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A FORM OF A GENERAL SOLUTION FOR A CLASS OF HIGH-ORDER HYPERBOLIC EQUATIONS AND SOME OF ITS APPLICATIONS

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Abstract. For one class of high-order hyperbolic equations with two independent variables, a general solution formula is obtained. This formula allows solving initial, initial-boundary and some non-local problems. In particular, in some cases solutions of Cauchy problems are written out in quadratures.

Keywords and phrases: High-order hyperbolic equations, general solution, Cauchy problem.

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In the theory of partial differential equations, an important role is assigned to the so-called obtaining a general solution for a given partial differential equation. In the paper, a general solution formula is obtained for one class of high-order hyperbolic equations, which can be used in the investigation of initial, initial-boundary, and non-local problems. In order to obtain the final form of the general solution, simple equations have been considered several times, and as a result of their gradual complication and generalization, the form of the general solution of the equation under consideration has been obtained. In particular, the solution of some special cases of the Cauchy problem is given in an explicit form.

Consider the following equation [1]:

$$Lu := \frac{\partial^n u}{\partial t^n} + a_{n-1}\frac{\partial^n u}{\partial t^{n-1}\partial x} + a_{n-2}\frac{\partial^n u}{\partial t^{n-2}\partial x^2} + ... + a_1\frac{\partial^n u}{\partial t\partial x^{n-1}} + a_0\frac{\partial^n u}{\partial x^n} = f(x,t), \qquad (1)$$

where $u = u(x,t)$ is an unknown two-variable function, $a_i = const, i = 0, , n-1$, are given constant coefficients, while $f(x,t)$ is the given function. Consider the characteristic equation of the given equation:

$$p(\lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + a_{n-2}\lambda^{n-2} + ... + a_1\lambda + a_0 = 0. \qquad (2)$$

Suppose that equation (1) is of hyperbolic type, which means that equation (2) has only real $\lambda_1, \lambda_2, ..., \lambda_l$ roots of multiplicity $k_1, k_2, ..., k_l$ respectively and, therefore, $P(\lambda)$ can be represented as follows

$$p(\lambda) = \prod_{i=1}^{l}(\lambda - \lambda_i)^{k_i}, \qquad \sum_{i=1}^{l} k_i = n. \qquad (3)$$

According to (3) equation (1) can be rewritten as follows [2]:

$$Lu := \prod_{i=1}^{l}\left(\frac{\partial}{\partial t} - \lambda_i\frac{\partial}{\partial x}\right)^{k_i} u = f(x,t). \qquad (4)$$

For some cases of equation (4) let us write its general solution.

1) For the following nonhomogeneous equation

$$\left(\frac{\partial}{\partial t} - \lambda_i \frac{\partial}{\partial x}\right) u = f(x + \lambda_j t), \quad \lambda_i \neq \lambda_j, \tag{5}$$

the general solution is given by formula:

$$u(x,t) = g(x + \lambda_j t) + \tau(x + \lambda_i t), \tag{6}$$

where $f = f(s)$ is a given continuous function, $g(s) = \dfrac{1}{\lambda_j - \lambda_i} \int_o^s f(\sigma) d\sigma$, while $\tau = \tau(s)$ is any one-variable function from the class C^1, and $\tau(x + \lambda_i t)$ is a general solution to its corresponding homogeneous equation.

2) Consider the nonhomogeneous equation of type (5) when $\lambda_i = \lambda_j$,, i.e.

$$\left(\frac{\partial}{\partial t} - \lambda_i \frac{\partial}{\partial x}\right) u = f(x + \lambda_i t). \tag{7}$$

For equation (7) a general solution is given by the formula:

$$u(x,t) = t f(x + \lambda_i t) + \tau(x + \lambda_i t). \tag{8}$$

Here, in contrast to equation (5) $f = f(s)$ is a given function of class C^1 and $\tau = \tau(s)$, as in (6), is an arbitrary one-variable function from the class C^1.

Based on representations (6) and (8), using the method of mathematical induction, the general solution of the following equation

$$\left(\frac{\partial}{\partial t} - \lambda_i \frac{\partial}{\partial x}\right)^n u = f(x + \lambda_j t), \tag{9}$$

is given by the formula:

$$u(x,t) = \frac{t^n}{n!} f(x + \lambda_i t) + \sum_{k=1}^{n} \frac{t^{n-k}}{(n-k)!} \tau_k(x + \lambda_i t), \tag{10}$$

when $\lambda_i = \lambda_j$, where $f = f(s)$ is given, and $\tau_k = \tau_k(s), \quad k = 1, ..., n, \quad C^n$ class arbitrary functions, while when $\lambda_i \neq \lambda_j$, the general solution of equation (9) is given by the formula

$$u(x,t) = \frac{1}{(\lambda_j - \lambda_i)^n} g(x + \lambda_j t) + \sum_{k=1}^{n} \frac{t^{n-k}}{(n-k)!} \tau_k(x + \lambda_i t). \tag{11}$$

Here $f = f(s)$ is the given continuous function, and $g = g(s)$ is the n-th order antiderivative function of $f = f(s)$:

$$g(s) = \frac{1}{(n-1)!} \int_0^s (s - \sigma)^{n-1} f(\sigma) d\sigma$$

(that is $g^n(s) = f(s), g^i(0) = 0, i = 1, 2, ..., n-1,$, and, therefore, $g(s)$ is the solution of the mentioned Cauchy problem), while $\tau_k = \tau_k(s)$ is an arbitrary function of class C^n.

Remark 1. As a result of the analysis of the output structure of the considered nonhomogeneous equations, we get that the smoothness of the right-hand side of the equation is closely related to $\lambda_i \neq \lambda_j$, or $\lambda_i = \lambda_j$ and also to the degree of iteration of the first-order operator $\left(\frac{\partial}{\partial t} - \lambda_i \frac{\partial}{\partial x}\right)$ included in the equation.

If we introduce the notation

$$L_i := \left(\frac{\partial}{\partial t} - \lambda_i \frac{\partial}{\partial x}\right)^{k_i},$$

then equation (4) can be rewritten in the following form:

$$Lu := \prod_{i=1}^{l} L_i u = f(x,t). \tag{12}$$

Consider the homogeneous equation corresponding to (12):

$$\prod_{i=1}^{l} L_i u = 0. \tag{13}$$

Lemma 1. *Any solution u of homogeneous equation (13) can be represented in the form:*

$$u = \sum_{i=1}^{l} u_i, \tag{14}$$

where u_i is a solution of the homogeneous equation $L_i u = 0$.

From Lemma 1 and representation (10) it follows

Theorem 1. *A general solution of homogeneous general type hyperbolic equation (13) with constant coefficients is given by the formula:*

$$u = \sum_{i=1}^{l} \sum_{j=1}^{k_i} \frac{t^{k_i - j}}{(k_i - j)!} \tau_{ij}(x + \lambda_i t), \tag{15}$$

where $\tau_{ij} = \tau_{ij}(s)$ are arbitrary functions from C^n, $i = 1, ..., l; j = 1, ..., k_i$.

From Theorem 1 it follows that the equation

$$\Box^n u := \left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\right)^n u = 0.$$

has a general solution given by formula:

$$u = \sum_{j=1}^{n} \frac{t^{n-j}}{(n-j)!} \tau_{1j}(x - t) + \sum_{j=1}^{n} \frac{t^{n-j}}{(n-j)!} \tau_{2j}(x + t).$$

Here $\tau_{1j} = \tau_{1j}(s)$, $\tau_{2j} = \tau_{2j}(s)$ are arbitrary functions from the class C^n. This formula follows from representation (15) and the following equality

$$\Box^n := \left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\right)^n = \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right)^n \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right)^n.$$

Remark 2. The representation (15) of the general solution of equation (13) can be used in solution of the Cauchy's problem posed for this equation with the following initial conditions:

$$\frac{\partial^i}{\partial t^i}u(x,0)=\varphi_i(x), \quad i=0,...,n-1, \tag{16}$$

where $\varphi_i, i=0,...,n-1$, are given functions of the class C^{n-i}.

Consider particular cases of the Cauchy's problem (13), (16), when equation (13) is strictly hyperbolic and n=2 or n=3.

In the case n=2 the Cauchy's problem (13), (16) can be written in the form:

$$\left(\frac{\partial}{\partial t}-\lambda_1\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t}-\lambda_2\frac{\partial}{\partial x}\right)u=0, \quad \lambda_1\neq\lambda_2, \tag{17}$$

$$u(x,0)=\varphi_0(x), \quad u_t(x,0)=\varphi_1(x). \tag{18}$$

According to Theorem 1 the general solution of equation (17) is given by the formula

$$u(x,t)=\tau_1(x+\lambda_1 t)+\tau_2(x+\lambda_2 t), \tag{19}$$

where τ_1 and τ_2 are arbitrary functions of the class C^2.Putting expression (19) into initial conditions (18) and solving obtained system of equations with respect to τ_1 and τ_2, and putting them afterwards into representation (19) we get the solution of the Cauchy problem (17), (18) by the formula

$$u(x,t)=\frac{1}{\lambda_1-\lambda_2}\int_{x+\lambda_2 t}^{x+\lambda_1 t}\varphi_1(\sigma)d\sigma+\frac{1}{\lambda_1-\lambda_2}(-\lambda_2\varphi_0(x+\lambda_1 t)+\lambda_1\varphi_0(x+\lambda_2 t)). \tag{20}$$

The formula (20) represents natural generalization of well-known D'Alembert's formula with $\lambda_1=1$ and $\lambda_2=-1$.

In the case n=3 the Cauchy problem (13), (16) can be written in the form:

$$\left(\frac{\partial}{\partial t}-\lambda_1\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t}-\lambda_2\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t}-\lambda_3\frac{\partial}{\partial x}\right)u=0, \qquad \lambda_i\neq\lambda_j, \tag{21}$$

$$u(x,0)=\varphi_0(x), \quad u_t(x,0)=\varphi_1(x), \quad u_{tt}(x,0)=\varphi_2(x). \tag{22}$$

According to Theorem 1 the general solution of equation (21) is given by the formula

$$u(x,t)=\tau_1(x+\lambda_1 t)+\tau_2(x+\lambda_2 t)+\tau_3(x+\lambda_3 t), \tag{23}$$

where τ_1,τ_2 and τ_3 are arbitrary functions of the class C^3. Putting expression (23) into initial conditions (22) and solving obtained system of equations with respect τ_1,τ_2 and τ_3, and putting them afterwards into expression (23)we get the following representation of the Cauchy problem (21), (22)

$$u(x,t)=\frac{\lambda_2\lambda_3\varphi_0(x+\lambda_1 t)}{(\lambda_2-\lambda_1)(\lambda_3-\lambda_1)}-\frac{\lambda_1\lambda_3\varphi_0(x+\lambda_2 t)}{(\lambda_2-\lambda_1)(\lambda_3-\lambda_2)}+\frac{\lambda_1\lambda_2\varphi_0(x+\lambda_3 t)}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)}$$

$$-\frac{(\lambda_3+\lambda_2)\int_0^{x+\lambda_1 t}\varphi_1(\sigma)d\sigma}{(\lambda_2-\lambda_1)(\lambda_3-\lambda_1)}+\frac{(\lambda_3+\lambda_1)\int_0^{x+\lambda_2 t}\varphi_1(\sigma)d\sigma}{(\lambda_2-\lambda_1)(\lambda_3-\lambda_2)}-\frac{(\lambda_2+\lambda_1)\int_0^{x+\lambda_3 t}\varphi_1(\sigma)d\sigma}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)}$$

$$+\frac{\int_0^{x+\lambda_1 t}(x+\lambda_1 t-\sigma)\varphi_2(\sigma)d\sigma}{(\lambda_2-\lambda_1)(\lambda_3-\lambda_1)}-\frac{\int_0^{x+\lambda_2 t}(x+\lambda_2 t-\sigma)\varphi_2(\sigma)d\sigma}{(\lambda_2-\lambda_1)(\lambda_3-\lambda_2)}+\frac{\int_0^{x+\lambda_3 t}(x+\lambda_3 t-\sigma)\varphi_2(\sigma)d\sigma}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)}.$$

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