)] \\ &- * F_{k-1}(y) \wedge \delta_y[|y-x^j|^{2-n-2k} w_h^{i_1,\dots,i_k}(y-x^j)] \Big] \\ &- \frac{1}{2} \left\{ \int_{+\Sigma} \Big[\phi_k(y) \wedge \underset{y}{*} d_y[|y-x^j|^{2-n-2k} w_h^{i_1,\dots,i_k}(y-x^j)] \right. \\ &- \delta_y[|y-x^j|^{2-n-2k} w_h^{i_1,\dots,i_k}(y-x^j)] \wedge \widetilde{\phi}_k(y) \\ &+ d_y[|y-x^j|^{2-n-2k} w_h^{i_1,\dots,i_k}(y-x^j)] \wedge \widetilde{\phi}_{k+2}(y) \\ &\left. -\phi_{k-2}(y) \wedge \underset{y}{*} \delta_y[|y-x^j|^{2-n-2k} w_h^{i_1,\dots,i_k}(y-x^j)] \Big] \right\} = 0, \qquad j = 1,\dots,m, \end{aligned}$$

for any $1 \le i_1 < \dots < i_k \le n, h = 1,2,\dots,\ k = 0,1,\dots,n$ ($\phi_k \equiv 0, k = -2,-1; \widetilde{\phi}_k \equiv 0, k = n+1, n+2; F_k \equiv 0, k = -1, n+1$). Here x^j is a fixed point in Ω_j $(j = 1,\dots,m)$.

The following result plays a key role in obtaining Theorems 7.1 and 7.2.
It concerns the following two spaces of non homogeneous differential forms:

$$\mathcal{U} = \left\{ U \in L^1(\Omega) : \exists\ \phi = \sum_{k=0}^{n-1} \phi_k,\ \widetilde{\phi} = \sum_{k=0}^{n-1} \widetilde{\phi}_{n-k} \in L^1(\Sigma),\ F = \sum_{k=0}^{n} F_k \in L^1(\Omega) \right.$$

$$\text{such that } \sum_{k=0}^{n-1} \int_\Omega dv_k \wedge *u_{k+1} - \sum_{k=1}^{n} \int_\Omega \delta v_k \wedge *u_{k-1} - \sum_{k=0}^{n} \int_\Omega v_k \wedge *F_k$$

$$= \sum_{k=0}^{n-1} \int_{+\Sigma} v_k \wedge \widetilde{\phi}_{k+1} + \sum_{k=1}^{n} \int_{+\Sigma} \phi_{k-1} \wedge *v_k \qquad \text{for any } \; V = \sum_{k=0}^{n} v_k \in C^1(\mathbb{R}^n) \Bigg\},$$

$$\mathcal{V} = \Bigg\{ U \in L^1(\Omega) : \exists\, \phi = \sum_{k=0}^{n-1} \phi_k,\ \widetilde{\phi} = \sum_{k=0}^{n-1} \widetilde{\phi}_{n-k} \in L^1(\Sigma)\ , F = \sum_{k=0}^{n} F_k \in L^1(\Omega)$$

$$\text{such that } \; - \int_{\Omega} [d_y s_k(x,y) \wedge *F_{k+1}(y) - \delta_y s_k(x,y) \wedge *F_{k-1}(y)]$$

$$+ \int_{+\Sigma} \Big[\phi_k(y) \wedge \underset{y}{*} d_y s_k(x,y) - \delta_y s_k(x,y) \wedge *\phi_k(y) + d_y s_k(x,y) \wedge *\phi_{k+2}(y)$$

$$- \phi_{k-2}(y) \wedge \underset{y}{*} \delta_y s_k(x,y) \Big] = \begin{cases} u_k(x) & x \in \Omega \\ 0 & x \notin \overline{\Omega} \end{cases} k = 0, \ldots, n$$

$$(\phi_k \equiv 0, k = -2, -1; \widetilde{\phi}_k \equiv 0, k = n+1, n+2; F_k \equiv 0, k = -1, n+1) \Bigg\}.$$

Roughly speaking the space $\mathcal{U}$ is given by the L^1 differential forms solutions of $dU - \delta U = F$ in Ω having L^1 traces in a weak sense (see (10)), while $\mathcal{V}$ is the space of the L^1 forms in Ω such that there exist L^1 forms on Σ for which the Cauchy integral formula holds. Actually we have:

Theorem 7.3 :

$$\mathcal{U} = \mathcal{V}.$$

8. The Dirichlet problem for the Cimmino system

In paper [13] Dragomir and Lanconelli studied the Cimmino system (3), that can be written in the following complex form

$$\begin{cases} u_{\overline{z}} + \overline{v}_w = 0, \\ u_{\overline{w}} - \overline{v}_z = 0. \end{cases}$$

The authors obtained, among many other results, a necessary condition for the resolubility of the Dirichlet problem for Cimmino system.

Theorem 8.1 : *Let $\Omega \subset \mathbb{C}^2$ be a bounded domain on which Green's formula holds and Σ its boundary; let $f,\ g \in L^2(\Omega)$, $F,\ G \in L^2(\Sigma)$. If there is a solution $u, v \in C^1(\Omega) \cap C^0(\overline{\Omega})$ to the boundary value problem*

$$\begin{cases} u_{\overline{z}} + \overline{v}_w = f, \quad u_{\overline{w}} - \overline{v}_z = g, & \text{in } \Omega, \\ u = F, \qquad v = G, & \text{on } \Sigma, \end{cases} \tag{14}$$

then (f, g, F, G) *satisfies the compatibility relations*

$$Re\left\{2\int_\Omega (f\overline{h}+g\overline{k})dV-\int_\Sigma \left\{F\left[(n_1+in_2)\overline{h}+(n_3+in_4)\overline{k}\,\right]\right.\right.$$
$$\left.\left.+G\left[(n_3+in_4)h-(n_1+in_2)k\right]\right\}d\sigma\right\}=0 \quad (15)$$

for any solution $h, k \in C^1(\Omega)\cap C^0(\overline{\Omega})$ *to*

$$h_z + k_w = 0, \qquad h_{\overline{w}} - k_{\overline{z}} = 0, \qquad in\ \Omega,$$

where (n_1, n_2, n_3, n_4) *is the outward unit normal on* Σ.

They left open the question of whether (15) is also sufficient for the solvability of (14).
More recently, Abreu Blaya et al. [1] studied (14) by means of quaternionic analysis. In particular, they found some different necessary and sufficient conditions involving some particular integral operators. From this they deduce that (15) are also sufficient when $f = g = 0$ and Ω is a simply connected domain.
In [3], Theorem 7.2 is used to obtain other necessary and sufficient conditions for the solvability of (14), also in the case of multiply connected domains. In what follows we illustrate these results.
Let us consider the problem (13) with the particular kind of data:

$$\phi = (\phi_0, 0, \phi_2, 0),\ \widetilde{\phi} = (-\phi_0, 0, \phi_2, 0) \in L^1(\Sigma),\ F = F_1 - *F_1 \in L^1(\Omega), \qquad (16)$$

where $F_1 = \gamma_k dx_k$. Applying Theorem 7.2, necessary and sufficient conditions for the solvability of (13) with these particular data are

$$\int_\Omega *F_1(y)\wedge dw_h(y) - \frac{1}{2}\int_{+\Sigma}[\phi_0(y)\wedge *dw_h(y) + dw_h(y)\wedge\phi_2(y)] = 0;$$
$$\int_\Omega *F_1(y)\wedge d_y[|y-x^j|^{-2}w_h(y-x^j)] - \frac{1}{2}\left\{\int_{+\Sigma}\Big[\phi_0(y)\wedge \underset{y}{*}d_y[|y-x^j|^{-2}w_h(y-x_j)]\right.$$
$$\left.+d_y[|y-x^j|^{-2}w_h(y-x^j)]\wedge\phi_2(y)\Big]\right\} = 0, \qquad j = 1,\dots,m; \quad (17)$$

$$\int_\Omega \Big[F_1(y)\wedge dw_h^{i_1,i_2}(y) - *F_1(y)\wedge\delta w_h^{i_1,i_2}(y)\Big] - \frac{1}{2}\left\{\int_{+\Sigma}\Big[\phi_2(y)\wedge *dw_h^{i_1,i_2}(y)\right.$$

$$\left.-\delta w_h^{i_1,i_2}(y)\wedge\phi_2(y) + dw_h^{i_1,i_2}(y)\wedge -\phi_0(y) - \phi_0(y)\wedge *\delta w_h^{i_1,i_2}(y)\Big]\right\} = 0;$$

$$\int_\Omega \left[F_1(y)\wedge d_y[|y-x^j|^{-6}w_h^{i_1,i_2}(y-x^j)] - *F_1(y)\wedge\delta_y[|y-x^j|^{-6}w_h^{i_1,i_2}(y-x^j)]\right]$$

$$-\frac{1}{2}\left\{\int_{+\Sigma}\left[\phi_2(y)\wedge \underset{y}{*}d_y[|y-x^j|^{-6}w_h^{i_1,i_2}(y-x^j)] - \delta_y[|y-x^j|^{-6}w_h^{i_1,i_2}(y-x^j)]\wedge\phi_2(y)\right.\right.$$

$$\left.\left.+d_y[|y-x^j|^{-6}w_h^{i_1,i_2}(y-x^j)]\wedge -\phi_0(y) - \phi_0(y)\wedge \underset{y}{*}\delta_y[|y-x^j|^{-6}w_h^{i_1,i_2}(y-x^j)]\right]\right\} = 0,$$

$$j=1,\dots,m, \quad (18)$$

for any $1\le i_1<i_2\le 4,\ h=1,2,\dots$. Here x^j is a fixed point in Ω_j $(j=1,\dots,m)$. Moreover the solution U can be written as

$$U = u_0+u_2+u_4,$$
$$u_0 = f_0, \quad u_2 = f_1(dx_1dx_2+dx_3dx_4) - f_2(dx_1dx_3+dx_4dx_2) + f_3(dx_1dx_4+dx_2dx_3), \quad u_4 = -f_0dx_1dx_2dx_3dx_4.$$

By exploiting the relation between self-conjugate differential forms and solutions of the Cimmino system, described in section 3, it is possible to prove that (13), with the particular data (16), is equivalent to the Dirichlet problem for the Cimmino system (14). Indeed, if $f=\frac{1}{2}(\gamma_1+i\gamma_2)$, $g=\frac{1}{2}(\gamma_3+i\gamma_4)$, $F=\alpha_0+i\alpha_1$ and $G=\beta_0+i\beta_1$ it suffices to take $\phi=(\alpha_0,0,\phi_2,0)$ and $\widetilde{\phi}=(-\alpha_0,0,\phi_2,0)$, where $\phi_2=\alpha_1(dx_1dx_2+dx_3dx_4)-\beta_0(dx_1dx_3-dx_2dx_4)-\beta_1(dx_1dx_4+dx_2dx_3)$. This equivalence applies to obtain the next claim.

Theorem 8.2 : *Let $\Omega\subset\mathbb{C}^2$ be a regular domain and Σ its boundary. Let $f,g\in L^1(\Omega)$ and $F,G\in L^1(\Sigma)$. Conditions* (17) *and* (18) *are necessary and sufficient for the resolubility of the boundary value problem for Cimmino system* (14).

The only point remaining concerns the relation between Dragomir and Lanconelli condition (15) and our conditions (17) and (18).
Let us consider the first of (17); it can be written as

$$\begin{aligned}
&\int_\Omega\left[\gamma_1\frac{\partial w_h}{\partial y_1}+\gamma_2\frac{\partial w_h}{\partial y_2}+\gamma_3\frac{\partial w_h}{\partial y_3}+\gamma_4\frac{\partial w_h}{\partial y_4}\right]dy\\
&-\frac{1}{2}\int_{+\Sigma}\left[\left(\alpha_0\frac{\partial w_h}{\partial y_1}+\alpha_1\frac{\partial w_h}{\partial y_2}-\beta_0\frac{\partial w_h}{\partial y_3}+\beta_1\frac{\partial w_h}{\partial y_4}\right)n_1\right.\\
&+\left(\alpha_0\frac{\partial w_h}{\partial y_2}-\alpha_1\frac{\partial w_h}{\partial y_1}+\beta_0\frac{\partial w_h}{\partial y_4}+\beta_1\frac{\partial w_h}{\partial y_3}\right)n_2 \qquad (19)\\
&+\left(\alpha_0\frac{\partial w_h}{\partial y_3}+\alpha_1\frac{\partial w_h}{\partial y_4}+\beta_0\frac{\partial w_h}{\partial y_1}-\beta_1\frac{\partial w_h}{\partial y_2}\right)n_3\\
&\left.+\left(\alpha_0\frac{\partial w_h}{\partial y_4}-\alpha_1\frac{\partial w_h}{\partial y_3}-\beta_0\frac{\partial w_h}{\partial y_2}-\beta_1\frac{\partial w_h}{\partial y_1}\right)n_4\right]d\sigma=0.
\end{aligned}$$

If we put $h_0=\dfrac{\partial w_h}{\partial y_1}, h_1=\dfrac{\partial w_h}{\partial y_2}, k_0=\dfrac{\partial w_h}{\partial y_3}, k_1=\dfrac{\partial w_h}{\partial y_4}$, we have that (15) implies (19). In a similar way, it is possible to prove that (15) implies (17) and (18). We

can conclude that (15) is not only necessary but also sufficient for the solvability of (14).

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