MAIN ARTICLES

THE NEUMANN PROBLEM FOR A DEGENERATE DIFFERENTIAL–OPERATOR EQUATION

Liparit Tepoyan

Yerevan State University, Faculty of mathematics and mechanics

Abstract. We consider the Neumann problem for a degenerate differential–operator equation of higher order. We establish some embedding theorems in weighted Sobolev space $W^m_\alpha$ and show existence and uniqueness of the generalized solution of the Neumann problem. We also give a description of the domain of definition and of the spectrum for the corresponding operator.

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1. Introduction

In present paper we consider the Neumann problem for the operator equation

$$Pu \equiv (-1)^m D_t^m(t^\alpha D_t^m)u + t^\alpha Au = f,$$

where $t \in (0,b)$, $\alpha \geq 0$, $D_t \equiv d/dt$, $f \in L_{2,\alpha}(0,b)$, and $A$ is a linear operator in Hilbert space $H$ and has a complete system $\{\varphi_k\}_{k=1}^\infty$ of the eigenfunctions, which form a Riesz’s basis in $H$. Note that the operator $A$ in general is an unbounded operator in $H$.

Our approach, similar to that used in [3], for the case $m = 1$ and in [11] for $m = 2$, is based on the consideration of the one-dimensional equation (1), i.e. when $A$ is the operator of multiplication by a number $a$, $a \in \mathbb{C}$, $Au = au$ (see [4]).

In Section 2 we define the weighted Sobolev space $W^m_\alpha$, describe the behavior of the functions from this space close to $t = 0$ (see [7], [8], [13]). We give the description of the domain of the definition $D(B)$ of the operator $B$ and prove that for $1 - \alpha \notin \sigma\mathbb{B}$ ($\sigma\mathbb{B}$ is the spectrum of the operator $\mathbb{B}$) the generalized solution of the Neumann problem for the one–dimensional equation (1) exists and is unique for every $f \in L_{2,\alpha}$.

In Section 3 under some conditions on the spectrum of the operator $A$ we prove unique solvability of the operator equation (1) for every $f \in L_{2,\alpha}(0,b)$ and give the description of the spectrum for the corresponding operator $\mathbb{P} = t^{-\alpha}P$. 
2. The One-dimensional Case

2.1. The space $W^m_\alpha$

Denote by $W^m_\alpha$ the completion of $C^m[0,b]$ in the norm

$$||u||^2_{W^m_\alpha} = \int_0^b \left( t^\alpha |u^{(m)}(t)|^2 + t^\alpha |u(t)|^2 \right) dt. \quad (2)$$

For the proofs of the Propositions 1, 2 and Remark 3 see [2] and [13].

**Proposition 1** For every $u \in W^m_\alpha$ close to $t = 0$ we have

$$|u^{(j)}(t)|^2 \leq (B_j + C_j t^{2m-2j-1-\alpha}) ||u||^2_{W^m_\alpha}, \quad (3)$$

where $\alpha \neq 1, 3, \ldots, 2m-1$, $j = 0, 1, \ldots, m-1$. For $\alpha = 2n+1$, $n = 0, 1, \ldots, m-1$ in (3) the factor $t^{2m-2j-1-\alpha}$ is to be replaced by $t^{2m-2j-2n-2}\ln|t|$, $j = 0, 1, \ldots, m-n-1$.

From Proposition 1 it follows that in the case $\alpha < 1$ (weak degeneracy) $u^{(j)}(0)$ exist for all $j = 0, 1, \ldots, m-1$, while for $\alpha \geq 1$ (strong degeneracy) not all $u^{(j)}(0)$ exist. More precisely, for $1 \leq \alpha < 2m-1$ the derivatives at zero $u^{(j)}(0)$ exist only for $j = 0, 1, \ldots, s_\alpha$, where $s_\alpha = m-1 - \lfloor \frac{\alpha+1}{2} \rfloor$ (here $\lfloor a \rfloor$ is the integer part of a number $a$) and for $\alpha \geq 2m-1$ all $u^{(j)}(0)$, $j = 0, 1, \ldots, m-1$, in general may be infinite.

**Proposition 2** The embedding

$$W^m_\alpha \subset L_{2,\alpha} \quad (4)$$

is compact for every $\alpha \geq 0$.

**Remark 3** The embedding

$$W^m_\alpha \subset L_{2,\beta} \quad (5)$$

is compact for every $\alpha > 2m-1$ and $\beta > \alpha - 2m$.

Observe that in the case $\beta = \alpha - 2m$ and $\alpha \leq 2m-1$ the embedding (5) fails (see [8]). For $\alpha \leq 2m-1$ we only have the embedding $W^m_\alpha \subset L_{2,\gamma}, \gamma > -1$. However, for $\alpha > 2m-1$ we have the embedding $W^m_\alpha \subset L_{2,\alpha-2m}$, which can be proved by using of the Hardy inequality (see [6] and [8]) and this embedding is not compact. Indeed, it is easy to verify, that for the bounded in $W^m_\alpha$ sequence $u_n(t) = n^{-\frac{1}{2}} t^{\frac{2m-\alpha-1}{2}} (\ln t)^{-\frac{1}{2}} \varphi(t)$, where $\varphi \in C^m[0,b]$, $\varphi(t) = 1$ for $t \in [0, \frac{\varepsilon}{2}], 0 < \varepsilon < \min \{1, b\}$ and $\varphi(t) = 0$ for $t \in [\varepsilon, b]$ doesn’t exist the convergent in $L_{2,\alpha-2m}$ subsequence (see [5], [12]).
2.2 One-dimensional Equation

Now we consider the Neumann problem for the special case $a = 1$ of the one-dimensional equation (1)

$$Bu \equiv (-1)^m (t^\alpha u^{(m)})^{(m)} + t^\alpha u = f, \quad f \in L_{2,-\alpha}. \tag{6}$$

**Definition 4** A function $u \in W^m_\alpha$ is called a generalized solution of the Neumann problem for the equation (6) if for every $v \in W^m_\alpha$ we have

$$(t^\alpha u^{(m)}, v^{(m)}) + (t^\alpha u, v) = (f, v), \tag{7}$$

where $(\cdot, \cdot)$ is the scalar product in $L_2(0, b)$.

**Proposition 5** The generalized solution of the Neumann problem for the equation (6) exists and is unique for every $f \in L_{2,-\alpha}$.

The uniqueness of the generalized solution immediately follows from Definition 4. To prove the existence we note that the linear functional $l_f(v) = (f, v)$ is continuous in $W^m_\alpha$ because

$$|l_f(v)| \leq ||f||_{L_{2,-\alpha}} ||v||_{L_{2,\alpha}} \leq ||f||_{L_{2,-\alpha}} ||v||_{W^m_\alpha}$$

and use Riesz’s lemma on the representation of continuous functionals.

If the generalized solution is classical then from (7) after integration by parts we get

$$(-1)^m ((t^\alpha u^{(m)}(t))^{(m)} v(t))^{(m)} + \sum_{j=0}^{m-1} (-1)^j ((t^\alpha u^{(m)}(t))^{(j)} v^{(m-j-1)}(t))\big|_0^b + (t^\alpha u, v) = (f, v).$$

Since the function $v \in W^m_\alpha$ is arbitrarily we conclude that the function $u(t)$ fulfills the following conditions (see [10])

$$(t^\alpha u^{(m)}(t))^{(j)}|_{t=0} = u^{(m+j)}(t)|_{t=b} = 0, \quad j = 0, 1, \ldots, m - 1. \tag{8}$$

For $\alpha = 0$ the conditions (8) are usual Neumann conditions, which are of Sturm type and, therefore, regular (see [9]).

**Definition 6** We say that $u \in W^m_\alpha$ belongs to $D(B)$, if the equality (7) is satisfied for some $f \in L_{2,-\alpha}$. In this case we will write $Bu = f$.

According to Definition 6 we have an operator

$$B : D(B) \subset W^m_\alpha \subset L_{2,\alpha} \to L_{2,-\alpha}.$$ 

If $u \in W^m_\alpha$ we know that $u^{(j)}(0), j = 0, 1, \ldots, m - 1$ exist only for $\alpha < 2m - 2j - 1$ (see Proposition 1). But for the generalized solution of the equation (7) we can improve it and give the following description of $u \in D(B)$.
Moreover, the inverse operator of the solutions for the equation \((-1)^m(t^\alpha u^{(m)})^{(m)} = f, f \in L_{2,-\alpha}\) near to the point \(t = 0\). Let \(\alpha \geq 2m - 1\). For the solution \(u(t)\) of this equation we have

\[
u^{(m)}(t) = t^{-\alpha}P_{m-1}(t) + \frac{(-1)^{m-1}t^{-\alpha}}{(m-1)!} \int_0^t (t - \tau)^{m-1} f(\tau) \, d\tau,
\]

where \(P_{m-1}(t) = a_0 + a_1t + \cdots + a_{m-1}t^{m-1}\) is the polynomial of the degree \(m - 1\). Since \(u \in W^m_\alpha\) we have that \(P_{m-1}(t) = 0\). Hence we can write

\[|u^{(m)}(t)| = \left| \frac{(-1)^{m-1}t^{-\alpha}}{(m-1)!} \int_0^t (t - \tau)^{m-1} \tau^a \tau^{-\frac{\alpha}{2}} f(\tau) \, d\tau \right| \leq c ||f||_{L_{2,-\alpha}} t^{2m-1-\alpha},\]

therefore, integrating \(u^{(m)}(t)\) \((m - j)\)-times, \(j = 0, 1, \ldots, m - 1\), we get for some polynomial \(Q_{m-j-1}\) of the degree \(m - j - 1\)

\[|u^{(j)}(t)| = \left| Q_{m-j-1}(t) + \frac{1}{(m-j-1)!} \int_0^t (t - \tau)^{m-j-1} u^{(m)}(t) \, d\tau \right| \leq c_j + d_j ||f||_{L_{2,-\alpha}} t^{2m-1-\alpha} + m-j.
\]

Let now \(2m-2j-1 \leq \alpha < 2m-2j+1, j = 1, 2, \ldots, m-1\). Then in the equality (9) \(a_0 = a_1 = \cdots = a_{m-j-1} = 0\) since \(u \in W^m_\alpha\). The second term in (9) we have already estimated and it exists for \(\alpha < 2m - 1\). Now it is enough to estimate only first term after integrating \(u^{(m)}(t)\) \((m - j)\)-times. The main term after integration remains \(c_{m-j}t^{2m-2j-\alpha}\), therefore for the existence of \(u^{(j)}(0)\) we get the condition \(2m - 2j - \alpha > 0\), i.e., \(\alpha < 2m - 2j\). Note that for other values of \(\alpha\) the existence of the values \(u^{(j)}(0)\) is proved in Proposition 1 (see [13]). Note also that the conditions in Proposition 7 are exact, i.e., if we take for example \(\alpha = 2m - 2j\), then the statement is false, the value \(u^{(j)}(0)\) in general doesn’t exist. □

To get an operator in the same space we set \(g(t) = t^{-\alpha}f(t)\). It is evident that \(g(t)\) belongs to \(L_{2,\alpha}\) and \(||f||_{L_{2,-\alpha}} = ||g||_{L_{2,\alpha}}\). Therefore, we get an operator \(B \equiv t^{-\alpha}B : D(B) = D(B) \subset W^m_\alpha \subset L_{2,\alpha} \rightarrow L_{2,\alpha}\) with \(Bu = g\).

**Proposition 8** The operator \(B : L_{2,\alpha} \rightarrow L_{2,\alpha}\) is positive and selfadjoint. Moreover, the inverse operator \(B^{-1} : L_{2,\alpha} \rightarrow L_{2,\alpha}\) is compact.

**Proof.** The self-adjointness of the symmetric operator \(B\) (symmetry and positivity of the operator \(B\) follow from the Definition 6) is a consequence of
the existence of the generalized solution for every \( f \in L_{2,-\alpha} \) (see [4]). Now using (2) and the equality (7) with \( v = u \) we get
\[
||u||_{W^{m}}^2 = (f,u) \leq ||f||_{L_{2,-\alpha}} ||u||_{L_{2,\alpha}} \leq ||g||_{L_{2,\alpha}} ||u||_{W^{m}}.
\]
and, therefore, we have
\[
||u||_{L_{2,\alpha}} \leq ||B u||_{L_{2,\alpha}}. \tag{10}
\]
The compactness of the operator \( B^{-1} : L_{2,\alpha} \to L_{2,\alpha} \) now follows from the inequality (10) and Proposition 2.

**Corollary 9** The operator \( B \) has a discrete spectrum, and the system of the corresponding eigenfunctions is dense in \( L_{2,\alpha} \).

This follows from the connection of the spectra of the operators \( B \) and \( B^{-1} \) and from the properties of compact selfadjoint operators (see [4]).

Note that if \( \lambda \) is an eigenvalue and \( u(t) \) a corresponding eigenfunction of the operator \( B \) then we have
\[
(-1)^m (t^\alpha u^{(m)})^{(m)} + t^\alpha u = \lambda t^\alpha u. \tag{11}
\]
It follows then from the inequality (10) and Definition 6 that \( \lambda \geq 1 \). Note that the number \( \lambda = 1 \) is an eigenvalue for the operator \( B \) with the multiplicity \( m \) since every polynomial of order \( m - 1 \) is an eigenfunction. Therefore, for the solvability of the equation
\[
(-1)^m (t^\alpha u^{(m)})^{(m)} = f, \quad f \in L_{2,-\alpha}, \tag{12}
\]
we get the following result:

**Proposition 10** The generalized solution of the Neumann problem for the equation (12) exists if and only if \( (f, P_{m-1}(t)) = 0 \) for any polynomial \( P_{m-1}(t) \) of order \( m - 1 \).

Here we have used both \( (g, P_{m-1}(t))_{\alpha} = (f, P_{m-1}(t)) \) since \( t^\alpha g(t) = f(t) \) ( \( (\cdot, \cdot)_{\alpha} \) is the scalar product in \( L_{2,\alpha} \)) and the definition of the operator \( B \). Note that the generalized solution of the Neumann problem for the equation (12) is unique up to an arbitrary additive polynomial of order \( m - 1 \).

Now we can consider the general case of the one-dimensional equation (1)
\[
P u \equiv (-1)^m (t^\alpha u^{(m)})^{(m)} + at^\alpha u = f, \quad f \in L_{2,-\alpha}, \tag{13}
\]
because the number \( 1 - a \) can be regarded as a spectral parameter for the operator \( B \). Therefore, we can state that if \( 1 - a \notin \sigma B \) then the equation (13) is uniquely solvable for every \( f \in L_{2,-\alpha} \).
3. The Operator Equation

In this section we consider the operator version of the equation (1)

\[ Pu \equiv (-1)^m D_t^m (t^\alpha D_t^m) u + t^\alpha Au = f, \quad f \in L_{2,-\alpha}((0, b), H), \quad \alpha \geq 0. \]  

(14)

Suppose that the operator \( A : H \to H \) has a complete system of eigenfunctions \( \{ \varphi_k \}_{k=1}^\infty, A \varphi_k = a_k \varphi_k, k \in \mathbb{N} \), forming a Riesz’s basis in \( H \) (see [4]), i.e., for every \( x \in H \) we have \( x = \sum_{k=1}^\infty x_k \varphi_k \), and there are some positive constants \( c_1 \) and \( c_2 \) such that

\[ c_1 \sum_{k=1}^\infty |x_k|^2 \leq \|x\|^2 \leq c_2 \sum_{k=1}^\infty |x_k|^2. \]  

(15)

Hence for every \( u \in L_{2,\alpha}((0, b), H), f \in L_{2,-\alpha}((0, b), H) \) we have

\[ u = \sum_{k=1}^\infty u_k(t) \varphi_k, \quad f = \sum_{k=1}^\infty f_k(t) \varphi_k, \quad k \in \mathbb{N}. \]  

(16)

Therefore, the operator equation (14) can be decomposed into an infinite chain of ordinary differential equations

\[ P_k u_k \equiv (-1)^m (t^\alpha u_k^{(m)})^{(m)} + a_k t^\alpha u_k = f_k, \quad f_k \in L_{2,-\alpha}, \quad k \in \mathbb{N}. \]  

(17)

For the equations (17) we can define the generalized solutions \( u_k(t), k \in \mathbb{N} \), of the Neumann problem (see Section 2).

**Definition 11** A function \( u \in L_{2,\alpha}((0, b), H) \) is called a generalized solution of the Neumann problem for the equation (14) if the functions \( u_k(t), k \in \mathbb{N} \), in the representation (16) are generalized solutions of the Neumann problem for the equations (17).

**Proposition 12** The operator equation (14) is uniquely solvable for every \( f \in L_{2,-\alpha}((0, b), H) \) if and only if the equations (17) are uniquely solvable for every \( f_k \in L_{2,-\alpha}, k \in \mathbb{N} \), and the inequalities

\[ \|u_k\|_{L_{2,\alpha}} \leq c \|f_k\|_{L_{2,-\alpha}} \]  

(18)

are satisfied uniformly with respect to \( k \in \mathbb{N} \).

For the proof of Proposition 12 see [4].

Let the numbers \( \lambda_1 = 1 < \lambda_2 < \cdots < \lambda_k < \cdots, \lambda_k \to +\infty \) when \( k \to \infty \), are the eigenvalues of the operator \( \mathbb{B} \) (see Section 2). Suppose that

\[ \rho(1 - a_k, \lambda_m) > \varepsilon, \quad k, m \in \mathbb{N}, \]  

(19)

where \( \varepsilon > 0 \) and \( \rho \) is the distance in the complex plane.
Theorem 13 Under the condition (19) the generalized solution of the Neumann problem for the operator equation (14) exists and is unique for every $f \in L_{2,-\alpha}((0,b),H)$.

First note that under the condition (19) the equations (17) are uniquely solvable for every $f_k \in L_{2,-\alpha}$, $k \in \mathbb{N}$ and the inequalities (18) are satisfied. Now the proof of Theorem 13 follows from Proposition 12.

Let $g = t^{-\alpha}f$, $f \in L_{2,-\alpha}((0,b),H)$. Then $g \in L_{2,\alpha}((0,b),H)$ and we define an operator

$$\mathbb{P} \equiv t^{-\alpha}P : D(\mathbb{P}) = D(P) \subset L_{2,\alpha}((0,b),H) \to L_{2,\alpha}((0,b),H),$$

with $\mathbb{P}u = g$ in $L_{2,\alpha}((0,b),H)$. It follows from the condition (19) that for the generalized solution of the Neumann problem we have

$$||u||_{L_{2,\alpha}((0,b),H)} \leq c||g||_{L_{2,\alpha}((0,b),H)}. \quad (20)$$

The operator $\mathbb{P}^{-1} : L_{2,\alpha}((0,b),H) \to L_{2,\alpha}((0,b),H)$ in general is not compact in contrast to Proposition 8 (it will be compact only in the case when the space $H$ is finite-dimensional). If the operator $A : H \to H$ is selfadjoint we can describe the spectrum of the operator $\mathbb{P}$.

Proposition 14 The spectrum of the operator $\mathbb{P}$ is equal to the closure of the direct sum of the spectra $\sigma B$ and $\sigma (A-I)$, i.e.,

$$\sigma \mathbb{P} = \sigma B + \sigma (A-I) \equiv \{\lambda_1 + \lambda_2 - 1 : \lambda_1 \in \sigma B, \lambda_2 \in \sigma A\}.$$  

The proof of Proposition 14 immediately follows from the equality

$$\mathbb{P} = B \otimes I_H + I_{L_{2,\alpha}} \otimes (A-I)$$

(here $\otimes$ means the tensor product of the operators). Note that here we use the assertion, that if $\lambda \in \sigma T$ for the selfadjoint operator $T$ in some separable Hilbert space $T : X \to X$, then there is some sequence $x_n \in D(T), n \in \mathbb{N}, ||x_n|| = 1$ such that $(T - \lambda)x_n \to 0, n \to \infty$ (see [1], [14]).

References


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