IMPLICIT DIFFERENCE SCHEME FOR THE NUMERICAL RESOLUTION OF THE CHARNEY-OBUKHOV EQUATION WITH VARIABLE COEFFICIENTS

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Abstract. In the present work first order accurate implicit difference scheme for the numerical solution of the nonlinear Charney-Obukhov equation with variable coefficients is constructed. On the basis of numerical calculations accomplished by means of this scheme, the dynamics of two-dimensional nonlinear solitary vortical structures at the presence of sheared flow is studied. For the considered equations the initial-boundary value problem is set when at the initial moment the solution in the form of different solitary structures are taken. The problem of stability for the first order accurate semi-discrete scheme is investigated. For solving of the considered difference scheme iteration method is offered. Convergence of this iteration method is proved. Suggested numerical method for investigation of dynamics of nonlinear Rossby waves propagation in the earth's neutral atmosphere under conditions of sheared zonal flows is used. Obtained results sufficiently well describe physical picture of phenomena.

Key words: Solitary Rossby wave, Charney-Obukhov equation, implicit difference scheme

AMS subject classification 2000: 35Q51, 35Q53

1. Statement of the problem

Modified by the presence of earth's inhomogeneous angular rotation and sheared zonal flow nonlinear Charney-Obukhov equation with variable coefficients in the frame of reference moving with velocity v along the x-axis has the following form [1]:

$$\frac{\partial}{\partial t}\Delta\psi - v\frac{\partial}{\partial x}\Delta\psi + J\left(\psi+\varphi_0, \Delta\psi+\varphi_0''+f_0\right) = 0\ . \tag{1}$$

This equation describes nonlinear interaction of Rossby waves with zonal shear flow in the earth's neutral atmosphere. Here $\psi(x,y,t)$ and $\varphi_0(y)$ are stream functions of medium and zonal shear flow, respectively. Operator $\triangle = \partial^2/\partial x^2 + \partial^2/\partial y^2$ is the two-dimensional Laplacian, $J(a,b) = \partial a/\partial x \partial b/\partial y - \partial a/\partial y \partial b/\partial x$ is the Jacobian, f_0 describes the influence of Coriolis force. Note that the presence of sheared zonal flow $\varphi_0(y)$ may be considered as the presence of external energy source (see the discussion). If $f_0 = \beta y$ and $\varphi_0 = const$, then we get the classical Charney-Obukhov equation.

If we introduce the generalized vorticity $W = \Delta\psi + f_0 + \varphi_0''$, Eq. (1) can be rewritten in terms of ψ and W as follows

$$\frac{\partial W}{\partial t} + J\left(\psi, W\right) - \left(v+\varphi_0'\right)\frac{\partial W}{\partial x} = 0, \tag{2}$$

$$\Delta\psi - \gamma\psi \;=\; W - (f_0 + \varphi_0''). \qquad (3)$$

Our aim is to solve system (2), (3) numerically in the cylindrical domain $Q_T = \Omega\times]0,T[$, where Ω area is the rectangle, $\Omega =]-a_1, a_1[\times]-a_2, a_2[$. Space variables x, y vary in the domain Ω, and the variable t varies in the interval $]0,T[$. As an initial condition at time $t = 0$ we take well known solitary dipole solution $\psi(x, y, 0) = \psi_0(x, y)$ (this function represents the stationary solution of the classical Charney-Obukhov and Hasageva-Mima equations).

Let us introduce a time step $\tau = T/m$ $(m > 1)$ and approximate Eq. (2) at the point (x, y, t_k), where $t_k = k\tau$ $(k = 1, ..., m)$, by the following semi-discrete scheme:

$$\frac{W^k - W^{k-1}}{\tau} + J\left(\psi^{k-1}, W^k\right) - (v + \varphi_0')\frac{\partial W^k}{\partial x} = 0. \qquad (4)$$

We assume that $W(t, x, y)$ and $\psi(t, x, y)$ are sufficiently smooth functions. Eq.(4) approximates Eq.(2) at the point (x, y, t_k) with an accuracy $O(\tau)$.

Let us cover area Ω by a grid and denote by h_1 a grid spacing in the x-direction and by h_2 in the y-direction, $h_1 = 2a_1/N_1$, $\quad h_2 = 2a_2/N_2$, where N_1 (> 1) and N_2 (> 1) are natural numbers. If in Eq. (4) we replace the first order derivatives with respect to spatial variables by central differences, we obtain the following difference equation:

$$\frac{W_{i,j}^k - W_{i,j}^{k-1}}{\tau} + F\left(\psi_{i,j}^{k-1}, \psi_{i,j}^k, W_{i,j}^{k-1}, W_{i,j}^k\right) = 0, \qquad (5)$$

where $i = 1, ..., N_1 - 1, \;\; j = 1, ..., N_2 - 1$,

$$F\left(\psi_{i,j}^{k-1}, W_{i,j}^{k-1}, W_{i,j}^k\right) = \widehat{J}\left(\psi_{i,j}^{k-1}, W_{i,j}^k\right) - (v + \varphi_0'(y_i))\,\delta_x W_{i,j}^k,$$

and $\widehat{J}(\psi_{i,j}, W_{i,j}) = (\delta_x\psi_{i,j})(\delta_y W_{i,j}) - (\delta_y\psi_{i,j})(\delta_x W_{i,j})$. The operator δ_x represents the central-difference analogy of the first order derivative with respect to x variable (similarly is defined the δ_y operator). The difference equation (5) approximates Eq.(2) with the accuracy $O(\tau + h_1^2 + h_2^2)$ at the point $(x_i, y_j, t_k) = (-a_1 + ih_1, -a_2 + jh_2, k\tau)$.

Reconstruction of the stream function ψ by means of the generalized vorticity W can be accomplished from the standard difference equation corresponding to Eq.(3).

It is known that the condition of existence of classical solution of initial-boundary value problem for evolution equation is the consistency of initial and boundary conditions. Therefore, we chose our boundary conditions taking into account the given initial condition.

We solve the system of difference equations (5) by the following iteration (in order to simplify writing we omit the index k in $W_{i,j}^k$):

$$\overset{n}{W}_{i,j} = W_{i,j}^{k-1} + \tau F\left(\psi_{i,j}^{k-1}, W_{i,j}^{k-1}, \overset{n-1}{W}_{i,j}\right), \qquad (6)$$

where n is an iteration index $(n = 1, 2, ...)$, $\overset{0}{W}_{i,j} = W_{i,j}^{k-1}$. The transition step τ from one time level to the next is subject to the condition:

$$\left(\frac{\tau}{h_2}c_1 + \frac{\tau}{h_1}\left(c_2 + v + \max_i |\varphi_0'(y_i)|\right)\right) < 1, \tag{7}$$

where c_1 and c_2 are maximums of $\left|\delta_x \psi_{i,j}^{k-1}\right|$ and $\left|\delta_y \psi_{i,j}^{k-1}\right|$, $(i = 1, ..., N_1 - 1,$ $j = 1, ..., N_2 - 1)$, respectively. Inequality (7) represents a sufficient condition of convergence of the iteration process (6).

Reconstruction of the stream function field ψ by means of generalized vorticity W can be accomplished from the following standard difference equation corresponding to Eq. (2)

$$\frac{\psi_{i+1,j}^k - 2\psi_{i,j}^k + \psi_{i-1,j}^k}{h_1^2} + \frac{\psi_{i,j+1}^k - 2\psi_{i,j}^k + \psi_{i,j-1}^k}{h_2^2} = W_{i,j}^k - \left(f_0(y_i) + \varphi_0''(y_i)\right), \tag{8}$$

where $i = 1, ..., N_1 - 1$ and $j = 1, ..., N_2 - 1$.

We solve the system of difference equations (8) with respect to the variable x by means of factorization method, and with respect to the variable y - by means of iteration (in order to simplify writing we omit the index k in $\psi_{i,j}^k$):

$$-\overset{n}{\psi}_{i+1,j} + a\overset{n}{\psi}_{i,j} - \overset{n}{\psi}_{i-1,j} = \alpha_0^2\left(\overset{n-1}{\psi}_{i,j+1} + \overset{n}{\psi}_{i,j-1}\right) - h_1^2\left(W_{i,j}^k - f_0(y_i) - \varphi_0''(y_i)\right), \tag{9}$$

where $i = 2, ..., N_1 - 1$ and $j = 2, ..., N_2 - 1$; n is a iteration index $(n = 1, 2, ...)$, $\alpha_0 = h_1/h_2$, $a = 2 + 2\alpha_0^2$ and $\overset{0}{W}_{i,j} = W_{i,j}^{k-1}$.

Remark. The convergence rate of the iteration process (6) and (9) is sufficiently high, as the fulfillment of the condition (7) provides the diagonal domination of matrix of the system (6), and the matrix of the system (9) automatically has the diagonal domination at the expense of the coefficient a. Another important factor for an increase of the convergence rate of the iteration process (6) and (9) is that we take the corresponding values obtained at the preceding time level $(k - 1)$ as the first approximation of mesh functions $\psi_{i,j}^k$ and $W_{i,.j}^k$.

Analogous approach was used by us for the numerical resolution of the classical Charney-Obukhov equation [2].

2. Results of numerical experiments and their analysis

It is well-known that the classical Charney-Obukhov equation (i.e. Eq. (1) with $\varphi_0 = 0$) describes the existence of solitary vortical structures. The modified Charney-Obukhov equation (1) under the action of sheared zonal flow describes the evolution of such vortical structures. Indeed from Eq. (1) it is easy to obtain the following dynamic equation

$$\frac{dE(t)}{dt} = -\iint\limits_{\Omega} \frac{\partial\psi}{\partial x}\frac{\partial\psi}{\partial y}\varphi_0''(y)\, dxdy\ , \tag{10}$$

where the energy of solitary vortical structures is defined as

$$E(t) = \int\int_{\Omega} \left[\left(\frac{\partial \psi}{\partial x} \right)^2 + \left(\frac{\partial \psi}{\partial y} \right)^2 \right] dxdy \, . \tag{11}$$

It is seen that in the absence of zonal flow ($\varphi_0 = 0$) the energy of vortical structures is conserved. Therefore the existence of sheared zonal flow may be considered as the presence of external energy source.

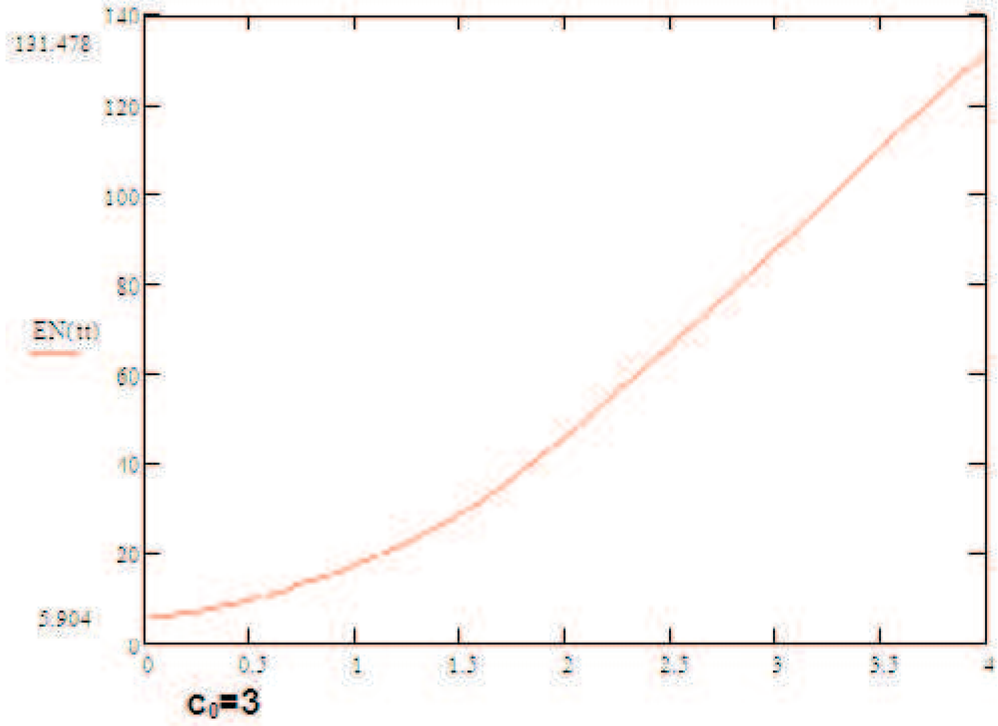

Fig. 1. Variation of system energy according to the time

The main goal of the presented paper is to construct such effective difference schemes for the numerical solutions of Eq.(1) which rightly reflect the physical sense of the problem. We have considered the case when $\varphi_0(y) = c_0 e^{-y^2}$ and $f_0(y) = \beta y$. Numerical calculations of Eq. (1) were accomplished for the following value of parameters $a_1 = a_2 = 20$, $N_1 = N_2 = 200$, $\tau = 0.00625$, $v = 1$, $\beta = 1$, $c_0 = 1, 2, 3, 4, 5$. The initial value of the stream function was taken solitary dipole solution [2].

In Fig. 1 the dependence of the energy of solitary vortical structures on time is given. It is seen that this energy increase in time. Increase of the energy of vortical structure causes a breaking in pieces of initial solitary dipole structure and appearance of new number of solitary structures as it seen from Fig. 2. Of course the more the amplitude c_0 is the earlier destruction process begin.

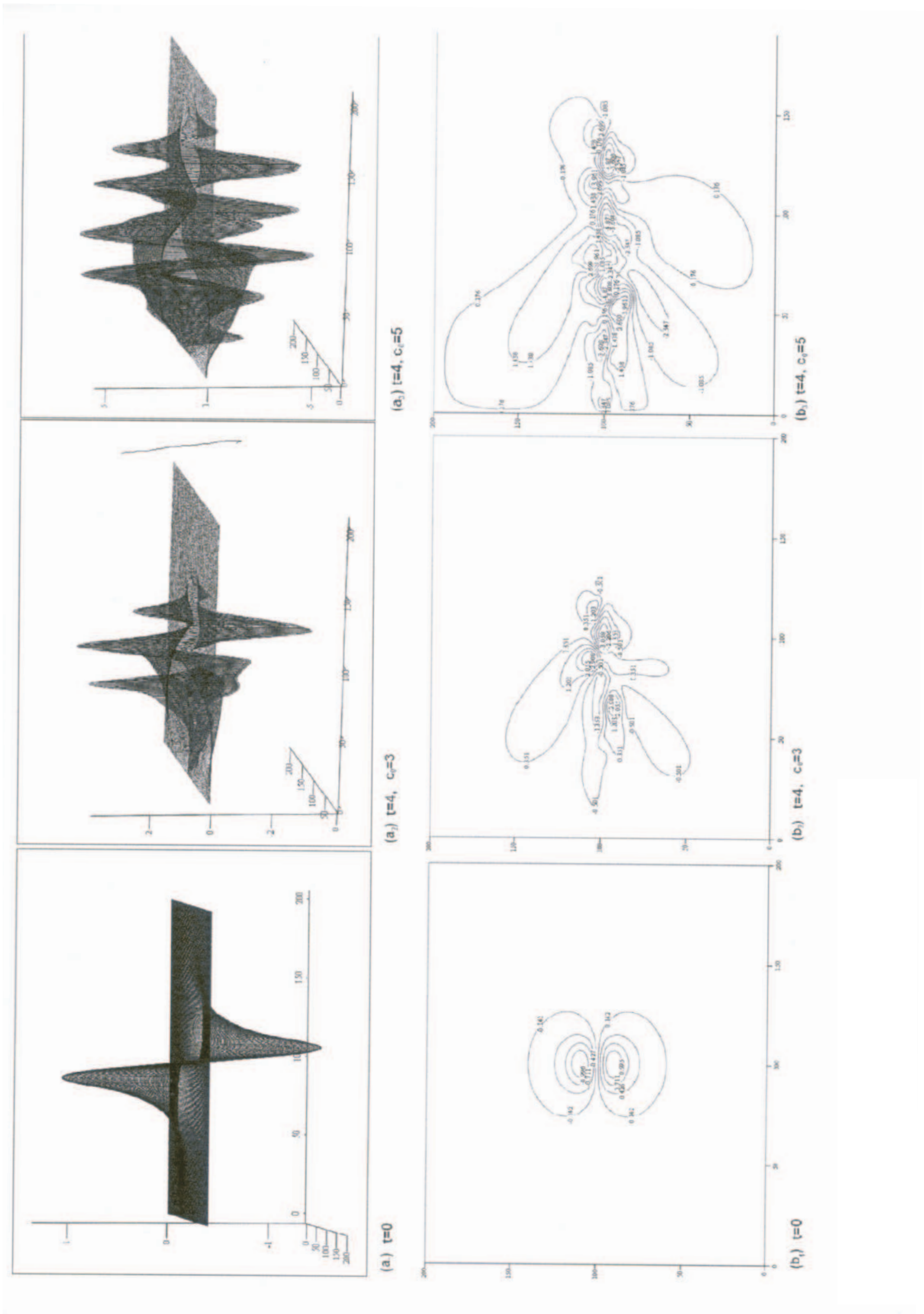

Fig. 2. Surface of the solution of eq. (1) and corresponding isolines

Note that if $f'_0(y), \varphi'_0(y)$ and $\varphi'''_0(y)$ functions are even and $\psi(x, y, 0)$ is odd with respect to the variable y, then the solution of Eq. (1) $\psi(x, y, t)$ is symmetric function with respect to y axe. This statement is confirmed by the numerical calculations (see. Fig. 3). In this case $\varphi_0(y) = c_0 y e^{-y^2}$, and $f_0(y)$ da $\psi(x, y, 0)$ is the same as in the previous case.

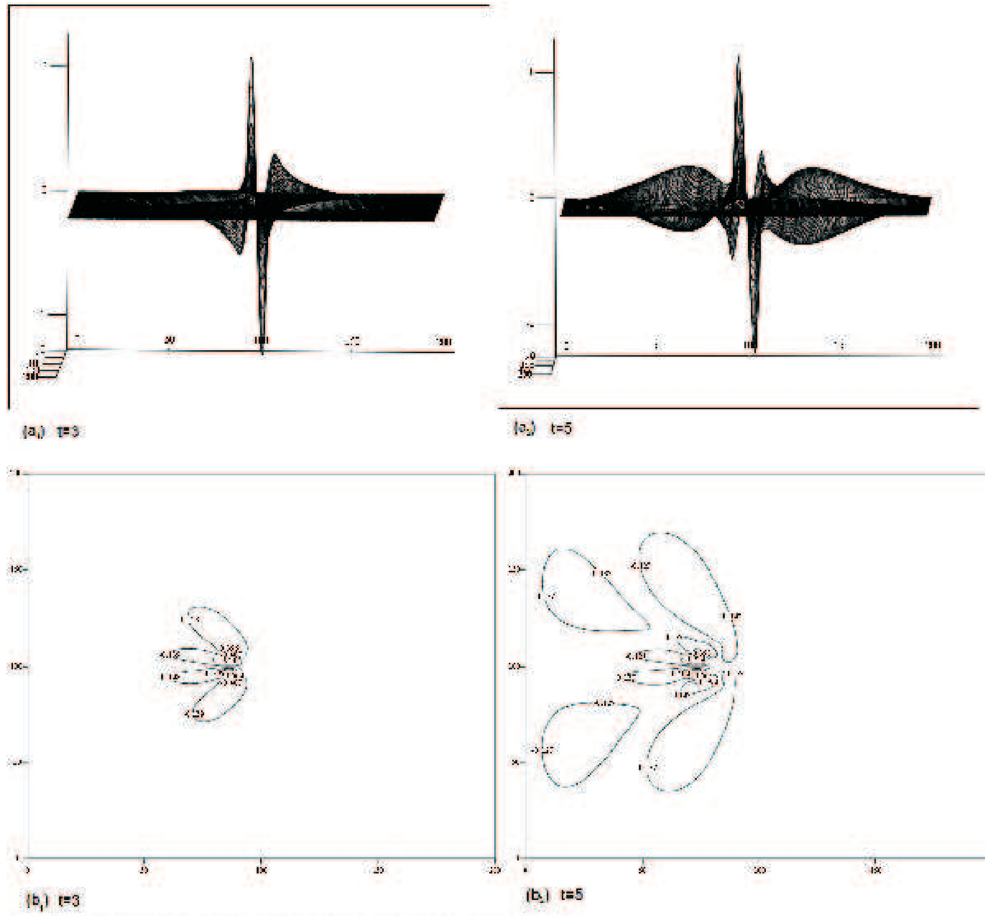

Fig. 3. Surface of the solution of eq. (1) and corresponding isolines in the symmetric case

Let us note that if $\varphi_0(y)$ is a linear function, then energy is conserved also.

It should be noted that a decrease of steps h and τ, beginning from the limiting value, does not affect essentially on the results, when the calculation time increases substantially. The fact that decrease of steps does not spoil the results confirms practically the stability of the presented scheme.

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