ON SOME APPLICATIONS OF SET-THEORETICAL METHODS IN EUCLIDEAN GEOMETRY

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The famous incompleteness theorem of Gödel proved by him in 1931 is a first fundamental result which directly indicates that even elementary arithmetic (treated as a rigorous formal system) contains many undecidable statements. Later on, more concrete combinatorial assertions of Ramsey type were pointed out, which also turn out to be undecidable within formal arithmetic, but can be completely solved by using various non-constructive methods, for instance, by applying König's lemma on countably infinite trees with finite levels (see, e.g., [1]). Moreover, it was vividly demonstrated that certain uncountable forms of the Axiom of Choice and the existence of nontrivial ultrafilters in the set ω of all natural numbers allow to establish deep combinatorial results motivated by various problems or questions from classical number theory (cf., for example, [2]).

Analogous situations can be observed in classical Euclidean geometry. As is well known, the earliest more or less serious investigations in the foundations of geometry were carried out without appealing to set-theoretical techniques. Furthermore, in his deep investigations of this subject, D. Hilbert tried to avoid any set-theoretical concepts, so he absolutely rejected an approach based on set theory (cf. the text of his Foundations of Geometry, 1899). But very soon it was recognized that it is much more convenient to inscribe all geometric ideas and notions in the general framework of the Zermelo-Fraenkel system of axioms of set theory (i.e., **ZFC** theory). In fact, almost all mathematicians consider now the Euclidean plane or a multi-dimensional Euclidean space as a certain set of points equipped with the vector structure over the field $\mathbf{R}$ of all real numbers and endowed with the corresponding inner product $< \cdot, \cdot >$. So, it is assumed that the set $\mathbf{R}$ is already given with some fragment of set theory and it is adopted that specific methods of this powerful theory may be applied in various geometric topics. Actually, in such a situation we are forced to appeal to second-order logic with respect to our geometric universe (= the Euclidean space). This kind of logic may be regarded, in a sense, as the "hidden" theory of sets. The examples given below serve to illustrate this important circumstance.

The fact just mentioned does not seem so surprising, because even some classical problems concerning geometrical constructions with the aid of a compass and a straight-edge implicitly lead to second-order logic. For instance, let us recall several famous geometric problems of this sort.

Example 1. The first legendary geometric problem that comes from ancient mathematics is formulated as follows: a cube with volume 1 is given;

it is required to construct a cube whose volume is equal to 2. As is widely known, the above-mentioned problem of elementary geometry is undecidable by using only a compass and a straight-edge. Analyzing the standard proof of this undecidability, one readily sees that the argument leaves the framework of elementary geometry and first-order logic. Indeed, to show that the needed construction cannot be carried out by a compass and a straight-edge, we suppose to the contrary that there exists a finite family of fields

$$\mathbf{Q} = P_0 \subset P_1 \subset P_2 \subset ... \subset P_k$$

such that every P_{i+1} is a simple extension of P_i obtained by adding some square root from an element of P_i and the last field P_k contains the unique real solution of the equation $x^3 = 2$. Assuming that k is the smallest natural number corresponding to a sequence of fields with this property, we finally come to a contradiction. Let us underline once more that the above-mentioned argument appeals to second-order logic, because:

(a) we deal here with a certain sequence of subfields (hence subsets) of $\mathbf{R}$;

(b) such a sequence can be of an arbitrarily long length.

More generally, if we have an equation

$$a_0x^3 + a_1x^2 + a_2x + a_3 = 0,$$

where $a_0 \neq 0$, $a_3 \neq 0$ and all coefficients a_0, a_1, a_2, a_3 are rational numbers, and if we a priori know that this equation is irreducible over $\mathbf{Q}$ (equivalently, has no solution belonging to $\mathbf{Q}$), then no solution of the same equation can be obtained by iterating finitely many times the process of constructing quadratic extensions of fields (starting with $\mathbf{Q}$). Indeed, it suffices to consider a particular case when the previous equation is of the following special form:

$$x^3 + px + q = 0,$$

where, as before, the coefficients p and q are rational numbers and $q \neq 0$. Suppose to the contrary that there exists a finite sequence

$$\mathbf{Q} = P_0 \subset P_1 \subset P_2 \subset ... \subset P_k$$

of subfields of $\mathbf{R}$ satisfying the following conditions:

(1) some solution of this equation belongs to P_k;

(2) for every natural number $i < k$, there exists an element $t_i \in P_i$ such that $\sqrt{t_i}$ does not belong to P_i and

$$P_{i+1} = \{c + d\sqrt{t_i} : c \in P_i, d \in P_i\}.$$

We may assume, without loss of generality, that k is the minimal natural number for which both conditions (1) and (2) are fulfilled. Then, denoting by x_0 a solution of our equation belonging to P_k, we must have $x_0 = c + d\sqrt{t_{k-1}}$

for some elements c, d and t_{k-1} from P_{k-1}. Our assumption on k implies that $c \neq 0$ and $d \neq 0$. Further, we may write

$$(c + d\sqrt{t_{k-1}})^3 + p(c + d\sqrt{t_{k-1}}) + q = 0,$$

from which it follows that

$$c^3 + 3cd^2 t_{k-1} + cp + q = 0, \quad 3c^2 + d^2 t_{k-1} + p = 0.$$

But these two relations yield at once that

$$(-2c)^3 + p(-2c) + q = 0,$$

which contradicts the minimality of k, because $-2c$ is an element of P_{k-1} and is also a solution of our equation.

In this manner, we easily obtain the non-decidability of another classical geometric problem - trisection of a given angle. As is well known, even $\pi/9$ (which equals the one third of an interior angle of any regular triangle) cannot be constructed with the aid of a compass and a straight-edge, because its construction is equivalent to solving the equation

$$x^3 + 3x^2 - 1 = 0,$$

which has no rational solutions.

The same conclusion should be made for the third classical problem in which it is required to construct (by using a compass and a straight-edge) a regular 7-gon inscribed in a given circle. Recall that the algebraic equivalent of this problem is expressed by the equation

$$x^3 + x^2 - 2x - 1 = 0,$$

which also has no rational roots. So we again easily come to the desired undecidability result.

Of course, the simple geometric problems discussed in Example 1 look as rather primitive ones and are very far from numerous much more complicated situations concerning deep facts and statements of Euclidean geometry, which have purely set-theoretical flavor. We considered the above-mentioned classical problems only as a trivial illustration of the necessity of second-order logic (or set theory) in various elementary constructions. Below, we will envisage several examples of assertions from Euclidean geometry, whose proofs are closely connected with essentially non-elementary reasoning. These assertions will be examined from the combinatorial and set-theoretical points of view. At present, it is already recognized that rather delicate set-theoretical techniques (e.g., uncountable forms of the Axiom of Choice, the Continuum Hypothesis or Martin's Axiom, the existence of large cardinals, etc.) are needed for establishing the validity of such assertions. Questions of this type are also important for the foundations of geometry. In particular, it is well known that there are topics in elementary geometry, which are connected with the most fundamental concepts of contemporary mathematics and, simultaneously, are of interest for a wide audience of mathematicians.

Among topics of this kind we may especially indicate the following:

1) the notions of volume and measure in elementary geometry (cf. the remarkable monograph by Hadwiger [12]);

2) equidecomposability theory of polyhedra and of more general geometric figures (cf. [3], [12], [23]);

3) elements of the theory of convex sets (see again [12] and many other text-books of convex geometry or convex analysis);

4) incidence and combinatorial properties of subsets of Euclidean space (including various combinatorial schemes motivated by problems and questions from convex, discrete, and projective geometry);

5) approaches to the foundations of geometry by starting with various primitive notions.

All of the above-mentioned topics have deep connections with purely logical and set-theoretical techniques. In our opinion, graduate students should be provided with information that highlights those important aspects of geometry, which are closely related to finite and infinite combinatorics, discrete mathematics, and general point set theory.

For instance, various combinatorial properties of convex polyhedra lying in Euclidean space are of interest for a broad audience of mathematicians, including under-graduate and graduate students. So, let us consider one typical question from the theory of convex polyhedra, which deserves to be discussed at the popular level. Extensive information on convex polyhedra is presented in the well-known monographs [6], [11] and [12], and in many other text-books of geometry. There are several interesting and important topics in equidecomposability theory of polyhedra, where essentially non-constructive methods found unexpected applications (cf. [3], [23]). We would like to touch upon some non-effective set-theoretical approaches which successfully work in elementary geometry.

Example 2. To be more concrete, recall Hilbert's third problem in which it is required to establish the non-equidecomposability of the three-dimensional unit cube and a regular tetrahedron of volume 1. This problem and many other similar questions, concerning equidecomposability of convex (and not necessarily convex) polyhedra, are closely connected with certain non-trivial solutions of the Cauchy functional equation

$$f(x+y) = f(x) + f(y) \quad (x \in \mathbf{R}, y \in \mathbf{R}).$$

It turned out that those solutions are very helpful and provide important invariants of the corresponding transformation groups (the group of all translations, the group of all motions, etc.). However, a nontrivial solution of the

Cauchy functional equation can be constructed only with the aid of the Axiom of Choice and any such solution is non-measurable in the Lebesgue sense (see [3] and [12] where a number of questions related to Hilbert's third problem are discussed in detail). Notice by the way that nontrivial solutions f of the Cauchy functional equation, which are exploited in equidecomposability theory of polyhedra, as a rule, satisfy the condition $f(\pi) = 0$. In order to construct such an f, we must first show that π is an irrational number. An elementary proof of this fact is not quite easy (as widely known, there are elementary proofs of the much stronger fact stating that π is a transcendental number).

Example 3. Considering more general classes of geometric figures, we come to various equidecomposability paradoxes. Recall that the most famous result in this direction is the Banach-Tarski paradox (see [23]), the validity of which is essentially implied by the Axiom of Choice and by some specific algebraic properties of the group of all rotations of the Euclidean space $\mathbf{R}^3$ about its origin. Here we can see a remarkable result of pure mathematics, which is valid because of deep relationships between geometry, set-theory and group theory (a detailed account is presented in [23], where many references to other works and sources are also given).

Example 4. In the theory of convex sets, sometimes it is useful to apply the Axiom of Choice, the method of transfinite induction or the Zorn lemma in order to obtain the desired result. For instance, let E be a vector space over the field $\mathbf{R}$ of all real numbers and let A and B be any two nonempty disjoint convex subsets of E. By using a simple geometric argument and the Zorn lemma, one shows that there always exist two convex subsets A' and B' of E such that:

(1) A' and B' are also disjoint and their union coincides with E;

(2) A is contained in A' and B is contained in B'.

Further, suppose that E is a Hausdorff topological vector space. Then, denoting by $cl(A')$ and $cl(B')$ the closures of the sets A' and B' respectively, it is not hard to see that the set

$$F = cl(A') \cap cl(B')$$

is an affine linear manifold in E. Moreover, if F differs from E, then F is an affine hyperplane in E separating the sets A' and B' and, hence, separating the initial convex sets A and B as well. We thus obtain the fundamental separation theorem for a pair of disjoint convex sets. This approach to separation theorems for disjoint convex sets is preferable in various respects. In particular, it vividly shows that, in the case of an infinite-dimensional real Hilbert space E, there exist two disjoint convex subsets A and B of E, both of which are everywhere dense in E (notice also that if E is separable, then the existence of A and B does not need the Axiom of Choice). In fact, if a Hausdorff topological vector space E is such that there exists at least one discontinuous linear functional on E, then the required decomposition of E into two convex everywhere dense subsets of E is guaranteed (the converse assertion is also true).

In this connection, it is interesting to mention that if any two disjoint convex polygons are given in the Euclidean plane $\mathbf{R}^2$, then there always exists a straight line separating these polygons and containing a side of one of them. In other words, for any two disjoint convex polygons a strengthened form of the separation principle is valid, which easily follows from the well-known Helly theorem [5]. However, the analogous result does not longer hold for two disjoint convex polyhedra in the three-dimensional Euclidean space $\mathbf{R}^3$.

Example 5. Set-theoretical methods also work when we are dealing with an uncountable version of the famous Erdös-Szekeres problem on the existence of a sufficiently large convexly independent subset of a given point-set (see [8]). Recall that, according to the Erdös-Szekeres theorem, if Z is an infinite subset of the Euclidean plane $\mathbf{R}^2$ and all points of Z are in general position, then there exists an infinite subset Z' of Z all points of which are in convex position. The natural question arises whether any uncountable set $U \subset \mathbf{R}^2$ of points in general position contains an uncountable subset U' of points in convex position. By applying a transfinite construction of Luzin type, it can be proved that the answer to this question is negative (for more details, see [14]).

Example 6. As for various incidence properties of families of sets in Euclidean spaces, the best known example of this kind is the Sylvester theorem on a finite family of collinear points in the Euclidean plane. This theorem admits a beautiful (in fact, set-theoretical) proof and is often presented in geometric text-books oriented to school pupils (another beautiful purely combinatorial proof of the Sylvester theorem is based on the Euler formula for planar graphs). However, it should be noticed here that, as a rule, even a three-dimensional version of this theorem, which is also true, is not discussed in those text-books or manuals.

Example 7. Certain subsets of the plane $\mathbf{R}^2$ with paradoxical incidence properties were constructed by using the Axiom of Choice. Among these sets a Mazurkiewicz set M should be mentioned especially, which is characterized by the property that every straight line of $\mathbf{R}^2$ meets M in exactly two points (see [19]). Similar sets can be constructed by replacing the family of all straight lines by the family of all circles lying in $\mathbf{R}^2$. In this way we come to a subset C of $\mathbf{R}^2$ which meets every circle of $\mathbf{R}^2$ in precisely three points. Let us observe that the descriptive structure of M (or of C) can be relatively good and relatively bad. For example, M can be of Lebesgue measure zero but also can be nonmeasurable in the Lebesgue sense. Further, some families of geometric figures having prescribed combinatorial properties were constructed by applying essentially non-elementary set-theoretical methods (see [4], [13], [15], and [18]). For instance, it was shown that there exists a family of pairwise congruent circles in $\mathbf{R}^2$ such that each point of $\mathbf{R}^2$ belongs to exactly three circles of this family. The construction of such a family may be regarded as dual to the construction of the set C mentioned above. Notice that, for any even $k > 0$, there exists an elementary example of a family of pairwise congruent circles in $\mathbf{R}^2$ such that each point of $\mathbf{R}^2$ belongs to exactly k circles of the family.

Example 8. It is well known that the most natural visual interpretation of various relationships between members of a given family of sets is provided by Euler-Venn diagrams. However, even in the simplest situation where an independent family of four sets is considered there exists no corresponding Euler-Venn diagram whose elements are disks lying in the Euclidean plane. If convex polygons are allowed to be elements of Euler-Venn diagrams, then the situation becomes much better and one may assert that there exists an infinite independent family of convex polygons in the plane. At the same time, it can be proved that there exists no uncountable independent family of polygons in $\mathbf{R}^2$. A class of geometric figures slightly more general than polygons enables us to prove the existence of uncountable independent families of such figures. For this purpose, let us call a quasi-polygon any compact subset of $\mathbf{R}^2$ whose interior is nonempty and whose boundary can be represented as the union of countably many non-degenerate line segments. By using the method of transfinite induction, it was shown in [17] that there exists an uncountable independent family of convex quasi-polygons in $\mathbf{R}^2$.

Example 9. As was demonstrated by Sierpiński (see, e.g., his remarkable monograph [21]), there are propositions of elementary geometry which are equivalent to the Continuum Hypothesis. For example, one of his old results states that the following two assertions are equivalent:

(i) the Continuum Hypothesis;

(ii) there exists a partition $\{X, Y, Z\}$ of the Euclidean space $\mathbf{R}^3$ such that X has finite intersection with any line parallel to x-axis, Y has finite intersection with any line parallel to y-axis, and Z has finite intersection with any line parallel to z-axis.

Sierpiński's above-mentioned result stimulated further investigations in this direction (Bagemihl, Davies, Erdös, Jackson, Mauldin and others).

Example 10. Let a and b be any two positive real numbers. It is easy to indicate a subset Z of the Euclidean plane $\mathbf{R}^2$, such that all horizontal sections of Z are line segments of length a and all vertical sections of Z are line segments of length b. In fact, Z can be taken as a strip in $\mathbf{R}^2$ whose boundary lines are expressible in the form of the equations

$$y = (b/a)x + c_1, \quad y = (b/a)x + c_2,$$

where $|c_1 - c_2| = b$. This construction is absolutely elementary and visual. But the natural question arises whether it is possible to construct a bounded set with analogous properties of its linear (horizontal and vertical) sections. More precisely, suppose that $0 \leq a \leq 1$ and $0 \leq b \leq 1$. Then one may ask whether there exists a set $W \subset [0,1]^2$ such that:

(1) all horizontal sections $([0,1] \times \{y\}) \cap W$, where $y \in [0,1]$, are of linear Lebesgue measure a;

(2) all vertical sections $(\{x\} \times [0,1]) \cap W$, where $x \in [0,1]$, are of linear Lebesgue measure b.

In the sequel, we shall say that $W \subset [0,1]^2$ is an (a,b)-homogeneous set in the unit square $[0,1]^2$ if both relations (1) and (2) are satisfied for W.

Notice that if $a = b$, then a set W with the above-mentioned property can be constructed effectively, i.e., without the aid of the Axiom of Choice. The main idea of such a construction is as follows. We first represent the given number $a \in [0,1]$ in the form

$$a = 1/2^{n_1} + 1/2^{n_2} + ... + 1/2^{n_k} + ...,$$

where $(n_1, n_2, ..., n_k, ...)$ is a strictly increasing sequence of positive integers, and then we define by recursion a sequence $(W_1, W_2, ..., W_k, ...)$ of subsets of $[0,1]^2$, which increases by the inclusion relation and, for each natural number $k > 0$, the horizontal and vertical sections of W_k by the line segments

$$[0,1] \times \{y\}, \quad [0,1] \times \{x\} \quad (x \in [0,1],\ y \in [0,1])$$

are of linear Lebesgue measure $1/2^{n_1} + 1/2^{n_2} + ... + 1/2^{n_k}$. Finally, we put

$$W = \cup\{W_k : 0 < k < \omega\}.$$

If $a \neq b$, then no effective construction of the required set W is possible, because according to the classical Fubini theorem, such a W must be non-measurable with respect to the two-dimensional Lebesgue measure λ_2 on the plane $\mathbf{R}^2$. Moreover, as follows from one result of Friedman [10], a set W with the desired properties cannot be constructed even within the Zermelo-Fraenkel set theory (cf. also [9]). However, by assuming the Continuum Hypothesis and applying Sierpiński's classical decomposition of the unit square $[0,1]^2$ (see [20], [21], [22]), it becomes possible to establish the existence of an (a,b)-homogeneous subset of the square $[0,1]^2$. For more details, we refer the reader to [16] where some three-dimensional analogues of such sets are also considered.

Finishing this brief survey, it is reasonable to recall that several eminent mathematicians (Tarski, Scott, Choquet and others) suggested their approaches to the foundations of geometry and investigated certain logical and set-theoretical aspects of the problem. An approach to various geometric constructions by means of the methods of non-standard models is also of special interest (cf. [7]).

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