MAIN ARTICLES

SHARP FUNCTION INEQUALITY FOR THE MULTILINEAR COMMUTATOR OF THE LITTLEWOOD-PALEY OPERATOR

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Abstract: In this paper, we obtain the sharp inequality for the multilinear commutator related to the Littlewood-Paley operator. Using the sharp inequality, the weighted L^p-norm inequality for the multilinear commutator for $1 < p < \infty$ is proved.

Key words: Multilinear commutator; Littlewood-Paley operator; Sharp inequality. *MSC 1991:* 42B20, 42B25.

1. Introduction

The commutators of singular integral operators have been well studied (see [1-4]). Let T be the Calderón-Zygmund singular integral operator. A classical result of Coifman, Rocherberg and Weiss (see [3]) states that commutator $[b,T](f) = T(bf) - bT(f)$ (where $b \in BMO(R^n)$ (see below)) is bounded on $L^p(R^n)$) for $1 < p < \infty$. In [8-10], the sharp estimates for some multilinear commutators of the Calderón-Zygmund singular integral operators are obtained. The Littlewood-Paley operator is an important operator in the harmonic analysis (see [12]). The main purpose of this paper is to prove the sharp inequality for the multilinear commutator related to the Littlewood-Paley operator. By using the sharp inequality, we obtain the weighted L^p-norm inequality for the multilinear commutator for $1 < p < \infty$.

2. Preliminaries and Theorems

First let us introduce some notations (see [4], [9], [11]). In this paper, Q will denote a cube of R^n with sides parallel to the axes. For a cube Q and a locally integrable function b, let $b_Q = |Q|^{-1} \int_Q b(x)dx$, the sharp function of b is defined by

$$b^{\#}(x) = \sup_{Q(x \in Q)} \frac{1}{|Q|} \int_Q |b(y) - b_Q| dy.$$

It is well-known that (see [4])

$$b^{\#}(x) = \sup_{Q(x \in Q)} \inf_{c \in \boldsymbol{C}} \frac{1}{|Q|} \int_Q |b(y) - c| dy.$$

We say that b belongs to $BMO(R^n)$ if $b^{\#}$ belongs to $L^\infty(R^n)$ and define $\|b\|_{BMO} := \|b^{\#}\|_{L^\infty}$. It is known as well (see [11]) that

$$\|b - b_{2^kQ}\|_{BMO} \le Ck\|b\|_{BMO}.$$

For $b_j \in BMO(R^n)$ $(j = 1, \cdots, m)$, set $\tilde{b} = (b_1, \cdots, b_m)$ and

$$\|\tilde{b}\|_{BMO} := \prod_{j=1}^{m} \|b_j\|_{BMO}.$$

For a given a positive integer m and $1 \le j \le m$, we denote by C_j^m the family of all finite subsets $\sigma := \{\sigma(1), \cdots, \sigma(j)\}$ of $\{1, \cdots, m\}$ of j different elements. For $\sigma \in C_j^m$, set $\sigma^c := \{1, \cdots, m\} \setminus \sigma$. For $\tilde{b} = (b_1, \cdots, b_m)$ and $\sigma := \{\sigma(1), \cdots, \sigma(j)\} \in C_j^m$, set $\tilde{b}_\sigma = (b_{\sigma(1)}, \cdots, b_{\sigma(j)})$, $b_\sigma := b_{\sigma(1)} \cdots b_{\sigma(j)}$ and $\|\tilde{b}_\sigma\|_{BMO} := \|b_{\sigma(1)}\|_{BMO} \cdots \|b_{\sigma(j)}\|_{BMO}$.

Let M be the Hardy-Littlewood maximal operator, i.e.,

$$M(f)(x) := \sup_{Q(x \in Q)} \frac{1}{|Q|} \int_Q |f(y)| dy;$$

further $M_p(f) := (M(|f|^p))^{\frac{1}{p}}$ for $1 < p < \infty$.

We denote the Muckenhoupt weights by A_1 (see [4]), i.e.,

$$A_1 := \{w : M(w)(x) \le Cw(x), a.e.\}.$$

Throughout this paper, we will study some multilinear commutators.

Definition. Let b_j $(j = 1, \cdots, m)$ be the fixed locally integrable functions on R^n. Let further $\varepsilon > 0$ and ψ be a fixed function which satisfies the following properties:

(1) $\int_{R^n} \psi(x) dx = 0$,

(2) $|\psi(x)| \le C(1 + |x|)^{-(n+1)}$,

(3) $|\psi(x + y) - \psi(x)| \le C|y|^\varepsilon (1 + |x|)^{-(n+1+\varepsilon)}$ when $2|y| < |x|$;

Denote $\Gamma(x) := \{(y, t) \in R_+^{n+1} : |x - y| \le t\}$ and the characteristic function of $\Gamma(x)$ by $\chi_{\Gamma(x)}$. The Littlewood-Paley multilinear commutator is defined by

$$S_\psi^{\tilde{b}}(f)(x) := \left[\int\int_{\Gamma(x)} |F_t^{\tilde{b}}(f)(x, y)|^2 \frac{dydt}{t^{n+1}} \right]^{\frac{1}{2}},$$

where

$$F_t^{\tilde{b}}(f)(x) = \int_{R^n} \left[\prod_{j=1}^{m} (b_j(x) - b_j(z)) \right] \psi_t(y - z) f(z) dz$$

and $\psi_t(x) = t^{-n}\psi\left(\frac{x}{t}\right)$ for $t > 0$. Set $F_t(f)(x) := \int_{R^n} \psi_t(x-y)f(y)dy$, we also introduce

$$S_\psi(f)(x) := \left[\int\int_{\Gamma(x)} |F_t(f)(x)|^2 \frac{dydt}{t^{n+1}}\right]^{\frac{1}{2}},$$

which is the Littlewood-Paley S operator (see [12]).

Let H be the space $H = \left\{h : \|h\| = \left(\int\int_{R_+^{n+1}} |h(y,t)|^2 \frac{dydt}{t^{n+1}}\right)^{\frac{1}{2}} < \infty\right\}$, then for each fixed $x \in R^n$, $F_t^{\tilde{b}}(f)(x,y)$ may be viewed as a mapping from $[0,+\infty)$ to H, and it is clear that

$$S_\psi(f)(x) = \left\|\chi_{\Gamma(x)} F_t(f)(y)\right\|$$

and

$$S_\psi^{\tilde{b}}(f)(x) = \left\|\chi_{\Gamma(x)} F_t^{\tilde{b}}(f)(x,y)\right\|.$$

Note that when $b_1 = \cdots = b_m$, $S_\psi^{\tilde{b}}$ is just the m order commutator(see[1], [6], [7]). It is well known that commutators are of great interest in harmonic analysis and have been widely studied by many authors (see [1-3], [5-10]). Our main purpose is to establish the sharp inequality for the multilinear commutator.

Now we state our theorems as follows.

Theorem 1. *Let $b_j \in BMO(R^n)$ for $j = 1, \cdots, m$. Then for any $1 < r < \infty$, there exists a constant $C > 0$ such that for any $f \in C_0^\infty(R^n)$ and any $x \in R^n$,*

$$(S_\psi^{\tilde{b}}(f))^\#(x) \le C\|\tilde{b}\|_{BMO}\left(M_r(f)(x) + \sum_{j=1}^m \sum_{\sigma\in C_j^m} M_r\left(S_\psi^{\tilde{b}_{\sigma^c}}(f)\right)(x)\right).$$

Theorem 2. *Let $b_j \in BMO(R^n)$ for $j = 1, \cdots, m$. Then $S_\psi^{\tilde{b}}$ is bounded on $L^p(w)$ for $w \in A_1$ and $1 < p < \infty$.*

3. Proof of Theorems

To prove the theorems, we need the following lemmas.

Lemma 1. (see [12]) *Let $w \in A_p$ and $1 < p < \infty$. Then S_ψ is bounded on $L^p(w)$.*

Lemma 2. *Let $1 < r < \infty$, $b_j \in BMO(R^n)$ for $j = 1, \cdots, k$ and $k \in N$. Then, we have*

$$\frac{1}{|Q|}\int_Q \prod_{j=1}^k |b_j(y) - (b_j)_Q|dy \le C\prod_{j=1}^k \|b_j\|_{BMO}$$

and

$$\left(\frac{1}{|Q|}\int\limits_Q \prod_{j=1}^{k}|b_j(y)-(b_j)_Q|^r dy\right)^{\frac{1}{r}} \leq C\prod_{j=1}^{k}\|b_j\|_{BMO}.$$

Proof. Choose $1 < p_j < \infty,\ j = 1, \cdots, m$, such that $\frac{1}{p_1}+\cdots+\frac{1}{p_m}=1$, by virtue of Hölder's inequality, we obtain

$$\begin{aligned}\frac{1}{|Q|}\int\limits_Q \prod_{j=1}^{k}|b_j(y)-(b_j)_Q|dy &\leq \prod_{j=1}^{k}\left(\frac{1}{|Q|}\int\limits_Q |b_j(y)-(b_j)_Q|^{p_j}dy\right)^{\frac{1}{p_j}}\\ &\leq C\prod_{j=1}^{k}\|b_j\|_{BMO}\end{aligned}$$

and

$$\begin{aligned}\left(\frac{1}{|Q|}\int\limits_Q \prod_{j=1}^{k}|b_j(y)-(b_j)_Q|^r dy\right)^{\frac{1}{r}} &\leq \prod_{j=1}^{k}\left(\frac{1}{|Q|}\int\limits_Q |b_j(y)-(b_j)_Q|^{p_j r}dy\right)^{\frac{1}{p_j r}}\\ &\leq C\prod_{j=1}^{k}||b_j||_{BMO}.\end{aligned}$$

Proof of Theorem 1. It suffices to prove that for $f \in C_0^\infty(R^n)$ and some constant C_0, the following inequality:

$$\frac{1}{|Q|}\int\limits_Q |S_\psi^{\tilde{b}}(f)(x)-C_0|dx \leq C||b||_{BMO}\left(M_r(f)(x)+\sum_{j=1}^{m}\sum_{\sigma\in C_j^m}M_r\left(S_\psi^{\tilde{b}_{\sigma^c}}(f)(x)\right)\right).$$

Fix a cube $Q = Q(x_0, d)$ and $\tilde{x} \in Q$. Set $f_1 = f\chi_{2Q}$ and $f_2 = f\chi_{2Q^c}$.

We first consider the case $m = 1$. Evidently,

$$\begin{aligned}F_t^{b_1}(f)(x,y) &= (b_1(x)-(b_1)_{2Q})F_t(f)(y)-F_t((b_1-(b_1)_{2Q})f_1)(y)\\ &\quad -F_t((b_1-(b_1)_{2Q})f_2)(y).\end{aligned}$$

Then,

$$\begin{aligned}&\left|S_\psi^{b_1}(f)(x)-S_\psi\left(\left(\left(b_1\right)_{2Q}-b_1\right)f_2\right)(x_0)\right|\\ =\ &\left|\left\|\chi_{\Gamma(x)}F_t^{b_1}(f)(x,y)\right\|-\left\|\chi_{\Gamma(x_0)}F_t\left(\left(\left(b_1\right)_{2Q}-b_1\right)f_2\right)(y)\right\|\right|\\ \leq\ &\left\|\chi_{\Gamma(x)}F_t^{b_1}(f)(x,y)-\chi_{\Gamma(x_0)}F_t\left(\left(\left(b_1\right)_{2Q}-b_1\right)f_2\right)(y)\right\|\\ \leq\ &\left\|\chi_{\Gamma(x)}(b_1(x)-(b_1)_{2Q})F_t(f)(y)\right\|+\left\|\chi_{\Gamma(x)}F_t((b_1-(b_1)_{2Q})f_1)(y)\right\|\\ &+\left\|\chi_{\Gamma(x)}F_t\left((b_1-(b_1)_{2Q})f_2\right)(y)-\chi_{\Gamma(x_0)}F_t((b_1-(b_1)_{2Q})f_2)(y)\right\|\\ =\ &A(x)+B(x)+C(x).\end{aligned}$$

For $A(x)$, in view of Hölder's inequality with the exponent $1/r + 1/r' = 1$, we get

$$\begin{aligned}
&\frac{1}{|Q|}\int\limits_{Q} A(x)dx \\
=\ &\frac{1}{|Q|}\int\limits_{Q} |b_1(x) - (b_1)_{2Q}||S_\psi(f)(x)|dx \\
\leq\ &C\left(\frac{1}{|2Q|}\int\limits_{2Q} |b_1(x) - (b_1)_{2Q}|^{r'}dx\right)^{\frac{1}{r'}}\left(\frac{1}{|Q|}\int\limits_{Q} |S_\psi(f)(x)|^r dx\right)^{\frac{1}{r}} \\
\leq\ &C\|b_1\|_{BMO} M_r(S_\psi(f))(\tilde{x}).
\end{aligned}$$

For $B(x)$, choose $1 < p < r$, by the boundedness of S_ψ on $L^p(R^n)$ and because of Hölder's inequality, we obtain

$$\begin{aligned}
&\frac{1}{|Q|}\int\limits_{Q} B(x)dx \\
\leq\ &\left(\frac{1}{|Q|}\int\limits_{R^n} |S_\psi((b_1 - (b_1)_{2Q})f_1)(x)|^p dx\right)^{\frac{1}{p}} \\
\leq\ &C\left(\frac{1}{|Q|}\int\limits_{2Q} |(b_1(x) - (b_1)_{2Q})f(x)|^p dx\right)^{\frac{1}{p}} \\
\leq\ &C\left(\frac{1}{|2Q|}\int\limits_{2Q} |f(x)|^r dx\right)^{\frac{1}{r}}\left(\frac{1}{|2Q|}\int\limits_{2Q} |b_1(x) - (b_1)_{2Q}|^{rp/(r-p)}dx\right)^{\frac{r-p}{rp}} \\
\leq\ &C\|b_1\|_{BMO} M_r(f)(\tilde{x}).
\end{aligned}$$

For $C(x)$, by virtue of Minkowski's inequality, we obtain

$$\begin{aligned}
&C(x) \leq \\
&\left[\int\int\limits_{R_+^{n+1}}\left(\int\limits_{(2Q)^c} |\chi_{\Gamma(x)} - \chi_{\Gamma(x_0)}||b_1(z) - (b_1)_{2Q}||\psi_t(y-z)||f(z)|\right)^2 \frac{dydt}{t^{n+1}}\right]^{\frac{1}{2}} \\
&\leq C\int\limits_{(2Q)^c} |b_1(z) - (b_1)_{2Q}||f(z)|\left|\int\int\limits_{|x-y|\leq t} \frac{t^{1-n}dydt}{(t + |y-z|)^{2n+2}}\right.
\end{aligned}$$

$$-\int\int_{|x_0-y|\le t}\frac{t^{1-n}dydt}{(t+|y-z|)^{2n+2}}\Bigg|^{\frac{1}{2}}dz$$

$$\le\int_{(2Q)^c}|b_1(z)-(b_1)_{2Q}||f(z)|\left(\int\int_{|y|\le t,|x+y-z|\le t}\Bigg|\frac{1}{(t+|x+y-z|)^{2n+2}}\right.$$

$$\left.-\frac{1}{(t+|x_0+y-z|)^{2n+2}}\Bigg|\frac{dydt}{t^{n-1}}\right)^{\frac{1}{2}}dz$$

$$\le\int_{(2Q)^c}|b_1(z)-(b_1)_{2Q}||f(z)|$$

$$\times\left(\int\int_{|y|\le t,|x+y-z|\le t}\frac{|x-x_0|t^{1-n}}{(t+|x+y-z|)^{2n+3}}dydt\right)^{\frac{1}{2}}dz.$$

Note that $2t+|x+y-z|\ge 2t+|x-z|-|y|\ge t+|x-z|$ when $|y|\le t$ and

$$\int_0^\infty\frac{tdt}{(t+|x-z|)^{2n+3}}=C|x-z|^{-2n-1}.$$

Then, for $x\in Q$,

$$\begin{aligned}C(x)&\le C\int_{(2Q)^c}|b_1(z)-(b_1)_{2Q}||f(z)|\\&\quad\times\left(\int\int_{|y|\le t}\frac{2^{2n+3}|x_0-x|t^{1-n}dydt}{(2t+2|x+y-z|)^{2n+3}}\right)^{\frac{1}{2}}dz\\&\le C\int_{(2Q)^c}|b_1(z)-(b_1)_{2Q}||f(z)||x-x_0|^{\frac{1}{2}}\\&\quad\times\left(\int\int_{|y|\le t}\frac{t^{1-n}dydt}{(2t+|x+y-z|)^{2n+3}}\right)^{\frac{1}{2}}dz\\&\le C\int_{(2Q)^c}|b_1(z)-(b_1)_{2Q}||f(z)||x-x_0|^{\frac{1}{2}}\\&\quad\times\left(\int\int_{|y|\le t}\frac{t^{1-n}dydt}{(t+|x-z|)^{2n+3}}\right)^{\frac{1}{2}}dz\end{aligned}$$

$$
\begin{aligned}
&\le C \int_{(2Q)^c} |b_1(z)-(b_1)_{2Q}||f(z)||x-x_0|^{\frac{1}{2}} \\
&\quad \times \left(\int_0^\infty \frac{tdt}{(t+|x-z|)^{2n+3}} \right)^{\frac{1}{2}} dz \\
&\le C \int_{(2Q)^c} |b_1(z)-(b_1)_{2Q}||f(z)| \frac{|x_0-x|^{\frac{1}{2}}}{|x_0-z|^{n+\frac{1}{2}}} dz \\
&\le C \sum_{k=1}^{\infty} \int_{2^{k+1}Q\setminus 2^kQ} |x_0-x|^{\frac{1}{2}} |x_0-z|^{-\left(n+\frac{1}{2}\right)} |b_1(z)-(b_1)_{2Q}||f(z)|dz \\
&\le C \sum_{k=1}^{\infty} 2^{-k/2} |2^{k+1}Q|^{-1} \int_{2^{k+1}Q} |b_1(z)-(b_1)_{2Q}||f(z)|dz \\
&\le C \sum_{k=1}^{\infty} 2^{-k/2} \left(\frac{1}{|2^{k+1}Q|} \int_{2^{k+1}Q} |f(z)|^r dz \right)^{\frac{1}{r}} \\
&\quad \times \left(\frac{1}{|2^{k+1}Q|} \int_{2^{k+1}Q} |b_1(z)-(b_1)_{2Q}|^{r'} dz \right)^{\frac{1}{r'}} \\
&\le C \sum_{k=1}^{\infty} 2^{-k/2} k \|b_1\|_{BMO} M_r(f)(\tilde{x}) \\
&\le C \|b_1\|_{BMO} M_r(f)(\tilde{x}).
\end{aligned}
$$

Thus,

$$
\frac{1}{|Q|} \int_Q C(x)dx \le C||b_1||_{BMO} M_r(f)(\tilde{x}).
$$

Now, we consider the case $m \ge 2$. For $b = (b_1, \cdots, b_m)$ we have

$$
F_t^{\tilde{b}}(f)(x,y) = \int_{R^n} \left[\prod_{j=1}^{m} (b_j(x)-b_j(z)) \right] \psi_t(y-z) f(z) dz
$$

$$
\begin{aligned}
&= \int_{R^n} \prod_{j=1}^{m} \left[(b_j(x)-(b_j)_{2Q}) - (b_j(z)-(b_j)_{2Q}) \right] \psi_t(y-z) f(z) dz \\
&= \sum_{j=0}^{m} \sum_{\sigma \in C_j^m} (-1)^{m-j} (b(x)-(b)_{2Q})_\sigma \int_{R^n} (b(z)-(b)_{2Q})_{\sigma^c} \psi_t(y-z) f(z) dz \\
&= (b_1(x)-(b_1)_{2Q}) \cdots (b_m(x)-(b_m)_{2Q}) F_t(f)(y) \\
&\quad + (-1)^m F_t((b_1-(b_1)_{2Q}) \cdots (b_m-(b_m)_{2Q}) f)(y)
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}(-1)^{m-j}(b(x)-(b)_{2Q})_\sigma\int\limits_{R^n}(b(z)-b(x))_{\sigma^c}\psi_t(y-z)f(z)dz\\
= \ & (b_1(x)-(b_1)_{2Q})\cdots(b_m(x)-(b_m)_{2Q})F_t(f)(y)\\
&+(-1)^m F_t((b_1-(b_1)_{2Q})\cdots(b_m-(b_m)_{2Q})f)(y)\\
&+\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}c_{m,j}(b(x)-(b)_{2Q})_\sigma F_t^{\tilde{b}_{\sigma^c}}(f)(x,y).
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&\left|S_\psi^{\tilde{b}}(f)(x)-S_\psi\left(\left(((b_1)_{2Q}-b_1)\cdots((b_m)_{2Q}-b_m)\right)f_2\right)(x_0)\right|\\
\leq \ & \left\|\chi_{\Gamma(x)}F_t^{\tilde{b}}(f)(x,y)-\chi_{\Gamma(x_0)}F_t\left(\left(((b_1)_{2Q}-b_1)\cdots((b_m)_{2Q}-b_m\right)f_2\right)(y)\right\|\\
\leq \ & \left\|\chi_{\Gamma(x)}(b_1(x)-(b_1)_{2Q})\cdots(b_m(x)-(b_m)_{2Q})F_t(f)(y)\right\|\\
+ \ & \sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}\left\|\chi_{\Gamma(x)}(\tilde{b}(x)-(b)_{2Q})_\sigma F_t^{\tilde{b}_{\sigma^c}}(f)(x,y)\right\|\\
+ \ & \left\|\chi_{\Gamma(x)}F_t((b_1-(b_1)_{2Q})\cdots(b_m-(b_m)_{2Q})f_1)(y)\right\|\\
+ \ & \left\|\chi_{\Gamma(x)}F_t\left(\prod_{j=1}^m(b_j-(b_j)_{2Q})f_2\right)(y)-\chi_{\Gamma(x_0)}F_t\left(\prod_{j=1}^m(b_j-(b_j)_{2Q})f_2\right)(y)\right\|\\
= \ & I_1(x)+I_2(x)+I_3(x)+I_4(x).
\end{aligned}
$$

For $I_1(x)$, according to Hölder's inequality with the exponent $1/p_1+\cdots+1/p_m+1/r=1$, where $1<p_j<\infty,\ j=1,\cdots,m$, we get

$$
\begin{aligned}
&\frac{1}{|Q|}\int\limits_Q I_1(x)dx\\
= \ & \frac{1}{|Q|}\int\limits_Q |b_1(x)-(b_1)_{2Q}|\cdots|b_m(x)-(b_m)_{2Q}||S_\psi(f)(x)|dx\\
\leq \ & \left(\frac{1}{|Q|}\int\limits_Q |b_1(x)-(b_1)_{2Q}|^{p_1}\right)^{\frac{1}{p_1}}\cdots\left(\frac{1}{|Q|}\int\limits_Q |b_m(x)-(b_m)_{2Q}|^{p_m}dx\right)^{\frac{1}{p_m}}\\
&\times\left(\frac{1}{|Q|}\int_Q|S_\psi(f)(x)|^r dx\right)^{\frac{1}{r}}\\
\leq \ & C\|\tilde{b}\|_{BMO}M_r(S_\psi(f))(\tilde{x}).
\end{aligned}
$$

For $I_2(x)$, in view of Minkowski's inequality and Lemma 2, we get

$$
\frac{1}{|Q|}\int\limits_Q I_2(x)dx
$$

$$\begin{aligned}
&\leq \sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}\frac{1}{|Q|}\int\limits_Q |(b(x)-(b)_{2Q})_\sigma||S_\psi^{\tilde{b}_{\sigma^c}}(f)(x)|dx \\
&\leq C\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}\left(\frac{1}{|2Q|}\int\limits_{2Q}|(b(x)-(b)_{2Q})_\sigma|^{r'}dx\right)^{\frac{1}{r'}} \\
&\quad\times\left(\frac{1}{|Q|}\int\limits_Q |S_\psi^{\tilde{b}_{\sigma^c}}(f)(x)|^r dx\right)^{\frac{1}{r}} \\
&\leq C\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}||\tilde{b}_\sigma||_{BMO}M_r(S_\psi^{\tilde{b}_{\sigma^c}}(f))(\tilde{x}).
\end{aligned}$$

For $I_3(x)$, choose $1 < p < r$, $1 < p_j < \infty$, $j = 1, \cdots, m$ such that $\frac{1}{p_1} + \cdots + \frac{1}{p_m} + \frac{p}{r} = 1$, by the boundedness of S_ψ on $L^p(R^n)$ and Hölder's inequality, we get

$$\begin{aligned}
&\frac{1}{|Q|}\int\limits_Q I_3(x)dx \\
&\leq \left(\frac{1}{|Q|}\int\limits_{R^n}|S_\psi((b_1-(b_1)_{2Q})\cdots(b_m-(b_m)_{2Q})f_1)(x)|^p dx\right)^{\frac{1}{p}} \\
&\leq C\left(\frac{1}{|Q|}\int\limits_{R^n}|(b_1(x)-(b_1)_{2Q})\cdots(b_m(x)-(b_m)_{2Q})|^p|f_1(x)|^p dx\right)^{\frac{1}{p}} \\
&\leq C\left(\frac{1}{|2Q|}\int\limits_{2Q}|f(x)|^r dx\right)^{\frac{1}{r}} \\
&\quad\times\left(\frac{1}{|2Q|}\int\limits_{2Q}|b_1(x)-(b_1)_{2Q}|^{p_1}dx\right)^{\frac{1}{pp_1}} \\
&\quad\times\cdots\left(\frac{1}{|2Q|}\int\limits_{2Q}|b_m(x)-(b_m)_{2Q}|^{pp_m}dx\right)^{\frac{1}{pp_m}} \\
&\leq C\|\tilde{b}\|_{BMO}M_r(f_1)(\tilde{x}).
\end{aligned}$$

For $I_4(x)$, similar to the proof in the case $m = 1$, we obtain

$$I_4(x) \le C \int\limits_{(2Q)^c} |x_0 - x|^{\frac{1}{2}} |x_0 - z|^{-(n+\frac{1}{2})} \left| \prod_{j=1}^{m} (b_j(z) - (b_j)_{2Q}) \right| |f(z)| dz,$$

taking $1 < p_j < \infty$ $j = 1, \cdots, m$ such that $1/p_1 + \cdots + 1/p_m + 1/r = 1$, then, for $x \in Q$

$$\begin{aligned} I_4(x) &\le C \sum_{k=1}^{\infty} \int\limits_{2^{k+1}Q \setminus 2^k Q} |x_0 - x|^{\frac{1}{2}} |x_0 - z|^{-(n+\frac{1}{2})} \\ &\quad \times \left| \prod_{j=1}^{m} (b_j(z) - (b_j)_{2Q}) \right| |f(z)| dz \\ &\le C \sum_{k=1}^{\infty} 2^{-k/2} |2^{k+1}Q|^{-1} \int\limits_{2^{k+1}Q} \left| \prod_{j=1}^{m} (b_j(z) - (b_j)_{2Q}) \right| |f(z)| dz \\ &\le C \sum_{k=1}^{\infty} 2^{-k/2} \left(\frac{1}{|2^{k+1}Q|} \int\limits_{2^{k+1}Q} |f(z)|^r dz \right)^{\frac{1}{r}} \\ &\quad \times \prod_{j=1}^{m} \left(\frac{1}{|2^{k+1}Q|} \int\limits_{2^{k+1}Q} |b_j(z) - (b_j)_{2Q}|^{p_j} dz \right)^{\frac{1}{p_j}} \\ &\le C \sum_{k=1}^{\infty} k^m 2^{-km} \prod_{j=1}^{m} \|b_j\|_{BMO} M_r(f)(\tilde{x}) \\ &\le C \|\tilde{b}\|_{BMO} M_r(f)(\tilde{x}). \end{aligned}$$

Thus

$$\frac{1}{|Q|} \int\limits_{Q} I_4(x) dx \le C \|\tilde{b}\|_{BMO} M_r(f)(\tilde{x}).$$

This completes the proof of the theorem.

Proof of Theorem 2. Choose $1 < r < p$ in Theorem 1 and using Lemma 1, we may get the conclusion of Theorem 2. This finishes the proof.

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