Bulletin of TICMI
Vol. 16, No. 1, 2012, 1–14

Boundary Value Problems for the Adjoint System of Differential Equations of the Thermoelasticity Theory of Hemitropic Solids

Ivanidze Diana* and Ivanidze Marekh

Georgian Technical University, Department of Mathematics, Kostava str. 77, Tbilisi 0175, Georgia

(*Received June 1, 2011; Accepted February 20, 2012*)

The differential operator, generated by the system of differential equations of the thermoelasticity theory of hemitropic solids, is not formally self-adjoint. In the study of boundary value problems (BVP) by the potential method, we have to consider adjoint integral operators which correspond to the adjoint differential operator. Therefore we need to investigate the BVPs for the adjoint differential operator. Due to the specific structure of the problems under consideration, we need to prove uniqueness of solutions in the spaces of vector functions bounded at infinity. This essentially complicates the study. We introduce a special class $Z^*(\Omega^-)$ of vector-functions, bounded at infinity in the case of an unbounded domain Ω^- and show that the layer potentials belong to this class. These results play an important role in the study of the direct boundary value problems for differential equations of the thermoelasticity theory of hemitropic solids.

Keywords: Elasticity theory, Elastic hemitropic materials, Potential theory.

AMS Subject Classification: 35J55, 35J55, 74A60, 74G05, 74G30, 74F05

The problems, treated in the paper, are closely related to the mathematical problems of the theory of thermoelastostatics for hemitropic continua. The boundary value problems for the adjoint operator arise naturally when the potential method is applied in the study of direct boundary value problems (for details and historical notes see [1], [2], [3], [8], [10], [11], [13] and the reference therein).

Let $\Omega^+ \subset \mathbb{R}^3$ be a bounded domain. Set $\partial\Omega^+ =: S \in C^{2,\kappa}$ with $0 < \kappa \leq 1$, $\overline{\Omega^+} = \Omega^+ \cup S$, and $\Omega^- = \mathbb{R}^3 \setminus \overline{\Omega^+}$.

The basic governing homogeneous equations of the theory of thermoelastostatics for hemitropic materials read as (see [13])

$$\begin{aligned}
&(\mu+\alpha)\Delta u(x) + (\lambda+\mu-\alpha)\operatorname{grad}\operatorname{div} u(x) + (\chi+\nu)\Delta\omega(x)\\
&\qquad +(\delta+\chi-\nu)\operatorname{grad}\operatorname{div}\omega(x) + 2\alpha\operatorname{curl}\omega(x) - \eta\operatorname{grad}\vartheta(x) = 0,\\
&(\chi+\nu)\Delta u(x) + (\delta+\chi-\nu)\operatorname{grad}\operatorname{div} u(x) + 2\alpha\operatorname{curl} u(x) + (\gamma+\varepsilon)\Delta\omega(x)\\
&\qquad +(\beta+\gamma-\varepsilon)\operatorname{grad}\operatorname{div}\omega(x) + 4\nu\operatorname{curl}\omega(x) - \zeta\operatorname{grad}\vartheta(x) - 4\alpha\omega(x) = 0,\\
&\kappa'\Delta\vartheta(x) = 0,
\end{aligned}$$

where $u = (u_1, u_2, u_3)^\top$ and $\omega = (\omega_1, \omega_2, \omega_3)^\top$ are the displacement vector and the microrotation vector respectively, ϑ is the temperature distribution function,

*Corresponding author. Email: Diana.ivanize@gmail.com

ISSN: 1512-0082 print

α, β, γ, δ, λ, μ, ν, χ, ε, η, ζ, and κ' are the material constants, $\partial = (\partial_1, \partial_2, \partial_3)$, $\partial_j = \partial/\partial x_j$, $j = 1, 2, 3$, the symbol $(\cdot)^\top$ denotes transposition.

The matrix differential operator, generated by these equations, is not formally self-adjoint and has the form

$$L(\partial) = \begin{bmatrix} L^{(1)}(\partial) & L^{(2)}(\partial) & L^{(5)}(\partial) \\ L^{(3)}(\partial) & L^{(4)}(\partial) & L^{(6)}(\partial) \\ [0]_{1\times 3} & [0]_{1\times 3} & \kappa^{'}\Delta \end{bmatrix}_{7\times 7}, \tag{1}$$

where

$$\begin{aligned} &L^{(1)}(\partial) := (\mu + \alpha)\Delta I_3 + (\lambda + \mu - \alpha)Q(\partial), \\ &L^{(2)}(\partial) = L^{(3)}(\partial) := (\chi + \nu)\Delta I_3 + (\delta + \chi - \nu)Q(\partial) + 2\alpha R(\partial), \\ &L^{(4)}(\partial) := [(\gamma + \varepsilon)\Delta - 4\alpha]I_3 + (\beta + \alpha - \varepsilon)Q(\partial) + 4\nu R(\partial), \\ &L^{(5)}(\partial) := -\eta\nabla^\top, \quad L^{(6)}(\partial) := -\zeta\nabla^\top, \\ &R(\partial) := [\,-\varepsilon_{pqj}\partial_j\,]_{3\times 3}, \quad Q(\partial) := [\,\partial_k\partial_j\,]_{3\times 3}. \end{aligned} \tag{2}$$

Here and in what follows ε_{pqj} denotes the permutation (Levi-Civitá) symbol and I_k stands for the $k \times k$ unit matrix . Throughout the paper summation over repeated indexes is meant from one to three if not otherwise stated.

Denote by $L^*(\partial)$ the operator, formally adjoint to $L(\partial)$: $L^*(\partial) := L^\top(-\partial)$. Introduce the generalized stress operators [9], [12], [13]

$$\mathcal{P}(\partial, n) = \begin{bmatrix} T^{(1)}(\partial, n) & T^{(2)}(\partial, n) & -\eta n^\top \\ T^{(3)}(\partial, n) & T^{(4)}(\partial, n) & -\zeta n^\top \\ [0]_{1\times 3} & [0]_{1\times 3} & \kappa^{'}\partial_n \end{bmatrix}_{7\times 7}, \tag{3}$$

$$\mathcal{P}^*(\partial, n) = \begin{bmatrix} T^{(1)}(\partial, n) & T^{(2)}(\partial, n) & [0]_{3\times 1} \\ T^{(3)}(\partial, n) & T^{(4)}(\partial, n) & [0]_{3\times 1} \\ [0]_{1\times 3} & [0]_{1\times 3} & \kappa^{'}\partial_n \end{bmatrix}_{7\times 7}, \tag{4}$$

where

$$\begin{aligned} &T^{(j)} = [T^{(j)}_{pq}]_{3\times 3}, \quad j = \overline{1,4}, \quad n = (n_1, n_2, n_3), \\ &T^{(1)}_{pq}(\partial, n) = (\mu + \alpha)\delta_{pq}\partial_n + (\mu - \alpha)n_q\partial_p + \lambda n_p\partial_q, \\ &T^{(2)}_{pq}(\partial, n) = (\chi + \nu)\delta_{pq}\partial_n + (\chi - \nu)n_q\partial_p + \delta n_p\partial_q - 2\alpha\varepsilon_{pqk}n_k, \\ &T^{(3)}_{pq}(\partial, n) = (\chi + \nu)\delta_{pq}\partial_n + (\chi - \nu)n_q\partial_p + \delta n_p\partial_q, \\ &T^{(4)}_{pq}(\partial, n) = (\gamma + \varepsilon)\delta_{pq}\partial_n + (\gamma - \varepsilon)n_q\partial_p + \beta n_p\partial_q - 2\nu\varepsilon_{pqk}n_k. \end{aligned} \tag{5}$$

Here $\partial_n = \partial/\partial n$ denotes the usual normal derivative.
Let us consider the following homogeneous "adjoint" equation

$$L^*(\partial)U(x) = \Phi(x), \quad x \in \Omega^\pm,$$

where

$$\Phi = (\widetilde{\Phi}, \Phi_7)^\top = (\Phi_1, \Phi_2, ..., \Phi_7)^\top \in [C^{0,\sigma}(\Omega^\pm)]^7, \qquad \widetilde{\Phi} = (\Phi_1, \Phi_2, ..., \Phi_6)^\top,$$

$$U = (u, \omega, \vartheta)^\top \in [C^{1,\sigma}(\overline{\Omega^\pm})]^7 \cap [C^{2,\sigma}(\Omega^\pm)]^7 \quad \text{with} \quad 0 < \sigma < 1.$$

Further, we introduce the "adjoint" layer potentials, associated with the operator $L^*(\partial)$,

$$V^*(g)(x) := \int_S \Gamma^*(x - y\, g(y)\, dS_y, \quad x \in \mathbb{R}^3 \setminus\ S, \tag{6}$$

$$W^*(g)(x) := \int_S \big[\mathcal{P}\big(\partial_y, n(y)\big)\big[\Gamma^*(x-y)\big]^\top\big]^\top g(y)\, dS_y, \quad x \in \mathbb{R}^3 \setminus\ S, \tag{7}$$

$$N^*_{\Omega^\pm}(h)(x) := \int_{\Omega^\pm} \Gamma^*(x-y)\, h(y)\, dy, \quad x \in \mathbb{R}^3, \tag{8}$$

where $g = (g_1, g_2, ..., g_7)^\top$ and $h = (h_1, h_2, ..., h_7)^\top$ are density vector-functions, defined on S and $\Omega^\pm$ respectively. We assume that in the case of an unbounded exterior domain Ω^- the support of the vector-function h is compact. By $\Gamma^*(x - y) = \Gamma^\top(y-x)$ is denoted the fundamental matrix of the operator $L^*(\partial)$ which is constructed explicitly and has the following form:

$$\Gamma(x) = \begin{bmatrix} [\Gamma^{(1)}_{pq}(x)]_{3\times3} & [\Gamma^{(2)}_{pq}(x)]_{3\times3} & [\Gamma^{(5)}_{pq}(x)]_{3\times1} \\ [\Gamma^{(3)}_{pq}(x)]_{3\times3} & [\Gamma^{(4)}_{pq}(x)]_{3\times3} & [\Gamma^{(6)}_{pq}(x)]_{3\times1} \\ [\Gamma^{(7)}_{pq}(x)]_{1\times3} & [\Gamma^{(8)}_{pq}(x)]_{1\times3} & \Gamma^{(9)}(x) \end{bmatrix}_{7\times7}$$

$$= \frac{1}{4\pi}\begin{bmatrix} \widetilde{\Psi}_1(x) I_3 & \widetilde{\Psi}_2(x) I_3 & [0]_{3\times1} \\ \widetilde{\Psi}_3(x) I_3 & \widetilde{\Psi}_4(x) I_3 & [0]_{3\times1} \\ [0]_{1\times3} & [0]_{1\times3} & \widetilde{\Psi}_5(x) \end{bmatrix}_{7\times7} - \frac{1}{4\pi}\begin{bmatrix} Q(\partial)\widetilde{\Psi}_6(x) & Q(\partial)\widetilde{\Psi}_7(x) & [0]_{3\times1} \\ Q(\partial)\widetilde{\Psi}_8(x) & Q(\partial)\widetilde{\Psi}_9(x) & [0]_{3\times1} \\ [0]_{1\times3} & [0]_{1\times3} & 0 \end{bmatrix}_{7\times7}$$

$$+ \frac{1}{4\pi}\begin{bmatrix} R(\partial)\Psi_{10}(x) & R(\partial)\Psi_{11}(x) & \nabla^\top\Psi_{14}(x) \\ R(\partial)\Psi_{12}(x) & R(\partial)\Psi_{13}(x) & \nabla^\top\Psi_{15}(x) \\ [0]_{1\times3} & [0]_{1\times3} & 0 \end{bmatrix}_{7\times7},$$

where

$$\Psi_1(x) = -\frac{\gamma+\varepsilon}{d_1|x|} - \frac{1}{d_1^2(\lambda_2^2 - \lambda_3^2)}\sum_{j=2}^{3}(-1)^j\Big\{4[\alpha d_1 + \alpha\mu(\gamma+\varepsilon) + 4\nu(\alpha\chi - \mu\nu)]$$

$$+ d_1(\gamma+\varepsilon)\lambda_1^2 + \frac{16\alpha^2\mu}{\lambda_j^2}\Big\}\frac{e^{i\lambda_j|x|} - 1}{|x|},$$

$$\Psi_2(x) = \Psi_3(x) = \frac{\chi+\nu}{d_1|x|} + \frac{1}{d_1^2(\lambda_2^2 - \lambda_3^2)}\sum_{j=2}^{3}(-1)^j\Big\{4\alpha[\mu(\chi+\nu) + 2(\alpha\chi - \mu\nu)]$$

$$+ d_1(\chi+\nu)\lambda_j^2\Big\}\frac{e^{i\lambda_j|x|}-1}{|x|}\,,$$

$$\Psi_4(x) = -\frac{\mu+\alpha}{d_1|x|} - \frac{\mu+\alpha}{d_1^2(\lambda_2^2-\lambda_3^2)}\sum_{j=2}^{3}(-1)^j(d_1\lambda_j^2+4\alpha\mu)\frac{e^{i\lambda_j|x|}-1}{|x|},$$

$$\Psi_5(x) = -\frac{1}{\kappa'|x|}\,,$$

$$\Psi_6(x) = -\frac{(\lambda+\mu)|x|}{2\mu(\lambda+2\mu)} + \frac{(\delta+2\chi)^2 d_2}{4\alpha(\lambda+2\mu)^2}\frac{e^{-\lambda_1|x|}-1}{|x|} + \frac{1}{\lambda_2^2-\lambda_3^2}\sum_{j=2}^{3}(-1)^j\Big\{\frac{\gamma+\varepsilon}{d_1}$$

$$+\frac{4}{d_1^2\lambda_j^2}[\alpha d_1 + \alpha\mu(\gamma+\varepsilon) + 4\nu(\alpha\chi-\mu\nu)] + \frac{16\alpha^2\mu}{d_1^2\lambda_j^4}\Big\}\frac{e^{i\lambda_j|x|}-1}{|x|}\,,$$

$$\Psi_7(x) = \Psi_8(x) = -\frac{\delta+2\chi}{4\alpha(\lambda+2\mu)}\frac{e^{-\lambda_1|x|}-1}{|x|} - \frac{1}{\lambda_2^2-\lambda_3^2}\sum_{j=2}^{3}(-1)^j\Big\{\frac{\chi+\nu}{d_1}$$

$$+\frac{4\alpha}{d_1^2\lambda_j^2}\big[\mu(\chi+\nu)+2(\alpha\chi-\mu\nu)\big]\Big\}\frac{e^{i\lambda_j|x|}-1}{|x|},$$

$$\Psi_9(x) = \frac{1}{4\alpha}\frac{e^{-\lambda_1|x|}-1}{|x|} + \frac{1}{\lambda_2^2-\lambda_3^2}\sum_{j=2}^{3}(-1)^j\,\frac{\mu+\alpha}{d_1^2}\Big(d_1+\frac{4\alpha\mu}{\lambda_j^2}\Big)\frac{e^{i\lambda_j|x|}-1}{|x|}\,,$$

$$\Psi_{10}(x) = \frac{4}{d_1^2(\lambda_2^2-\lambda_3^2)}\sum_{j=2}^{3}(-1)^j\Big[\nu d_1 + (\gamma+\varepsilon)(\alpha\chi-\mu\nu) + \frac{4\alpha^2\chi}{\lambda_j^2}\Big]\frac{e^{i\lambda_j|x|}-1}{|x|}\,,$$

$$\Psi_{11}(x) = \Psi_{12}(x) = \frac{2}{d_1^2(\lambda_2^2-\lambda_3^2)}\sum_{j=2}^{3}(-1)^j\Big[2(\chi+\nu)(\mu\nu-\alpha\chi) - \alpha d_1$$

$$-\frac{4\alpha^2\mu}{\lambda_j^2}\Big]\frac{e^{i\lambda_j|x|}-1}{|x|}\,,$$

$$\Psi_{13}(x) = \frac{4(\mu+\alpha)(\alpha\chi-\mu\nu)}{d_1^2(\lambda_2^2-\lambda_3^2)}\,\frac{e^{i\lambda_2|x|}-e^{i\lambda_3|x|}}{|x|},$$

$$\Psi_{14}(x) = \frac{1}{\kappa'}\Big\{-\frac{\eta|x|}{2(\lambda+2\mu)} + \big[\zeta(\lambda+2\mu)-\eta(\delta+2\chi)\big]\frac{\delta+2\chi}{4\alpha(\lambda+2\mu)^2}\frac{e^{-\lambda_1|x|}-1}{|x|}\Big\}\,,$$

$$\Psi_{15}(x) = \frac{\eta(\delta+2\chi)-\zeta(\lambda+2\mu)}{4\kappa'\alpha(\lambda+2\mu)}\,\frac{e^{-\lambda_1|x|}-1}{|x|};$$

Evidently, the layer potentials solve the homogeneous equation $L^*(\partial)V^*(x) = L^*(\partial)W^*(x) = 0$, $x \in \Omega^{\pm}$.

Due to the specific structure of the problems under consideration, we need to prove uniqueness of solutions in the spaces of vector functions, bounded at infinity. In this respect, we introduce a special class $Z^*(\Omega^-)$ of vector-functions bounded at infinity in the case of an unbounded domain Ω^- and show that the potentials belong to this class.

Definition 1: A vector-function $U^* = (u^*, \omega^*, \vartheta^*)^\top$ is said to belong to the class $Z^*(\Omega^-)$ if it is continuous in a neighbourhood of infinity and satisfies the following asymptotic conditions

$$\text{(i)} \quad u^*(x) = \mathcal{O}(|x|^{-1}), \quad \omega^*(x) = \mathcal{O}(|x|^{-2}), \quad \vartheta^*(x) = \mathcal{O}(1), \tag{9}$$

$$\text{(ii)} \quad \lim_{R\to\infty} \frac{1}{4\pi R^2} \int\limits_{\Sigma(0,R)} \vartheta^*(x)\, d\Sigma(0,R) = 0, \quad x \in \Omega^-, \tag{10}$$

where $\Sigma(0,R)$ is a sphere centered at the origin and radius R.

Let $\widetilde{U} = (u, \omega, \vartheta)^\top$, $\widetilde{U}' = (u', \omega', \vartheta')^\top \in [C^2(\overline{\Omega^+})]^7$. Then the following Green formula holds

$$\int\limits_{\Omega^+} \big[\, U' \cdot L(\partial) U - L^*(\partial) U' \cdot U \big]\, dx$$

$$= \int\limits_{\partial\Omega^+} \big[\{U'\}^+ \cdot \{\mathcal{P}(\partial, n) U\}^+ - \{\mathcal{P}^*(\partial, n) U'\}^+ \cdot \{U\}^+ \big]\, dS, \tag{11}$$

where the operators $L(\partial)$, $L^*(\partial) = L^\top(-\partial)$, $\mathcal{P}(\partial, n)$, and $\mathcal{P}^*(\partial, n)$ are defined by relations (1), (3), and (4). Here and in the sequel, the symbols $\{\,\cdot\,\}^\pm$ denote the limiting values on $\partial\Omega^\pm$ from $\Omega^\pm$ respectively and the central dot stands for the scalar product of two vectors.

Theorem 2: *Let $\partial\Omega^+ = S \in C^{1,\kappa}$, $0 < \kappa \leqslant 1$, and U be a regular vector-function from the space $[C^2(\overline{\Omega^+})]^7$. Then the following integral representation formula holds*

$$W^*\big(\{U\}^+\big)(x) - V^*\big(\{\mathcal{P}U\}^+\big)(x) + N^*_{\Omega^+}\big(L^*(\partial)U\big)(x) = \begin{cases} U(x), & x \in \Omega^+, \\ 0, & x \in \Omega^-. \end{cases} \tag{12}$$

Proof: It is standard and follows from Green's formula (11). □

Lemma 3: *The single and double layer potentials $V^*(g)$ and $W^*(g)$, defined by formulae* (6) *and* (7)*, belong to the class $Z^*(\Omega^-)$ and the following operators*

$$\begin{aligned} V^* \ &: \ [C^{k,\sigma}(S)]^7 \to [C^{k+1,\sigma}(\overline{\Omega^\pm})]^7, \\ W^* \ &: \ [C^{k,\sigma}(S)]^7 \to [C^{k,\sigma}(\overline{\Omega^\pm})]^7 \end{aligned} \tag{13}$$

are continuous, provided $S \in C^{k+1,\kappa}$, where $k \geqslant 0$ is an intiger number and $0 < \sigma < \kappa \leqslant 1$.

Proof: The mapping properties (13) can be established by standard arguments, applied, e.g. in the references [4], [5], [6], [7], [9].

To prove the properties (9)-(10) for the layer potentials, note that in view of the equality $\Gamma^*(x-y) = \Gamma^\top(y-x)$, in a vicinity of infinity the following asymptotic relation

$$\Gamma(x-y)=\begin{bmatrix} [\mathcal{O}(|x|^{-1})]_{3\times 3} & [\mathcal{O}(|x|^{-2})]_{3\times 3} & [0]_{3\times 1} \\ [\mathcal{O}(|x|^{-2})]_{3\times 3} & [\mathcal{O}(|x|^{-2})]_{3\times 3} & [0]_{3\times 1} \\ [\chi_0 \frac{x_j}{|x|} + \mathcal{O}(|x|^{-2})]_{1\times 3} & [\mathcal{O}(|x|^{-2})]_{1\times 3} & \mathcal{O}(|x|^{-1}) \end{bmatrix}_{7\times 7} \quad \text{as } |x|\to\infty$$

holds with $\chi_0 = -\dfrac{\eta}{2(\lambda+2\mu)}$, whence we conclude that

$$\begin{aligned} \big[V^*(g)(x)\big]_k &= \int\limits_S \Gamma^*_{kj}(x-y)\, g_j(y)\, dS_y \\ &= \begin{cases} \mathcal{O}(|x|^{-1}), & k=1,2,3, \\ \mathcal{O}(|x|^{-2}), & k=4,5,6, \\ \mathcal{O}(|x|^{-1}) + \chi_0\, \dfrac{x_k}{|x|} \displaystyle\int\limits_S g_7(y)\, dS_y, & k=7. \end{cases} \end{aligned} \tag{14}$$

Thus, if $V^*(g) =: (u^*, \omega^*, \vartheta^*)^\top$, then

$$u^*(x) = \mathcal{O}(|x|^{-1}),\ \ \omega^*(x) = \mathcal{O}(|x|^{-2}),\ \ \ \vartheta^*(x) = \mathcal{O}(1) = \chi_1\, \frac{x_k}{|x|} + \mathcal{O}(|x|^{-1}),$$

with a constant factor χ_1, defined by the equality

$$\chi_1 = \chi_0 \int\limits_S g_7(y)\, dS\, .$$

Now, we show that the function $V_7^*(g) = \vartheta^*$ satisfies the condition (10). Indeed, using the third asymptotic relation in (14), we get

$$\begin{aligned} \lim_{R\to\infty} \frac{1}{4\pi R^2} \int\limits_{\Sigma(0,R)} \vartheta^*(x)\, d\Sigma(0,R) &= \lim_{R\to\infty} \frac{\chi_1}{4\pi R^2} \int\limits_{\Sigma(0,R)} \frac{x}{R}\, d\Sigma(0,R) \\ &= \lim_{R\to\infty} \frac{\chi_1}{4\pi R^2} \int\limits_{\Sigma(0,R)} n(x) d\Sigma(0,R), \end{aligned}$$

where $n(x) = \frac{x}{R}$ is the exterior unit normal vector to the spherical surface $\Sigma(0,R)$ at the point $x \in \Sigma(0,R)$. Due to the Gauss formula we have

$$\int\limits_{\partial\Omega} n_k(x) dS = \int\limits_{\Omega} \frac{\partial\, 1}{\partial x_k} dx = 0, \qquad k=1,2,3,$$

for arbitrary bounded domain Ω with $n = (n_1, n_2, n_3)$ being the exterior unit normal vector to the boundary surface $\partial\Omega$. Consequently,

$$\int\limits_{\Sigma(0,R)} n(x)\, d\Sigma(0,R) = 0$$

and the relation (10) follows. Since the conditions (9) are satisfied automatically for the components of the vector $V^*(g)$, finally we get $V^*(g) \in Z^*(\Omega^-)$.

Now we show that $W^*(g) \in Z^*(\Omega^-)$. To this end, let us note that the following asymptotic relation

$$\Big[\mathcal{P}(\partial, n)\big(\Gamma^*(x-y)\big)^\top\Big]^\top = \begin{bmatrix} [\mathcal{O}(|x|^{-2})]_{3\times3} & [\mathcal{O}(|x|^{-2})]_{3\times3} & [0]_{3\times1} \\ [\mathcal{O}(|x|^{-2})]_{3\times3} & [\mathcal{O}(|x|^{-2})]_{3\times3} & [0]_{3\times1} \\ [\mathcal{O}(|x|^{-1})]_{1\times3} & [\mathcal{O}(|x|^{-1})]_{1\times3} & \mathcal{O}(|x|^{-2}) \end{bmatrix}_{7\times7}$$

holds for sufficiently large $|x|$. In accordance with (7), the last equality implies

$$\big[W^*(g)(x)\big]_k = \begin{cases} \mathcal{O}(|x|^{-2}), & k = 1,2,3, \\ \mathcal{O}(|x|^{-2}), & k = 4,5,6, \\ \mathcal{O}(|x|^{-1}), & k = 7, \end{cases}$$

whence the inclusion $W^*(g) \in Z^*(\Omega^-)$ follows. $\square$

Further we describe the jump relations for the layer potentials.

Theorem 4: *Let $S \in C^{1,\kappa}$, $g \in [C^{0,\sigma}(S)]^7$ and $h \in [C^{1,\sigma}(S)]^7$ with $0 < \sigma < \kappa \leqslant 1$. Then for all points $x \in S$ the following relations hold true:*

$$\{V^*(g)(x)\}^\pm = V^*(g)(x) = \mathcal{H}^* g(x), \tag{15}$$

$$\{\mathcal{P}^*(\partial_x, n(x))V^*(g)(x)\}^\pm = [\mp 2^{-1} I_7 + \mathcal{K}^*]g(x), \tag{16}$$

$$\{W^*(g)(x)\}^\pm = [\pm 2^{-1} I_7 + \mathcal{N}^*]g(x), \tag{17}$$

$$\{\mathcal{P}^*(\partial_x, n(x))W^*(h)(x)\}^+ = \{\mathcal{P}^*(\partial_x, n(x))W^*(h)(x)\}^- = \mathcal{L}^* h(x), \quad S \in C^{2,\kappa}, \tag{18}$$

where the operators $\mathcal{H}^*$, $\mathcal{K}^*$, $\mathcal{N}^*$, and $\mathcal{L}^*$ are pseudodifferential operators of order -1, 0, 0, and 1, respectively, and are defined by the formulae

$$\mathcal{H}^* g(x) := \int\limits_S \Gamma^*(x-y) g(y) dS_y, \tag{19}$$

$$\mathcal{K}^* g(x) := \int\limits_S \Big[\mathcal{P}^*\big(\partial_x, n(x)\big)\Gamma^*(x-y)\Big] g(y) dS_y, \tag{20}$$

$$\mathcal{N}^* g(x) := \int\limits_S \Big[\mathcal{P}\big(\partial_y, n(y)\big)\big[\Gamma^*(x-y)\big]^\top\Big]^\top g(y) dS_y, \tag{21}$$

$$\mathcal{L}^* h(x) := \lim_{\Omega^\pm \ni z \to x \in S} \mathcal{P}^*\big(\partial_z, n(x)\big) \int\limits_S \Big[\mathcal{P}\big(\partial_y, n(y)\big)\big[\Gamma^*(z-y)\big]^\top\Big]^\top g(y)dS_y. \quad (22)$$

Proof: The jump relations (19)-(21) can be proved by standard arguments, described, e.g., in the references [4], [5], [6], [7], [9].

The equality (22), i.e., the relation

$$\big\{\mathcal{P}^*\big(\partial_x, n(x)\big)W^*(h)(x)\big\}^+ = \big\{\mathcal{P}^*\big(\partial_x, n(x)\big)W^*(h)(x)\big\}^-, \quad x \in S,$$

is called the Lyapunov-Tauber type theorem for the double layer potential. We demonstrate here the simplest method of the proof of the property (22). We set $U^* \equiv W^*(h)$. Since $S \in C^{2,\kappa}$ and $h \in [C^{1,\sigma}(S)]^7$ we have $U^* = W^*(h) \in [C^{1,\sigma}(\overline{\Omega^\pm})]^7 \cap [C^\infty(\Omega^\pm)]^7$ in view of Lemma 3. Moreover, U^* solves the homogeneous equation $L^*(\partial)U^* = 0$ in $\Omega^\pm$ and $U^* \in Z(\Omega^-)$. It can be shown that then the integral representation formula, the so called Green's third formula holds for the domains Ω^+ and Ω^- (see Theorem 2):

$$W^*\big(\{U^*\}^+\big)(x) - V^*\big(\{\mathcal{P}U^*\}^+\big)(x) = \begin{cases} U^*(x), & x \in \Omega^+, \\ 0, & x \in \Omega^-, \end{cases}$$

$$-W^*\big(\{U^*\}^-\big)(x) + V^*\big(\{\mathcal{P}U^*\}^-\big)(x) = \begin{cases} 0, & x \in \Omega^+, \\ U^*(x), & x \in \Omega^-. \end{cases}$$

Take the sum of these equalities to obtain

$$U^*(x) = W^*\big([U^*]_S\big)(x) - V^*\big([\mathcal{P}U^*]_S\big)(x), \qquad x \in \Omega^\pm, \quad (23)$$

where

$$[U^*]_S := \{U^*\}^+ - \{U^*\}^-, \qquad [\mathcal{P}U^*]_S := \{\mathcal{P}U^*\}^+ - \{\mathcal{P}U^*\}^- \quad \text{on} \quad S.$$

Evidently, $[U^*]_S := \{U^*\}^+ - \{U^*\}^- = \{W^*(h)\}^+ - \{W^*(h)\}^- = h$ on S due to (17) and from (23) with $U^* \equiv W^*(h)$ we get

$$W^*(h)(x) = W^*(h)(x) - V\big(\psi\big)(x), \quad x \in \Omega^+ \cup \Omega^-, \quad (24)$$

where

$$\psi := [\mathcal{P}^*(\partial_x, n(x))W^*(h)]_S = \{\mathcal{P}W^*(h)\}^+ - \{\mathcal{P}W^*(h)\}^- \quad \text{on} \quad S. \quad (25)$$

Equation (24) yields

$$V\big(\psi\big)(x) = 0, \quad x \in \Omega^+ \cup \Omega^-,$$

and by the jump relations (16) we arrive at the equality

$$0 = \big\{\mathcal{P}^*\big(\partial_x, n(x)\big)V^*(\psi)\big\}^- - \big\{\mathcal{P}^*\big(\partial_x, n(x)\big)V^*(\psi)\big\}^+ = \psi,$$

which along with the relation (25) completes the proof. □

The main goal of our investigation is to study the null spaces of the operators $\mp 2^{-1} I_7 + \mathcal{K}^*$ and $\pm 2^{-1} I_7 + \mathcal{N}^*$, which appear naturally in the analysis of the integral operators corresponding to the direct boundary value problems of the thermoelastostatics for hemitropic solids. To this end, we have to study the homogeneous boundary value problems for the differential operator $L^*(\partial)$.

Let us start with the "adjoint" interior Neumann type boundary value problem: Find a regular solution vector $U^* = (\widetilde{U}^*, \vartheta^*)^\top = (u^*, \omega^*, \vartheta^*)^\top \in [C^1(\overline{\Omega^+})]^7 \cap [C^2(\Omega^+)]^7$ to the differential equation

$$L^*(\partial) U^*(x) = 0, \qquad x \in \Omega^+, \tag{26}$$

satisfying the following boundary condition

$$\big\{\mathcal{P}^*(\partial, n) U^*(x)\big\}^+ = 0, \qquad x \in S. \tag{27}$$

Taking into account the structures of the differential operators $L^*(\partial)$ and $\mathcal{P}^*(\partial, n)$, equations (26) and (27) can be rewritten as follows

$$\begin{aligned} &\widetilde{L}^*(\partial) \widetilde{U}^*(x) = 0, \qquad x \in \Omega^+, \\ &-\eta \operatorname{div} u^*(x) - \zeta \operatorname{div} \omega^*(x) + \kappa' \Delta \vartheta^*(x) = 0, \qquad x \in \Omega^+, \end{aligned}$$

and

$$\begin{aligned} &\big\{T(\partial, n) \widetilde{U}^*(x)\big\}^+ = 0, \qquad x \in S, \\ &\kappa' \frac{\partial \vartheta^*(x)}{\partial n} = 0, \qquad x \in S, \end{aligned}$$

where

$$\widetilde{L}^*(\partial) = \begin{bmatrix} L^{(1)}(\partial) \; L^{(2)}(\partial) \\ L^{(3)}(\partial) \; L^{(4)}(\partial) \end{bmatrix}_{6\times 6},$$

while

$$T(\partial, n) = \begin{bmatrix} T^{(1)}(\partial, n) \; T^{(2)}(\partial, n) \\ T^{(3)}(\partial, n) \; T^{(4)}(\partial, n) \end{bmatrix}_{6\times 6},$$

with $L^{(j)}(\partial)$ and $T^{(j)}(\partial, n)$, $j = \overline{1,4}$, defined in (2) and (5). The operators $\widetilde{L}^*(\partial)$ and $T(\partial, n)$ correspond to the hemitropic elasticity when thermal effects are not taken into consideration (see [9]). Note that the operator $\widetilde{L}^*(\partial)$ is self-adjoint and $\widetilde{L}^*(\partial) = \widetilde{L}(\partial)$.

As we see, the boundary value problem for the vector-function $\widetilde{U}^* = (u^*, \omega^*)^\top$ is separated and reads as

$$\widetilde{L}^*(\partial) \widetilde{U}^*(x) \equiv \widetilde{L}(\partial) \widetilde{U}^*(x) = 0, \qquad x \in \Omega^+, \tag{28}$$

$$\big\{T(\partial, n) \widetilde{U}^*(x)\big\}^+ = 0, \qquad x \in S. \tag{29}$$

It is shown in [9] that the general solution to the problem (28)-(29) has the following form

$$\widetilde{U}^* = (u^*, \omega^*)^\top = \big([a \times x] + b, a\big)^\top, \tag{30}$$

$$u^*(x) = [a \times x] + b, \qquad \omega^*(x) = a, \tag{31}$$

where a and b are arbitrary three-dimensional real constant vectors.

For the vectors u^* and ω^*, defined by equations (31), we have $\operatorname{div} u^* = 0$ and $\operatorname{div} \omega^* = 0$, and therefore for the temperature function ϑ^* we obtain the following Neumann type boundary value problem for Laplace equation

$$\Delta\vartheta^*(x) = 0, \quad x \in \Omega^+, \tag{32}$$

$$\big\{\partial_n \vartheta^*(x)\big\}^+ = 0, \quad x \in S. \tag{33}$$

Consequently,

$$\vartheta^*(x) = const = c, \quad x \in \Omega^+.$$

Thus, we have proved the following assertion.

Theorem 5: *The vector-function* $U^* = \big(\widetilde{U}^*, \vartheta^*\big)^\top = ([a \times x] + b, a, c)^\top$, *where* a *and* b *are arbitrary three-dimensional real constant vectors, while* c *is an arbitrary real scalar constant, is a general solution to the homogeneous boundary value problem* (26)-(27).

Next we consider the exterior Neumann type boundary value problem: Find a regular solution vector $U^* = (\widetilde{U}^*, \vartheta^*)^\top = (u^*, \omega^*, \vartheta^*)^\top \in [C^1(\overline{\Omega^-})]^7 \cap [C^2(\Omega^-)]^7 \cap Z^*(\Omega^-)$ to the differential equation

$$L^*(\partial)U^*(x) = 0, \qquad x \in \Omega^-, \tag{34}$$

satisfying the homogeneous boundary condition

$$\big\{\mathcal{P}^*(\partial, n)U^*(x)\big\}^- = 0, \qquad x \in S. \tag{35}$$

As in the case of the interior problem, this problem is decomposed into two boundary value problems, the exterior counterparts of the problems (28)-(29) and (32)-(33). Applying the corresponding uniqueness results to the exterior Neumann type boundary value problem in the class of functions, satisfying the asymptotic decay conditions, shown in (9) we conclude that (see [9])

$$u^*(x) = 0, \quad \omega^*(x) = 0, \qquad x \in \Omega^-.$$

Therefore the temperature function ϑ^* solves the classical exterior Neumann type boundary value problem

$$\Delta\vartheta^*(x) = 0, \qquad x \in \Omega^-,$$

$$\big\{\partial_n \vartheta^*(x)\big\}^- = 0, \quad x \in S,$$

in the space of bounded functions, satisfying the asymptotic constraints

$$\vartheta^*(x) = \mathcal{O}(1) \quad \text{as} \quad |x| \to \infty,$$
$$\lim_{R\to\infty} \frac{1}{4\pi R^2} \int\limits_{\Sigma(0,R)} \vartheta^*(x)\, d\Sigma(0,R) = 0.$$

It can be shown that this problem possesses only the trivial solution and we arrive at the following uniqueness result.

Theorem 6: *The "adjoint" exterior Neumann type boundary value problem* (34)-(35) *possesses only the trivial solution in the space of regular vector-functions* $[C^1(\overline{\Omega^-})]^7 \cap [C^2(\Omega^-)]^7 \cap Z^*(\Omega^-)$.

Similar theorems hold also for the Dirichlet problems.

Theorem 7: *The "adjoint" interior and exterior Dirichlet type homogeneous boundary value problems,*

$$L^*(\partial)U^*(x) = 0, \qquad x \in \Omega^\pm,$$
$$\big\{U^*(x)\big\}^\pm = 0, \qquad x \in S,$$

possess only the trivial solution in the space of regular vector-functions $[C^1(\overline{\Omega^\pm})]^7 \cap [C^2(\Omega^\pm)]^7$), *provided* $U^* \in Z^*(\Omega^-)$ *in the case of the exterior problem.*

Now we are in a position to investigate the null spaces of the boundary integral operators, generated by the single and double layer potentials. In particular, the above formulated uniqueness results yield the following assertions. We start with auxiliary lemmata.

Lemma 8: *The following equalities hold in appropriate function spaces:*

$$\begin{aligned} &\mathcal{N}^*\mathcal{H}^* = \mathcal{H}^*\mathcal{K}^*, &&\mathcal{L}^*\mathcal{N}^* = \mathcal{K}^*\mathcal{L}^*, \\ &\mathcal{H}^*\mathcal{L}^* = -4^{-1}\, I_7 + [\mathcal{N}^*]^2, &&\mathcal{L}^*\mathcal{H}^* = -4^{-1}\, I_7 + [\mathcal{K}^*]^2. \end{aligned} \tag{36}$$

Lemma 9: *Let* $S \in C^{2,\kappa}$ *and* $0 < \sigma < \kappa \leqslant 1$. *The integral operator*

$$\mathcal{H}^* \;:\; [C^{0,\sigma}(S)]^7 \to [C^{1,\sigma}(S)]^7 \tag{37}$$

is invertible and

$$[\mathcal{H}^*]^{-1} \;:\; [C^{1,\sigma}(S)]^7 \to [C^{0,\sigma}(S)]^7 \tag{38}$$

is a pseudodifferential operator of order 1, *more precisely, it is a singular integro-differential operator.*

Proof: It is word for word of the proof of Theorem 6.6 in [13]. □

Theorem 10: *Let* $S \in C^{2,\kappa}$ *and* $0 < \sigma < \kappa \leqslant 1$. *The null spaces of the singular integral operators*

$$2^{-1}I_7 + \mathcal{K}^* \;:\; [C^{0,\sigma}(S)]^7 \to [C^{0,\sigma}(S)]^7, \tag{39}$$

$$2^{-1}I_7 + \mathcal{N}^* \ : \ [C^{0,\sigma}(S)]^7 \to [C^{0,\sigma}(S)]^7, \tag{40}$$

are trivial, while the null spaces of the singular integral operators

$$\begin{aligned} -\,2^{-1}I_7 + \mathcal{K}^* \ &: \ [C^{0,\sigma}(S)]^7 \to [C^{0,\sigma}(S)]^7, \\ -\,2^{-1}I_7 + \mathcal{N}^* \ &: \ [C^{0,\sigma}(S)]^7 \to [C^{0,\sigma}(S)]^7, \end{aligned} \tag{41}$$

have the dimension, equal to 7. Moreover, the vectors

$$\begin{gathered} \Psi^{(1)}(x) = (0, -x_3, x_2, 1, 0, 0, 0)^\top, \qquad \Psi^{(2)}(x) = (x_3, 0, -x_1, 0, 1, 0, 0)^\top, \\ \Psi^{(3)}(x) = (-x_2, x_1, 0, 0, 0, 1, 0)^\top, \qquad \Psi^{(4)}(x) = (1, 0, 0, 0, 0, 0, 0)^\top, \\ \Psi^{(5)}(x) = (0, 1, 0, 0, 0, 0, 0, 0)^\top, \qquad \Psi^{(6)}(x) = (0, 0, 1, 0, 0, 0, 0)^\top, \\ \Psi^{(7)}(x) = (0, 0, 0, 0, 0, 0, 0, 1)^\top, \end{gathered} \tag{42}$$

restricted onto the surface S represent a basis $\big\{\Psi^{(k)}(x)\big\}_{k=1}^{k=7}$ of the null space of the operator $-2^{-1}I_7 + \mathcal{N}^*$.

Proof: Let $g \in [C^{0,\sigma}(S)]^7$ be a solution of the homogeneous integral equation

$$2^{-1}\, g(x) + \mathcal{K}^* g(x) = 0, \qquad x \in S.$$

Then the vector

$$U^*(x) := V^*(g)(x), \qquad x \in \Omega^\pm,$$

solves the exterior homogeneous Neumann type boundary value problem and since

$$U^* = V^*(g) \in [C^{1,\sigma}(\overline{\Omega^\pm})]^7 \cap [C^2(\Omega^\pm)]^7 \cap Z(\Omega^-),$$

by Theorem 6 we conclude $U^*(x) = V^*(g) = 0$ in Ω^-. Further, in view of (15), we see that $U^*(x) = V^*(g)$ solves the interior Dirichlet type problem in Ω^+ and due to Theorem 7 it vanishes identically in the interior domain: $U^*(x) = V^*(g) = 0$ in Ω^+. Therefore, in accordance with the jump relations (16), we finally get

$$\{\mathcal{P}^*(\partial_x, n(x))V^*(g)(x)\}^- - \{\mathcal{P}^*(\partial_x, n(x))V^*(g)(x)\}^+ = g(x) = 0, \qquad x \in S,$$

which proves that the kernel of the operator (39) is trivial.

For the operator (40) we proceed as follows. Let $g \in [C^{0,\sigma}(S)]^7$ be a solution of the homogeneous integral equation

$$2^{-1}\, g(x) + \mathcal{N}^* g(x) = 0, \qquad x \in S.$$

Due to the embedding theorems for solutions to the singular integral equations (see e.g., [6], Ch. IV), we actually have $g \in [C^{1,\sigma}(S)]^7$ and the vector

$$U^*(x) := W^*(g)(x), \qquad x \in \Omega^\pm,$$

solves the interior homogeneous Dirichlet type boundary value problem. Since

$$U^* = W^*(g) \in [C^{1,\sigma}(\overline{\Omega^\pm})]^7 \cap [C^2(\Omega^\pm)]^7 \cap Z(\Omega^-),$$

by Theorem 7 we conclude $U^*(x) = W^*(g)(x) = 0$ in Ω^+. Further, in view of (18), we see that $U^* = W^*(g)$ solves the exterior Neumann type problem in Ω^- and due to Theorem 6 it vanishes identically in the exterior domain: $U^*(x) = W^*(g)(x) = 0$ in Ω^-. Therefore, in accordance with the jump relations (17), we finally get

$$\{W^*(g)(x)\}^+ - \{W^*(g)(x)\}^- = g(x) = 0, \qquad x \in S,$$

which proves that the kernel of the operator (40) is trivial.

Next we show that the vectors (42) represent a basis in the null space of the operator (41). On the one hand, using the integral representation formula (12) for the vectors $\Psi^{(k)}(x)$, $k = \overline{1,7}$, and taking into account that $\{\mathcal{P}^*(\partial_x, n(x))\Psi^{(k)}(x)\}^+ = 0$ on S, it is easy to show that

$$\Psi^{(k)}(x) = W^*(h^{(k)})(x), \qquad x \in \Omega^+,$$

with

$$h^{(k)}(x) = \{\Psi^{(k)}(x)\}^+ = \Psi^{(k)}(x), \qquad x \in S, \quad k = \overline{1,7},$$

whence the inclusion $h^{(k)} \in \ker\{-2^{-1}I_7 + \mathcal{N}^*\}$ follows immediately in view of the jump relation (17). Consequently, $\dim\ker\{-2^{-1}I_7 + \mathcal{N}^*\} \geqslant 7$, since the system of vectors $\{\Psi^{(k)}(x)\}_{k=1}^{k=7}$ on S are linearly independent. On the other hand, for an arbitrary solution $h \in [C^{0,\sigma}(S)]^7$ of the homogeneous integral equation

$$-\,2^{-1}\,h(x) + \mathcal{N}^* h(x) = 0, \qquad x \in S,$$

as above, we conclude that $h \in [C^{1,\sigma}(S)]^7$ and the vector $U^* := W^*(h)$ vanishes identically in Ω^-, since $U^* = W^*(h) \in [C^{1,\sigma}(\overline{\Omega^\pm})]^7 \cap [C^2(\Omega^\pm)]^7 \cap Z(\Omega^-)$ and solves the exterior homogeneous Dirichlet type problem. Further, by the Lyapunov-Tauber type theorem (see (18)), we see that the vector-function $U^* = W^*(h)$ solves the homogeneous interior Neumann type problem and therefore by Theorem 5 we have

$$U^*(x) = W^*(h) = ([a \times x] + b, a, c)^\top, \qquad x \in \Omega^+,$$

with some constant vectors a and b, and a scalar constant c. Then it follows that

$$h(x) = \{W^*(h)(x)\}^+ - W^*(h)(x)\}^- = ([a \times x] + b, a, c)^\top, \qquad x \in S,$$

which implies that h belongs to linear span of the vectors $\{\Psi^{(k)}(x)\}_{k=1}^{k=7}$, $x \in S$, and consequently $\dim\ker\{-2^{-1}I_7 + \mathcal{N}^*\} \leqslant 7$. This proves that the system $\{\Psi^{(k)}(x)\}_{k=1}^{k=7}$ is a basis of the null space $\ker\{-2^{-1}I_7 + \mathcal{N}^*\}$.

By the similar arguments it can be shown that the system of vector-functions defined on S $\{[\mathcal{H}^*]^{-1}\Psi^{(k)}(x)\}_{k=1}^{k=7}$, where $[\mathcal{H}^*]^{-1}$ is the operator inverse to $[\mathcal{H}^*]$ (see

(37) and (38)) represents a basis of the null space $\ker\{-2^{-1}I_7+\mathcal{K}^*\}$. Indeed, by Lemmata 8 and 9 from the first relation in (36), we find that

$$[\mathcal{H}^*]^{-1}\mathcal{N}^* = \mathcal{K}^*[\mathcal{H}^*]^{-1}.$$

Therefore,

$$\big[-2^{-1}I_7+\mathcal{K}^*\big]\,[\mathcal{H}^*]^{-1}\Psi^{(k)} = [\mathcal{H}^*]^{-1}\big[-2^{-1}I_7+\mathcal{N}^*\big]\,\Psi^{(k)} = 0$$

since $\Psi^{(k)} \in \ker\{-2^{-1}I_7+\mathcal{N}^*\}$, which completes the proof. □

Remark 1 : More detailed analysis, based on the results in [13], shows that the singular integral operators

$$\begin{aligned}
&\pm\, 2^{-1}I_7+\mathcal{K}^* \ :\ [H_2^{-\frac{1}{2}}(S)]^7 \to [H_2^{-\frac{1}{2}}(S)]^7,\\
&\pm\, 2^{-1}I_7+\mathcal{N}^* \ :\ [H_2^{\frac{1}{2}}(S)]^7 \to [H_2^{\frac{1}{2}}(S)]^7,
\end{aligned}$$

are Fredholm with zero index and have the same null spaces, described in Theorem 10.

References

[1] J. Dyszlewicz, *Micropolar Theory of Elasticity,* Lecture Notes in Applied and Computational Mechanics, **15**, Springer-Verlag, Berlin, 2004

[2] A.C. Eringen, *Microcontinuum Field Theories. I: Foundations and Solids.* Springer-Verlag, New York, 1999

[3] R. Gachechiladze, I. Gwinner, D. Naroshvili, *A boundary variational inequality approach to unilateral contact with hemitropic materials*, Memoirs on Differential Equations and Mathematical Physics, **39** (2006), 69-103

[4] L. Jentsch and D. Natroshvili, *Three-dimensional mathematical problems of thermoelasticity of anisotropic bodies.* Part I. Memoirs on Differential Equations and Mathematical Physics, **17** (1999), 7–127

[5] L. Jentsch and D. Natroshvili, *Three-dimensional mathematical problems of thermoelasticity of anisotropic bodies*, Part II. Memoirs on Differential Equations and Mathematical Physics, **18** (1999), 1–50

[6] V.D. Kupradze, T.G. Gegelia, M.O. Basheleishvili, and T.V. Burchuladze, *Three Dimensional Problems of the Mathematical Theory of Elasticity and Thermoelasticity* (in Russian), Nauka, Moscow, 1976 (English translation: North Holland Series in Applied Mathematics and Mechanics **25**, North Holland Publishing Company, Amsterdam, New York, Oxford, 1979)

[7] D. Natroshvili, A. Djagmaidze and M. Svanadze, *Problems of the Linear Theory of Elastic Mixtures*, Tbilisi University, Tbilisi, 1986

[8] D. Natroshvili, R. Gachechiladze, A. Gachechiladze, and I.G. Stratis, *Transmission problems in the theory of elastic hemitropic materials*, Applicable Analysis, **86**, 12 (2007), 1463-1508

[9] D. Natroshvili, L. Giorgashvili, and I.G. Stratis, *Mathematical problems of the theory of elasticity of chiral materials*, Applied Mathematics, Informatics, and Mechanics, **8**, No. 1 (2003), 47–103

[10] D. Natroshvili, L. Giorgashvili, and I.G. Stratis, Representation formulas of general solutions in the theory of hemitropic elasticity, *Quart. J. Mech. Appl. Math.,* **59** (2006), 451–474

[11] D. Natroshvili and I.G. Stratis, *Mathematical problems of the theory of elasticity of chiral materials for Lipschitz domains*, Mathematical Methods in the Applied Sciences, **29**, Issue 4 (2006), 445-478

[12] D. Natroshvili, L. Giorgashvili, and S. Zazashvili, *Steady state oscillation problems of the theory of elasticity of chiral materials*, Journal of Integral Equations and Applications, **17**, No. 1, Spring (2005), 19-69

[13] D. Natroshvili, L. Giorgashvili, and S. Zazashvili, *Mathematical problems of thermoelasticity for hemitropic solids*, Memoirs on Differential Equations and Mathematical Physics, **48**(2009), 97-174