Seminar of I. Vekua Institute
of Applied Mathematics
REPORTS, Vol. 49, 2023

OSCILLATION CRITERIA FOR HIGHER ORDER EMDEN-FOWLER TYPE DIFFERENCE EQUATIONS WITH DELAY ARGUMENT

Koplatadze R.

Abstract. In the work, the oscillatory properties of solutions of the Emden-Fowler type difference equation

$$\Delta^{(n)}u(k)+p(k)\,\big|u(\tau(k))\big|^{\lambda}\operatorname{sign}u\big(\tau(k)\big)=0,$$

are investigated, where $n\geq 2$, $0<\lambda<1$, $p:\mathbb{N}\to\mathbb{R}$, $\tau:\mathbb{N}\to\mathbb{N}$ and $\lim\limits_{k\to+\infty}\tau(k)=+\infty$. The sufficient conditions for oscillation of solutions are established.

Keywords and phrases: Proper solution, property **A**, property **B**.

AMS subject classification (2010): 39A11.

1. Introduction

This work is devoted to the study of oscillatory properties of the difference equation

$$\Delta^{(n)}u(k)+p(k)\,\big|u(\tau(k))\big|^{\lambda}\operatorname{sign}u\big(\tau(k)\big)=0, \tag{1.1}$$

where $n\geq 2$, $p:\mathbb{N}\to\mathbb{R}$, $\tau:\mathbb{N}\to\mathbb{N}$ and

$$0<\lambda<1,\quad \lim_{k\to+\infty}\tau(k)=+\infty. \tag{1.2}$$

Here $\Delta^{(0)}u(k)=u(k)$, $\Delta^{(1)}u(k)=u(k+1)-u(k)$, $\Delta^{(i)}u(k)=\Delta^{(1)}\circ\Delta^{(i-1)}u(k)$ $(i=1,\dots,n)$. It will always be assumed that either the condition

$$p(k)\geq 0\quad\text{for}\quad k\in\mathbb{N}, \tag{1.3}$$

or the condition

$$p(k)\leq 0\quad\text{for}\quad k\in\mathbb{N} \tag{1.4}$$

are fulfilled.

The following notation will be used throughout the work.

Let $k_0\in\mathbb{N}$. By $\mathbb{N}^+_{k_0}$ $(\mathbb{N}^-_{k_0})$ we denote a set of natural numbers $\mathbb{N}^+_{k_0}=\{k_0,k_0+1,\dots\}$ $\big(\mathbb{N}^-_{k_0}=\{1,2,\dots,k_0\}\big)$.

Definition 1.1. Let $k_0\in\mathbb{N}$ and $k_*=\inf\big\{\min(k,\tau(k)):k\in\mathbb{N}^+_{k_0}\big\}$. We call a function $u:\mathbb{N}^+_{k_*}\to\mathbb{R}$ a proper solution of equation (1.1), if it satisfies (1.1) on $\mathbb{N}^+_{k_0}$ and

$$\sup\big\{\big|u(i)\big|:i\in\mathbb{N}^+_{k}\big\}>0\ \text{ for any }\ k\in\mathbb{N}^+_{k_0}.$$

Definition 1.2. We say that a proper solution $u:\mathbb{N}^+_{k_0}\to(0,+\infty)$ of equation (1.1) is oscillatory if, for any $k\in\mathbb{N}^+_{k_0}$, there exist $k_1;k_2\in\mathbb{N}^+_k$ such that $u(k_1)\,u(k_2)\leq 0$. Otherwise, the solution is called nonoscillatory.

Definition 1.3. We say that equation (1.1) has Property **A** if any of its proper solutions is oscillatory when n is even and either is oscillatory or satisfies

$$\big|\Delta^{(i)}u(k)\big|\downarrow 0\quad\text{as}\quad k\uparrow+\infty,\quad k\in\mathbb{N}\quad(i=0,\dots,n-1), \tag{1.5}$$

when n is odd.

Definition 1.4. We say that equation (1.1) has Property **B** if any of its proper solutions is either oscillatory, or satisfies (1.5), or

$$\left|\Delta^{(i)}u(k)\right| \uparrow +\infty, \quad k \in \mathbb{N} \quad (i = 0, \dots, n-1), \tag{1.6}$$

when n is even and either is oscillatory, or satisfies (1.6) when n is odd.

The sufficient conditions of higher order Emden-Fowler type difference equations to have Property **A** and **B**, when $0 < \lambda < 1$, $\tau(k) \geq k+1$, $\lambda > 1$ and $\lim\limits_{k \to +\infty} \tau(k) = +\infty$, can be found in [1–4]. The problem of establishing sufficient conditions for the oscillation of all solutions to the second order linear and nonlinear difference equations is considered in [5–7]. Analogous results for linear ordinary and nonlinear functional differential equations are given in [8–16].

2. On some classes of nonoscillatory discrete functions

Lemma 2.1. *Let $n \geq 2$, $k_0 \in \mathbb{N}$, $u : \mathbb{N}^+_{k_0} \to \mathbb{R}$ and $u(k) > 0$, $\Delta^{(n)}u(k) \leq 0$ ($\Delta^{(n)}u(k) \geq 0$) for $k \in \mathbb{N}^+_{k_0}$, $\Delta^{(n)}u(k) \not\equiv 0$ for any $s \in \mathbb{N}^+_{k_0}$ and $k \in \mathbb{N}^+_s$. Then there exist $k_1 \in \mathbb{N}^+_{k_0}$ and $\ell \in \{0, \dots, n\}$ such that $\ell + n$ is odd ($l + n$ is even) and*

$$\begin{cases} \Delta^{(i)}u(k) > 0 & \text{for} \quad k \in \mathbb{N}^+_{k_1} \quad (i = 0, \dots, \ell), \\ (-1)^{i+\ell}\Delta^{(i)}u(k) > 0 & \text{for} \quad k \in \mathbb{N}^+_{k_1} \quad (i = \ell, \dots, n-1), \\ (-1)^{n+\ell}\Delta^{(n)}u(k) \geq 0 & \text{for} \quad k \in \mathbb{N}^+_{k_1}. \end{cases} \tag{2.1}$$

The lemma follows immediately from the fact that if $u(k) > 0$ and $\Delta^{(2)}u(k) \leq 0$ for $k \in \mathbb{N}^+_{k_0}$, then there exists $k_1 \in \mathbb{N}^+_{k_0}$ such that $\Delta^{(1)}u(k) > 0$ for $k \in \mathbb{N}^+_{k_1}$.

Lemma 2.2 ([3]). *Let $u : \mathbb{N} \to \mathbb{R}$ and let for some $k \in \mathbb{N}$ and $\ell \in \{1, \dots, n-1\}$ (2.1) be fulfilled. Then*

$$\sum_{k=1}^{+\infty} k^{n-\ell-1}\left|\Delta^{(n)}u(k)\right| < +\infty \tag{2.2}$$

and there exists $k_1 \in \mathbb{N}^+_{k_0}$ such that

$$\begin{cases} \left|\Delta^{(i)}u(k)\right| \geq \frac{1}{(n-i-1)!} \sum_{j=k}^{+\infty} \prod_{r=1}^{n-i-1}(j+r-k-1)\left|\Delta^{(n)}u(j)\right| \\ \qquad \textit{for} \quad k \in \mathbb{N}^+_{k_1} \quad (i = \ell, \dots, n-1), \end{cases} \tag{2.3}$$

$$\begin{cases} \Delta^{(i)}u(k) \geq \Delta^{(i)}u(k_1) + \frac{1}{(\ell-i-1)(n-\ell-1)!} \sum_{s=k_1}^{k-1} \prod_{r=1}^{\ell-i-1}(k+r-s-1) \\ \qquad \times \sum_{j=k}^{+\infty} \prod_{r=1}^{n-\ell-1}(j+r-s-1)\left|\Delta^{(n)}u(j)\right| \\ \qquad \textit{for} \ k \in \mathbb{N}^+_{k_1} \quad (i = 0, \dots, \ell-1). \end{cases} \tag{2.4}$$

If, in addition,

$$\sum_{k=1}^{+\infty} k^{n-\ell}\left|\Delta^{(n)}u(k)\right| = +\infty, \tag{2.5}$$

then

$$\frac{\Delta^{(\ell-i)}u(k)}{\prod\limits_{r=0}^{i-1}(k-r)}\downarrow, \qquad \frac{\Delta^{(\ell-i)}u(k)}{\prod\limits_{r=1}^{i-1}(k-r)}\uparrow \tag{2.6}$$

for large k

$$u(k)\ge \frac{1+o(1)}{\ell!}\,k^{\ell-1}\Delta^{(\ell-1)}u(k) \tag{2.7}$$

and

$$\begin{cases}\Delta^{(\ell-1)}u(k) & \ge \frac{k}{(n-\ell-1)!}\sum_{i=k}^{+\infty} i^{n-\ell-1}\big|\Delta^{(n)}u(k)\big| \\ & +\frac{1}{(n-\ell-1)!}\sum_{i=k_1}^{k} i^{n-\ell}\big|\Delta^{(n)}u(i)\big| \quad \text{for} \quad k\in\mathbb{N}^+_{k_1}.\end{cases} \tag{2.8}$$

3. Necessary condition for the existence of solutions of type (2.1)

The results of this section play an important role in establishing sufficient conditions for equation (1.1) to have Properties **A** and **B**.

Let $k_0\in\mathbb{N}$ and $\ell=\{1,\dots,n-1\}$. By U_{ℓ,k_0} we denote a set of all solutions of equation (1.1) satisfying condition (2.1).

Theorem 3.1. *Let conditions* (1.2), (1.3) ((1.4)) *be fulfilled,* $\ell\in\{1,\dots,n-1\}$ *with* $\ell+n$ *odd* ($\ell+n$ *even*) *and*

$$\sum_{k=1}^{+\infty} k^{n-\ell}\big(\tau(k)\big)^{\lambda(\ell-1)}\big|p(k)\big|=+\infty. \tag{3.1}$$

Moreover, if $U_{\ell,k_0}\neq\varnothing$ *for some* $k_0\in\mathbb{N}$, *then*

$$\sum_{k=1}^{+\infty} k^{n-\ell-1}\big(\tau(k)\big)^{\lambda(\ell-1)}\big(\widetilde{\tau}(k)\big)^{\lambda}\big|p(k)\big|<+\infty, \tag{3.2}$$

where

$$\widetilde{\tau}(k)=\begin{cases}\tau(k), & \tau(k)\le k,\\ k, & \tau(k)\ge k.\end{cases} \tag{3.3}$$

Proof. Let $k_0\in\mathbb{N}$, $\ell\in\{1,\dots,n-1\}$, $\ell+n$ be odd ($\ell+n$ be even) and $U_{\ell,k_0}\neq\varnothing$. By the definition of the set U_{ℓ,k_0}, equation (1.1) has a proper solution $u\in U_{\ell,k_0}$ satisfying condition (2.1). By (2.1) and (3.1), it is clear that condition (2.5) holds. Thus by Lemma 2.2, (2.2)–(2.8) are fulfilled and owing to (1.1) and (2.7), from (2.8), we get

$$\begin{cases}\Delta^{(\ell-1)}u(k) & \ge \frac{k}{(n-\ell-1)!}\sum_{i=k}^{+\infty} i^{n-\ell-1}\big|p(k)\big|u^{\lambda}\big(\tau(k)\big) \\ & \ge \frac{k}{\ell!(n-\ell)!}\sum_{i=k}^{+\infty} i^{n-\ell-1}\big(\tau(i)\big)^{\lambda(\ell-1)}\big(\Delta^{(\ell-1)}(\tau(i))\big)^{\lambda} \ \text{ for } \ k\in\mathbb{N}_{k_1}.\end{cases} \tag{3.4}$$

Since $\Delta^{(\ell-1)}u(k)\uparrow$ and by (3.3), $\widetilde{\tau}(k)\le k$, from (3.4), we have

$$y(k)\ge \frac{\widetilde{\tau}(k)}{\ell!(n-\ell)!}\sum_{i=k}^{+\infty} i^{n-\ell-1}\big|p(i)\big|\big(\tau(i)\big)^{\lambda(\ell-1)}\big(y(i)\big)^{\lambda} \quad k\in\mathbb{N}_{k_1}, \tag{3.5}$$

where $y(k) = u(\widetilde{\tau}(k))$. From (3.5), we get

$$y^\lambda(k) \geq \left(\frac{\widetilde{\tau}(k)}{\ell!(n-\ell)!}\right)^\lambda \left(\sum_{i=k}^{+\infty} i^{n-\ell-1}\big|p(i)\big|\big(\tau(i)\big)^{\lambda(\ell-1)} y^\ell(i)\right)^\lambda \quad k \in \mathbb{N}_{k_1}.$$

Therefore

$$\frac{y^\lambda(k)k^{n-\ell-1}\big|p(k)\big|\big(\tau(k)\big)^{\lambda(\ell-1)}}{\left(\sum\limits_{i=k}^{+\infty} i^{n-\ell-1}\big|p(i)\big|\big(\tau(i)\big)^{\lambda(\ell-1)} y(i)\right)^\lambda} \geq \left(\frac{\widetilde{\tau}(k)}{\ell!(n-\ell)!}\right)^\lambda k^{n-\ell-1}\big|p(k)\big|\big(\tau(k)\big)^{\lambda(\ell-1)}.$$

From the last inequality, we have

$$\begin{cases} \sum_{s=k_1}^k & \dfrac{y^\lambda(s)s^{n-\ell-1}\big|p(s)\big|\big(\tau(s)\big)^{\lambda(\ell-1)}}{\left(\sum\limits_{i=s}^{+\infty} i^{n-\ell-1}\big|p(i)\big|\big(\tau(i)\big)^{\lambda(\ell-1)}\big(y(i)\big)^\lambda\right)^\lambda} \\ & \geq \dfrac{1}{\big(\ell!(n-\ell)!\big)^\lambda} \sum_{s=k_1}^k \big(\tau(s)\big)^{\lambda(\ell-1)} s^{n-\ell-1}\big|p(s)\big|\big(\widetilde{\tau}(s)\big)\big)^\lambda. \end{cases} \tag{3.6}$$

Denote

$$a(k) = \sum_{i=k}^{+\infty} i^{n-\ell-1}\big(\tau(i)\big)^{\lambda(\ell-1)}\big|p(i)\big|\,\big(y(i)\big)^\lambda.$$

From (3.6), we get

$$\left\{\sum_{j=k_1}^k \frac{a_s - a_{s+1}}{(a_s)^\lambda} \geq \frac{1}{\ell!(n-\ell)!} \sum_{s=k_1}^k s^{n-\ell-1}\big|p(s)\big|\,\big(\tau(s)\big)^{\lambda(\ell-1)}\big(\widetilde{\tau}(s)\big)^\lambda. \right. \tag{3.7}$$

Since

$$\begin{aligned} \sum_{s=k_1}^k \frac{a_s - a_{s+1}}{(a_s)^\lambda} &= \sum_{s=k_1}^k (a_s)^{-\lambda} \int_{a_{s+1}}^{a_s} dt \\ &\leq \sum_{s=k_1}^k \int_{a_{s+1}}^{a_s} t^{-\lambda} dt \leq \int_0^{a_{k+1}} t^{-\lambda} dt = \frac{a_{k_1}^{1-\lambda}}{1-\lambda}\,, \end{aligned}$$

inequality (3.7), yields

$$\sum_{s=k_1}^k s^{n-\ell-1}\big|p(s)\big|\big(\tau(s)\big)^{\lambda(\ell-1)}\big(\widetilde{\tau}(s)\big)^\lambda \leq \big(\ell!(n-\ell)!\big)^\lambda \frac{a_{k_1}^{1-\lambda}}{1-\lambda}\,.$$

Thus from the last inequality, we have

$$\sum_{s=k_1}^{+\infty} s^{n-\ell-1}\big|p(s)\big|\big(\tau(s)\big)^{\lambda(\ell-1)}\big(\widetilde{\tau}(s)\big)^\lambda < +\infty,$$

which proves the validity of the theorem.

4. Sufficient conditions of nonexistence of solutions of type (2.1)

Theorem 4.1. *Let conditions* (1.3), ((1.4)) *be fulfilled,* $\ell \in \{1, \dots, n-1\}$ *with* $\ell+n$ *odd* ($\ell + n$ *even*) *and* (3.1) *hold. If, moreover,*

$$\sum_{k=1}^{+\infty} k^{n-\ell-1} \big(\widetilde{\tau}(k)\big)^{\lambda(\ell-1)} \big(\widetilde{\tau}(k)\big)^{\lambda} \big|p(k)\big| = +\infty, \tag{4.1}$$

then for any $k_0 \in \mathbb{N}$, $U_{\ell,k_0} = \varnothing$, *where* $\widetilde{\tau}$ *is defined by* (3.3).

Proof. Assume the contrary. Let there exist $k_0 \in \mathbb{N}$ such that $U_{\ell,k_0} \neq \varnothing$. Thus equation (1.1) has a proper solution $u : \mathbb{N}^+_{k_0} \to (0, \infty)$ satisfying conditions (2.1).

Since the conditions of Theorem 3.1 are fulfilled, (3.2) holds, that contradicts (4.1). The obtained contradiction proves the validity of the theorem.

From the theorem if $\tau(k) \leq k$, there immediately follows the following

Corollary 4.1. Let conditions (1.3), ((1.4)) be fulfilled, $\ell \in \{1, \dots, n-1\}$ with $\ell + n$ odd ($\ell + n$ even) and

$$\sum_{k=1}^{+\infty} k^{n-\ell-1} \big(\tau(k)\big)^{\lambda\ell} \, \big|p(k)\big| = +\infty. \tag{4.2}$$

Then $U_{\ell,k_0} = \varnothing$, for any $k_0 \in \mathbb{N}$.

5. Difference equations with Property A

Theorem 5.1. *Let condition* (1.3) *be fulfilled and for any* $\ell \in \{1, \dots, n-1\}$ *with* $\ell + n$ *odd,* (4.1) *hold as well, and when* n *is odd,*

$$\sum_{k=1}^{n-1} k^{n-1} \big|p(k)\big| = +\infty. \tag{5.1}$$

Then equation (1.1) *has Property* **A**.

Proof. Let equation (1.1) have a proper nonoscillatory solution $u : \mathbb{N}^+_{k_0} \to (0, +\infty)$ (the case $u(k) < 0$ is similar). Then by (1.1), (1.3) and Lemma 2.1, there exists $\ell \in \{0, \dots, n-1\}$ such that $\ell+n$ is odd and condition (2.1) holds. Since the conditions of Theorem 4.1 are fulfilled, for any $\ell \in \{1, \dots, n-1\}$ with $\ell + n$ odd, we have $\ell \notin \{1, \dots, n-1\}$. Therefore n is odd and $\ell = 0$. Then we can show that conditions (1.5) hold.

If that is not the case, there exists $c > 0$ such that $u(k) \geq c$ for a sufficiently large k. According to (2.1), with $\ell = 0$, from (1.1), we have

$$\sum_{i=k_0}^{k} i^{n-1} \Delta^{(n)} u(i) + c \sum_{i=k_0}^{k} i^{n-1} p(i) \leq 0 \quad \text{for} \quad k \in \mathbb{N}_{k_0}, \tag{5.2}$$

where $k_u \in \mathbb{N}$ is a sufficiently large natural number.

By the identity

$$\sum_{i=k_0}^{k} i^{n-1}\Delta^{(n)}u(i) = k^{n-1}\Delta^{(n-1)}u(k+1) - (k_0-1)^{n-1}\Delta^{(n-1)}u(k_0) - \sum_{i=k_0}^{k}\Delta^{(n-1)}u(i)\Delta(i-1)^{n-1}$$

it is easy to show that

$$\sum_{i=k_0}^{k} i^{n-1}\Delta^{(n)}u(i) = \sum_{i=0}^{n-1}(-1)^i\Delta^{(i)}(k-i)^{n-1}\Delta^{(n-i-1)}u(k+1) - \sum_{i=0}^{n-1}(-1)^i(k_0-i-1)^{(n-i-1)}\Delta^{(n-i-1)}u(k_0).$$

Since $(-1)^i\Delta^{(i)}u(k) \geq 0$, from (5.2), we have

$$c^\lambda\sum_{i=k_0}^{k} i^{n-1}p(i) \leq \sum_{i=0}^{n-1}(k_0-i-1)^{n-i-1}\big|\Delta^{(n-i-1)}u(k_0)\big|.$$

Therefore

$$\sum_{i=1}^{+\infty} i^{n-1}p(i) < +\infty,$$

which contradicts condition (5.1). The contradiction proves the validity of the theorem.

Corollary 5.1. Let conditions (1.2), (1.3) be fulfilled, $\ell \in \{1,\dots,n-1\}$, $\ell+n$ be odd, (4.2) and when n is odd (5.1) hold. Moreover, let

$$\tau(k) \leq k \quad \text{for} \quad k \in \mathbb{N}. \tag{5.3}$$

Then equation (1.1) has Property **A**.

Theorem 5.2. *Let conditions* (1.2), (1.3) *be fulfilled and*

$$\liminf_{k\to+\infty}\frac{\tau^\lambda(k)}{k} > 0. \tag{5.4}$$

Then the condition

$$\sum_{k=1}^{+\infty} k^{n-2}\big(\widetilde{\tau}(k)\big)^\lambda p(k) = +\infty, \tag{5.5}$$

for even n and conditions (5.1) *and*

$$\sum_{k=1}^{+\infty} k^{n-3}\big(\tau(k)\big)^\lambda\big(\widetilde{\tau}(k)\big)^\lambda p(k) = +\infty, \tag{5.6}$$

for odd n are sufficient for equation (1.1) *to have Property* **A**.

Proof. It is obvious that, according to (5.4) and (5.5) ((5.1), (5.4), (5.6)), for even n (for odd n), where $\ell+n$ is odd, all conditions of Theorem 5.1 are fulfilled, which proves the validity of the theorem.

Corollary 5.2. Let conditions (1.2), (1.3), (5.4) be fulfilled, and

$$\sum_{k=1}^{+\infty} k^{n-2}\big(\tau(k)\big)^{\lambda}\,p(k)=+\infty$$

for even n and conditions (5.1) and

$$\sum_{k=1}^{+\infty} k^{n-3}\big(\tau(k)\big)^{2\lambda}\,p(k)=+\infty$$

for odd n. Then equation (1.1) has Property **A**.

Theorem 5.3. *Let conditions* (1.2), (1.3) *be fulfilled and*

$$\limsup_{k\to+\infty}\frac{\tau^{\lambda}(k)}{k}<+\infty. \tag{5.7}$$

Then for equation (1.1) *to have Property* **A**, *it suffices that*

$$\sum_{k=1}^{+\infty}\big(\tau(k)\big)^{\lambda(n-2)}\big(\widetilde{\tau}(k)\big)^{\lambda}p(k)=+\infty \tag{5.8}$$

for even n and conditions (5.1) *and* (5.2) *for odd n.*

Corollary 5.3. Let conditions (1.2), (1.3), (5.3), (5.7) be fulfilled. Then (1.1) to have Property **A**, it suffices that

$$\sum_{k=1}^{+\infty}\big(\tau(k)\big)^{\lambda(n-1)}\,p(k)=+\infty \tag{5.9}$$

for even n and conditions (5.1) and (5.9) for odd n.

6. Difference equations with Property B

Theorem 6.1. *Let conditions* (1.2), (1.4) *be fulfilled and for any $\ell\in\{1,\dots,n-2\}$ with $\ell+n$ even, let as well as* (4.1) *hold and*

$$\sum_{k=1}^{+\infty}\big(\tau(k)\big)^{\lambda(n-1)}\big|p(k)\big|=+\infty \tag{6.1}$$

If moreover, for even n,

$$\sum_{k=1}^{+\infty}k^{n-1}\big|p(k)\big|=+\infty, \tag{6.2}$$

then equation (1.1) *has Property* **B**.

Proof. Let equation (1.1) have a proper nonoscillatory solution $u : \mathbb{N}_{k_0} \to (0, +\infty)$. Then by (1.1), (1.3) and Lemma 2.1, there exists $\ell \in \{0, \dots, n\}$ such that $\ell + n$ is even and condition (2.1) holds. Since the conditions of Theorem 4.2 are fulfilled, for any $\ell \in \{1, \dots, n-2\}$ with $\ell + n$ even, we have $\ell \notin \{1, \dots, n-2\}$. Therefore $\ell = n$, or $\ell = 0$ and n is even.

Assume that $\ell = n$. To complete the proof, it suffices to show that (1.6) is valid. From (2.1), when $\ell = n$, we have $u(\tau(k)) \geq c(\tau(k))^{n-1}$ for $k \in \mathbb{N}^+_{k_1}$, where $c > 0$ and $k_1 \in \mathbb{N}_{k_0}$ is a sufficiently large natural number. Therefore by (1.2), (6.1) and (2.1), when $\ell = n$, from (1.1), we get

$$\Delta^{(n-1)}u(k) \geq \Delta^{(n-1)}u(k_1) + c^\lambda \sum_{i=k_1}^{k} |p(j)| \big(\tau(i)\big)^{\lambda(n-1)} \to +\infty \ \text{ for } \ k \to +\infty.$$

Now assume that condition (1.5) holds. Therefore equation (1.1) has Property **B**.

Corollary 6.1. Let conditions (1.2), (1.4), (5.3) be fulfilled, $\ell \in \{1, \dots, n-2\}$, $\ell + n$ be odd, (4.2) and, when n is even, (5.2) hold. Then equation R(1.1) has Property **B**.

Theorem 6.2. *Let conditions* (1.2), (1.4), (5.4) *be fulfilled. Then the condition*

$$\sum_{k=1}^{+\infty} k^{n-2} \big(\widetilde{\tau}(k)\big)^\lambda |p(k)| = +\infty \tag{6.3}$$

for odd n and conditions (6.2) *and*

$$\sum_{k=1}^{+\infty} k^{n-3} \big(\tau(k)\big)^\lambda \big(\widetilde{\tau}(k)\big)^\lambda |p(k)| = +\infty \tag{6.4}$$

for even n are sufficient for equation (1.1) *to have Property* **B**, *where $\widetilde{\tau}$ is definition by* (3.3).

Proof. It is obvious that by (5.4), (6.3) and (6.4), all conditions of Theorem 6.1 are fulfilled, which proves the validity of the theorem.

Corollary 6.2. Let conditions (1.2), (1.4), (5.3), (5.4) be fulfilled. Then

$$\sum_{k=1}^{+\infty} k^{n-2} \big(\tau(k)\big)^\lambda |p(k)| = +\infty$$

for odd n and the condition

$$\sum_{k=1}^{+\infty} k^{n-3} \big(\tau(k)\big)^{2\lambda} |p(k)| = +\infty$$

is sufficient for even n. Then equation (1.1) to have Property **B**.

Theorem 6.3. *Let conditions* (1.2), (1.4) (5.7) (5.1) *be fulfilled and*

$$\sum_{k=1}^{+\infty} k \big(\tau(k)\big)^{\lambda(n-3)} \big(\widetilde{\tau}(k)\big)^\lambda |p(k)| = +\infty.$$

Then equation (1.1) *has Property* **B**.

Proof. According to (5.7) and (6.5) it follows from the above that for any $\ell \in \{1, \dots, n-2\}$, $\ell + n$ even, condition (4.1) is fulfilled. That is, all conditions of Theorem 6.1 are fulfilled, which proves the validity of the theorem.

Corollary 6.3. Let conditions (1.2), (1.4), (5.7), (5.1), (5.3) be fulfilled and

$$\sum_{k=1}^{+\infty} k\big(\tau(k)\big)^{\lambda(n-2)} \big|p(k)\big| = +\infty.$$

Then equation (1.1) has Property **B**.

R E F E R E N C E S

1. Koplatadze R., Khachidze N. Asymptotic behavior of solutions of n-th order Emden-Fowler difference equations with advanced argument. *J. Contemporary Mathematical Analysis of Sciences*, **55**, 4 (2021), 201–213.

2. Koplatadze R. On higher order nonlinear difference equations with property B. *Semin. I. Vekua Inst. Appl. Math. Rep.*, **47** (2021), 42-56.

3. Koplatadze R. Asymptotic behavior of solutions of n-th order Emden-Fowler type differential equations with deviating argument. *Semin. I. Vekua Inst. Appl. Math. Rep.*, **48** (2022), 15-27.

4. Koplatadze R., Khachidze N. Nonlinear difference equations with properties A and B. *Funct. Differ. Equ.*, **25**, 1-2 (2018), 91-95.

5. Koplatadze R., Kvinikadze G., and Stavroulakis I. P. Oscillation of second order linear difference equations with deviating arguments. *Adv. Math. Sci. Appl.*, **12**, 1 (2002), 217-226.

6. Koplatadze R., Pinelas S. Oscillation of nonlinear difference equations with delayed argument. *Commun. Appl. Anal.* **16**, 1 (2012), 87-95.

7. Koplatadze R., Pinelas S. Oscillation criteria for first order linear difference equations with several delay arguments. *Nonlinear Oscillations*, **17**, 2 (2014), 247–267; *J. Math. Sci.* (N.Y.), **208**, 5 (2015), 571-592.

8. Koplatadze R., Chanturia T. On oscillatory properties of differential equations with a deviating argument (Russian). *Izdat. Tbilis. Univ., Tbilisi*, 1977, 115 pp.

9. Koplatadze R. On oscillatory properties of solutions of functional differential equations. *Mem. Diff. Eq. Math. Phys.*, **3** (1994), 3-179.

10 Koplatadze R., Graef J., and Kvinikadze G. Nonlinear functional differential equations with Properties A and B. *J. Math. Anal. Appl.*, **306** (2005), 136-160.

11. Koplatadze R. Quasi-linear functional differential equations with property A. *J. Math. Anal. Appl.*, **330** (2007), 483-510.

12. Koplatadze R., Litsyn E, Oscillation criteria for higher order “almost linear” functional differential equation. *Funct. Differ. Equ.*, **16**, 3 (2009), 387-434.

13. Domoshnitsky A., Koplatadze R. On asymptotic behavior of solutions of generalized Enmden-Fowler differential equations with delay argument. *Abstract and Applied Analysis*, 2014, Art. ID 168425, 13 pp.

14. Domoshnitsky A., Koplatadze R. On higher order generalized Emden-Fowler differential equations with delay argument. *Reprinted in J. Math. Sci.* (N.Y.), **220**, 4 (2017), 461-482. *Nelnn Koliv.*, **18**, 4 (2015), 507-526.

15. Koplatadze R. Almost linear functional differential equations with properties A and B. *Trans. A. Razmadze Math. Inst.* **170**, 2 (2016), 215-242.

16. Berezansky L., Domoshnitsky A., Koplatadze R. Oscillation, nonoscillation, stability and asymptotic properties for second and higher order functional differential equations. *Taylor and Francs Group*, 2020.

Received 31.07.2023; revised 20.08.2023; accepted 10.09.2023

Author's address:

R. Koplatadze
Department of Mathematics of
I. Javakhishvili Tbilisi State University &
I. Vekua Institute of Applied Mathematics
11, University St., Tbilisi 0186
Georgia
E-mail: r_koplatadze@yahoo.com; roman.koplatadze@tsu.ge