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Summation of Walsh-Fourier Series, Convergence and Divergence

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In this paper the author gives a short rèsume of the recent achievements with respect to the convergence and divergence of some summation methods of the one and two dimensional Walsh Fourier series. The discussion of Fejér, Cèsaro and Riesz's logarithmic means are included. One of the most celebrated results of Levan Zhizhiashvili is the almost everywhere convergence of the Marcinkiewicz means of the trigonometric series of two variable integrable functions. We discuss a recent generalization of the result of Zhizhiashvili with respect to the Walsh system.

Keywords: Walsh system, One and two dimensional Fourier series, Fejér, Cèsaro and Riesz's logarithmic means, Almost everywhere convergence, Divergence, Marcinkiewicz means.

AMS Subject Classification: 42C10.

1. Introduction

Let the numbers $n \in \mathbf{N}$ and $x \in I := [0, 1)$ be expanded with respect to the binary number system:

$$n = \sum_{k=0}^{\infty} n_k 2^k, \quad x = \sum_{k=0}^{\infty} x_k 2^{-k-1},$$

where if x is a dyadic rational, that is an element of the set $\{k/2^n : k, n \in \mathbf{N}\}$, then we choose the finite expansion. Let $(\omega_n, n \in \mathbf{N})$ represent the Walsh-Paley system. That is, the n-th Walsh-Paley function is

$$\omega_n(x) := \prod_{k=0}^{\infty} (-1)^{n_k x_k}.$$

The n-th Walsh-Fourier coefficient of the integrable function $f \in L^1(I)$ is

$$\hat{f}(n) := \int_I f(x)\omega_n(x)dx.$$

The n-th partial sum of the Walsh-Fourier series of the integrable function $f \in$

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$L^1(I)$:

$$S_n f(y) := \sum_{k=0}^{n-1} \hat{f}(k)\omega_k(y).$$

The n-th Fejér or $(C,1)$ mean of the function f is

$$\sigma_n f := \frac{1}{n}\sum_{k=1}^{n} S_k f.$$

2. Results - one dimension

In 1955 Fine proved [4] for the Walsh-Paley system the wellknown Fejér-Lebesgue theorem. Namely, for every integrable function f we have the a.e. relation

$$\sigma_n f \to f.$$

Let us have a look at the situation with the (C,α) means. What are they? Let $A_n^\alpha := \frac{(1+\alpha)\dots(n+\alpha)}{n!}$, where $n \in \mathbf{N}$ and $\alpha \in \mathbf{R}$ ($-\alpha \notin \mathbf{N}$). It is known, that $A_n^\alpha \sim n^\alpha$.

The n-th (C,α) mean of the function $f \in L^1(I)$:

$$\sigma_{n+1}^\alpha f := \frac{1}{A_n^\alpha}\sum_{k=0}^{n} A_{n-k}^{\alpha-1} S_k f.$$

In 1975 Schipp proved [25], that $\sigma_n^\alpha f \to f$ a.e. for each $f \in L^1(I)$ and $\alpha > 0$.

What can be said in the case of the Walsh-Kaczmarz system? What is this Walsh-Kaczmarz system? This is nothing else, but a rearrangement of the Walsh-Paley system. Introduce it as follows. If $n > 0$, then let $|n| := \max(j \in \mathbf{N} : n_j \neq 0)$. The n-th Walsh-Kaczmarz function is

$$\kappa_n(x) := r_{|n|}(x)(-1)^{\sum_{k=0}^{|n|-1} n_k x_{|n|-1-k}},$$

as if $n > 0$, $\kappa_0(x) := 1, x \in I$. Then the elements of the a Walsh-Kaczmarz system and the Walsh-Paley system are dyadic blockwise rearrangements of each other. This means that

$$\{\kappa_n : 2^k \le n < 2^{k+1}\} = \{\omega_n : 2^k \le n < 2^{k+1}\}.$$

In 1998 Gát proved [6] the Fejér-Lebesgue theorem for the Walsh-Kaczmarz system. That is, $\sigma_n f \to f$ a.e. for each $f \in L^1(I)$. In 2004 Simon [27] generalized the result of Gát above for (C,α) summation methods. In other words, the maximal convergence space of the (C,α) means is the L^1 Lebesgue space, that is, the largest one.

It is also of prior interest what can be said - with respect to this reconstruction issue (that is, the reconstruction of the function from the partial sums of its Fourier series)- if we have only a subsequence of the partial sums. In 1936 Zalcwasser [34] asked how "rare" can be the sequence of integers $a(n)$ such that

$$\frac{1}{N}\sum_{n=1}^{N} S_{a(n)}f \to f. \tag{1}$$

This problem with respect to the trigonometric system was completely solved for continuous functions (uniform convergence) in [1, 3, 24, 29]. That is, if the sequence a is convex, then the condition $\sup_n n^{-1/2}\log a(n) < +\infty$ is necessary and sufficient for the uniform convergence for every continuous function. For the time being, this issue with respect to the Walsh-Paley system has not been solved. Only a sufficient condition is known, which is the same as in the trigonometric case. The paper about this is written by Glukhov [16]. See the more dimensional case also by Glukhov [17].

With respect to convergence almost everywhere, and integrable functions the situation is more complicated. Belinsky proved [2] for the trigonometric system the existence of a sequence $a(n) \sim \exp(\sqrt[3]{k})$ such that the relation (1) holds a.e. for every integrable function. In this paper Belinsky also conjectured that if the sequence a is convex, then the condition $\sup_n n^{-1/2}\log a(n) < +\infty$ is necessary and sufficient again. So, that would be the answer for the problem of Zalcwasser [34] in this point of view (trigonometric system, a.e. convergence and L^1 functions). Gát proved [9] that this is not the case for the Walsh-Paley system. See below Theorem 2.1. On the other hand, differences between the Walsh-Paley and the trigonometric system are not so surprising. For example Totik [28] proved for the trigonometric system that for any subsequence $a(n)$ of the natural numbers there exists an integrable function f such that $\sup_n |S_{a(n)}f| = \infty$ everywhere. On the other hand, let $v(n) := \sum_{i=0}^{\infty} |n_i - n_{i+1}|, (n = \sum_{i=0}^{\infty} n_i 2^i)$ be the variation of the natural number n expanded in the number system based 2. It is a well-known result in the literature that for each sequence a tending strictly monotone increasing to plus infinity with the property $\sup_n v(a(n)) < +\infty$ we have the a.e. convergence $S_{a(n)}f \to f$ for all integrable functions f. Is it also a necessary condition? This question of Balashov was answered by Konyagin [18] in the negative. He gave an example. That is, a sequence a with property $\sup_n v(a(n)) = +\infty$ and he proved that $S_{a(n)}f \to f$ a.e. for every integrable function f.

In [9] the author of the present paper proved (see Theorem 2.1) that for each lacunary sequence a (that is $a(n+1)/a(n) \geq q > 1$) and each integrable function f the relation (1) holds a.e. This may also be interesting from the following point of view. If the sequence a is lacunary, then the a.e. relation $S_{a(n)}f \to f$ holds for all functions f in the Hardy space H. The trigonometric and the Walsh-Paley case can be found in [36] (trigonometric case) and [19] (Walsh-Paley case). But, the space H is a proper subspace of L^1. Therefore, it is of interest to investigate relation (1) for L^1 functions and lacunary sequence a.

In paper [9] it is also proved (Theorem 2.2) that for any convex sequence a (with $a(+\infty) = +\infty$ - of course) and for each integrable function the Riesz's logarithmic means of the function converges to the function almost everywhere. That is, the Riesz's logarithmic summability method can reconstruct the corresponding integrable function from any (convex) subsequence of the partial sums in the Walsh-Paley situation. For the time being there is no result known with respect to a.e. convergence of logarithmic means of subsequences of partial sums, neither in the trigonometric nor in the Walsh-Kaczmarz case.

Theorem 2.1 : *Let $a : \mathbf{N} \to \mathbf{N}$ be a sequence with property $\frac{a(n+1)}{a(n)} \geq q > 1$ ($n \in \mathbf{N}$). Then for all integrable functions $f \in L^1(I)$ we have the a.e. relation*

$$\frac{1}{N} \sum_{n=1}^{N} S_{a(n)} f \to f.$$

Theorem 2.2 : *Let $a : \mathbf{N} \to \mathbf{N}$ be a convex sequence with property $a(+\infty) = +\infty$. Then for each integrable function f we have the a.e. relation*

$$\frac{1}{\log N} \sum_{n=1}^{N} \frac{S_{a(n)} f}{n} \to f.$$

3. Results - two dimension

What can be said in the two dimensional situation? This is quite a different story. Define the two-dimensional Walsh-Paley functions in the following way:

$$\omega_n(x) := \omega_{n_1}(x^1)\omega_{n_2}(x^2),$$

where $n = (n_1, n_2) \in \mathbf{N}^2$, $x = (x^1, x^2) \in I^2$. Let f be an integrable function. The Fourier coefficients, the rectangular partial sums of its Fourier series:

$$\hat{f}(n) := \int_{I^2} f(x)\omega_n(x)dx,$$

$$S_{n_1,n_2} f := \sum_{k_1=0}^{n_1-1} \sum_{k_2=0}^{n_2-1} \hat{f}(k_1, k_2)\omega_{k_1,k_2}.$$

Moreover, the two-dimensional Fejér or $(C, 1)$ means of the function $f \in L^1(I^2)$:

$$\sigma_{n_1,n_2} f := \frac{1}{n_1 n_2} \sum_{k_1=1}^{n_1} \sum_{k_2=1}^{n_2} S_{k_1,k_2} f \quad (n \in \mathbf{P}^2).$$

In 1931 Marczinkiewicz and Zygmund proved for the two-dimensional trigonometric system [21], and in 1992 Móricz, Schipp and Wade verified [22] for the two-dimensional Walsh-Paley system, that for every $f \in L \log^+ L(I^2)$

$$\sigma_{n_1,n_2} f \to f$$

a.e. as $\min\{n_1, n_2\} \to \infty$, that is, in the Pringsheim sense.

Since $L \log^+ L(I^2) \subsetneq L^1(I^2)$, then it would be interesting to "enlarge" the convergence space, if possible. In 2000 Gát proved [7], that it is impossible. That is, for each measurable function $\delta : [0, +\infty) \to [0, +\infty)$, $\delta(\infty) = 0$, (that is vanishing at plus infinity) there exists a function

$$f \in L \log^+ L\delta(L) \quad \text{such that} \quad \sigma_{n_1,n_2} f \not\to f$$

a.e. (in the Pringsheim sense).

However, what "positive" can be said about the function of the class $L^1(I^2)$ in spite of the fact that the two-dimensional Fejér means are not convergent a.e. in the Pringsheim sense? That could be the so called restricted convergence. For the two-dimensional trigonometric system Marcinkiewicz and Zygmund proved [20] in 1939, that

$$\sigma_{n_1,n_2} f \to f$$

a.e. for every $f \in L^1(I^2)$ as if $\min\{n_1, n_2\} \to \infty$, provided that

$$2^{-\alpha} \le \frac{n_1}{n_2} \le 2^{\alpha}$$

for some $\alpha \ge 0$. In other words, the set of admissible indices (n_1, n_2) remains in some cone. This theorem for the two-dimensional Walsh-Paley system was verified by Móricz, Schipp and Wade in 1992 in the case when n_1, n_2 both are powers of two.

$$\sigma_{2^{n_1},2^{n_2}} f \to f$$

a.e. for every $f \in L^1(I^2)$ as if $\min\{n_1, n_2\} \to \infty$, provided that $|n_1 - n_2| \le \alpha$ for some $\alpha \ge 0$.

The proof of the Marcinkiewicz-Zygmund theorem [20] (with respect to the Walsh-Paley system) for arbitrary set of indices remaining in some cone is due to Gát and Weisz [5, 30], separately in 1996.

It is an interesting question whether it is possible to weaken somehow the "cone restriction" in a way that a.e. convergence remains for each function in L^1. Maybe for some "interim space" if not for space L^1. The answer is negative both from the point of view of space and from the point of view of restriction. Namely, in 2001 Gát proved [8] the theorem below:

Let $\delta : [0, +\infty) \to [0, +\infty)$ be measurable, $\delta(+\infty) = 0$ and let $w : \mathbf{N} \to [1, +\infty)$ be an arbitrary increasing function such that

$$\sup_{x \in \mathbf{N}} w(x) = +\infty.$$

Moreover, $\vee n := \max(n_1, n_2)$, $\wedge n := \min(n_1, n_2)$. Then, there exists a function $f \in L \log^+ L\delta(L)$ such that

$$\sigma_{n_1,n_2} f \not\to f$$

a.e. as $\wedge n \to \infty$ such that the restriction condition $\frac{\vee n}{\wedge n} \le w(\wedge n)$ is also fulfilled. That is, there is no "interim" space. Either we have space $L \log^+ L$ and "no restriction at all", or the "cone restriction" and then the maximal convergence space is

L^1. As a consequence of this we have

$$\sigma_{n_1,n_2} f \to f$$

a.e. for each $f \in L(I^2)$ as $\min\{n_1, n_2\} \to \infty$, provided that

$$\frac{\vee n}{\wedge n} \leq w(\wedge n)$$

if and only if

$$\sup w(x) < \infty.$$

Another question. What is the situation with the (C, α) summation of 2-dimensional Walsh-Fourier series?

$$\sigma^{\alpha}_{n_1+1,n_2+1} f = \frac{1}{A^{\alpha}_{n_1} A^{\alpha}_{n_2}} \sum_{k_1=0}^{n_1} \sum_{k_2=0}^{n_2} A^{\alpha-1}_{n_1-k_1} A^{\alpha-1}_{n_2-k_2} S_{k_1,k_2} f.$$

In 1999 Weisz proved [31], that

$$\sigma^{\alpha}_{n_1,n_2} f \to f$$

a.e. as $\min\{n_1, n_2\} \to \infty$ for each $f \in L \log^+ L(I^2)$ and $\alpha > 0$.

The question is the same again. That is, is it possible to give a "larger" convergence space for the (C, α) summability method $(\alpha > 0)$? Is there such an α? If $\alpha \leq 1$, then there is not. Because for the $(C, 1)$ method one can not give such a "larger" space.

On the other hand, what is the situation with the (C, α) methods, for $\alpha > 1$?

What can be said in the case of the Walsh-Kaczmarz system? In 2001 Simon proved [26], that $\sigma_{n_1,n_2} f \to f$ a.e. as if $\min\{n_1, n_2\} \to \infty$ (in the Pringsheim sense) for every $f \in L \log^+ L(I^2)$. He also proved the restricted "cone" convergence for functions belonging to $L^1(I^2)$. The divergence result with respect to the two-dimensional Walsh-Kaczmarz-Fejér means, that is, the fact that the maximal convergence space for the Pringsheim sense a.e. convergence is the space $L \log^+ L$ is due to Getsadze [12]. Although, it is an open question the case of (C, α) summation with respect to the Kaczmarz system.

4. The Marcinkiewicz means - generalization of the result of Zhizhiashvili

This is another story and also very interesting to discuss the almost everywhere convergence of the Marcinkiewicz means $\frac{1}{n} \sum_{j=0}^{n-1} S_{j,j} f$ of integrable functions with respect to orthonormal systems. Although, this mean is defined for two-variable functions, in the view of almost everywhere convergence there are similarities with the one-dimensional case. On the one side, the maximal convergence space for two dimensional Fejér means (no restriction on the set of indices other than they have to converge to $+\infty$) is $L \log^+ L$ ([7, 10]), and on the other side, for the Marcinkiewicz means we have a.e. convergence for every integrable functions (for the trigonometric, Walsh Paley systems).

We mention that the first result is due to Marcinkiewicz [21]. But he proved "only" for functions in the space $L\log^+ L$ the a.e. relation $t_n f \to f$ with respect to the trigonometric system. The "L^1 result" for the trigonometric, Walsh-Paley, and the so called Walsh-Kaczmarz systems see the papers of Zhizhiasvili [35] (trigonometric system), Weisz [33] (Walsh system), Goginava [13, 14] (Walsh system) and Nagy [23] (Walsh-Kaczmarz system).

After then, we turn our attention to the generalization of Marcinkiewicz means. Let $\alpha = (\alpha_1, \alpha_2) : \mathbf{N}^2 \to \mathbf{N}^2$ be a function. Define the following Marcinkiewicz-like means:

$$t_n^\alpha(x) := \frac{1}{n}\sum_{k=0}^{n-1} S_{\alpha_1(|n|,k),\alpha_2(|n|,k)} f(x^1, x^2), \quad (f \in L^1(I^2), n \in \mathbf{P}).$$

The following properties will play a prominent role in the a.e. convergence of these generalized means. ($\#B$ denotes the cardinality of set B.) Roughly speaking they will be necessary and sufficient conditions.

$$\#\{l \in \mathbf{N} : \alpha_j(|n|, l) = \alpha_j(|n|, k), l < n\} \le C \quad (k < n, n \in \mathbf{P}, j = 1, 2) \tag{2}$$

$$\max\{\alpha_j(|n|, k) : k < n\} \le Cn \quad (n \in \mathbf{P}, j = 1, 2). \tag{3}$$

More precisely, we proved in [11] the ,,theorem of convergence":

Theorem 4.1: *Let α satisfy (2) and (3). Then we have $t_n^\alpha f \to f$ for each $f \in L^1(I^2)$.*

Condition (2) is clearly a necessary one in the following sense. Let $\alpha_1(|n|, k) = 0$, $\alpha_2(|n|, k) = k$ for every $n, k \in \mathbf{N}$. Then (3) is satisfied and (2) is not. It is very simple to give a function $f \in L^1(I^2)$ such as $t_n^\alpha f \to f$ fails to hold a.e. To construct an α with (2) which fails to satisfy (3) and a $f \in L^1(I^2)$ such that $t_n^\alpha f$ does not converge to f a.e. is more complicated.

The "theorem of divergence" aims to show that (3) is also a necessary condition in a certain sense. That is, we proved [11]:

Theorem 4.2: *Let $\gamma : \mathbf{N} \to \mathbf{N}$ be any function with property $\gamma(+\infty) = +\infty$. Then there exists a function α satisfying (2),*

$$\max\{\alpha_1(|n|, k) : k < n\} \le Cn, \quad \max\{\alpha_2(|n|, k) : k < n\} \le Cn\gamma(n) \quad (n \in \mathbf{P})$$

and $f \in L^1(I^2)$ such that $\limsup_{n\in\mathbf{N}} |t_n^\alpha f| = +\infty$ almost everywhere.

Of course it would have been possible to write the conditions as $\alpha_1(n) \le Cn\gamma(n)$ and $\alpha_2(n) \le Cn$. We gave in [11] a corollary of Theorem 4.1.

Corollary 4.3: *Let (a_n) be a lacunary sequence of reals, i.e. $a_{n+1} \ge a_n q$ for some $q > 1$ ($n \in \mathbf{N}$) and α satisfy condition (2) and $\alpha_j(n, k) \le Ca_n$ ($k < a_n$, $j = 1, 2$) (modified version of condition (3)). Then for every integrable function $f \in L^1(I^2)$*

we have

$$\frac{1}{a_n}\sum_{k=0}^{a_n-1} S_{\alpha_1(n,k),\alpha_2(n,k)}f(x) \to f(x)$$

for a.e. $x \in I^2$.

The triangular partial sums of the two-dimensional Walsh-Fourier series are defined as

$$S_k^{\triangle}f(x,y) := \sum_{i=0}^{k-1}\sum_{j=0}^{k-i-1}\hat{f}(i,j)\omega_i(x)\omega_j(y).$$

Denote by

$$D_k^{\triangle}(x,y) := \sum_{i=0}^{k-1}\sum_{j=0}^{k-i-1}\omega_i(x)\omega_j(y)$$

the n-th triangular Walsh-Dirichlet kernel. For $n \in \mathbf{P}$ and an integrable function f the triangular Fejér means of order n of the two-dimensional Walsh–Fourier series of a function f is given by

$$\sigma_n^{\triangle}f(x,y) := \frac{1}{n}\sum_{j=0}^{n-1}S_j^{\triangle}f(x,y).$$

It is easy to show that

$$\sigma_n^{\triangle}f(x,y) = \int_{I^2} f(s,t)K_n^{\triangle}(x+s,y+t)d\mu(s,t),$$

where

$$K_n^{\triangle}(x,y) := \frac{1}{n}\sum_{j=0}^{n-1}D_j^{\triangle}(x,y).$$

This triangular summability method is rarely investigated in the literature (see the references in [32]). In [15] it is proved that the maximal operator $\sigma_{\#}^{\triangle}f := \sup_n \left|\sigma_{2^n}^{\triangle}f\right|$ of the Fejér means of the triangular partial sums of the double Walsh-Fourier series is bounded from the dyadic Hardy space $H_p(I^2)$ to the $L_p(I^2)$ if $p > 1/2$, is bounded from $H_{1/2}(I^2)$ to the space weak- $L_{1/2}(I^2)$ and it is not bounded from $H_{1/2}(I^2)$ to $L_{1/2}(I^2)$. As a consequence of these assumptions it is proved in [15] the a.e. convergence $\sigma_{2^n}^{\triangle}f \to f$ for each integrable function f. We remark that Corollary 4.1. immediately gives the generalization of this result. Namely,

$$\sigma_{a(n)}^{\triangle}f \to f$$

for every lacunary sequence $a(n)$ and integrable function f.

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