Seminar of I. Vekua Institute of Applied Mathematics
REPORTS, Vol. 49, 2023

ON THE ACCURACY OF THE DIFFERENCE SCHEME FOR A NONLINEAR MODEL OF THE DYNAMIC BEAM

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Abstract. The initial boundary value problem is posed for a nonlinear integro-differential inhomogeneous equation that describes the dynamic behaviour of the beam. To approximate the solution with respect to a time variable the Crank–Nicolson type difference scheme is used, the error of which is estimated.

Keywords and phrases: Dynamic beam equation, Crank–Nicolson scheme, error estimate.

AMS subject classification (2010): 35L20, 65M06, 65M15.

1. Statement of the problem

Let us consider the beam oscillation problem

$$\frac{\partial^2 u}{\partial t^2}(x,t)+\frac{\partial^4 u}{\partial x^4}(x,t)-h\,\frac{\partial^4 u}{\partial x^2\partial t^2}(x,t)$$
$$-\left(\lambda+\int_0^L\left(\frac{\partial u}{\partial x}(x,t)\right)^2dx\right)\frac{\partial^2 u}{\partial x^2}(x,t)=f(x,t), \tag{1.1}$$
$$0<x<L,\quad 0<t\le T,$$

$$\begin{gathered} u(x,0)=u^0(x),\quad \frac{\partial u}{\partial t}(x,0)=u^1(x),\\ u(0,t)=u(L,t)=0,\quad \frac{\partial^2 u}{\partial x^2}(0,t)=\frac{\partial^2 u}{\partial x^2}(L,t)=0, \end{gathered} \tag{1.2}$$

where h is a non-negative and λ a positive constant, u^0, $u^1(x)$ and $f(x,t)$ are the given sufficiently smooth functions and $u(x,t)$ is the unknown function.

Equation (1.1), the general form of which was written by Henriques de Brito [5], describes the oscillation of a beam. For the case where $f(x,t)=0$, $\lambda=0$, equation (1.1) is derived by Menzala and Zuazua [8] as a limit of one-dimensional Karman model. Numerical methods for the integro-differential beam equations with the same nonlinearity as that of (1.1) are investigated in [1, 2, 3, 4, 9, 10].

2. Algorithm

Let us approximate the solution of problem (1.1), (1.2) with respect to the variable x. For this we use the Galerkin method [6]. The solution is represented as a finite series

$$u_n(x,t)=\sum_{i=1}^n u_{ni}(t)\sin\frac{i\pi x}{L}\,,$$

where the coefficients $u_{ni}(t)$ are defined from the following system of nonlinear differential equations

$$\Big(1+h\Big(\frac{j\pi}{L}\Big)^2\Big)u''_{ni}(t)+\Big(\frac{i\pi}{L}\Big)^4u_{ni}(t)$$
$$+\Big(\lambda+\frac{L}{2}\sum_{j=1}^{n}\Big(\frac{j\pi}{L}\Big)^2u_{nj}^2(t)\Big)\Big(\frac{i\pi}{L}\Big)^2u_{ni}(t)=f_i(t), \tag{2.1}$$
$$i=1,2,\dots,n,\ \ 0<t\le T,$$

under the initial conditions

$$u_{ni}(0)=u_i^0,\quad u'_{ni}(0)=u_i^1,\ \ i=1,2,\dots,n. \tag{2.2}$$

Here

$$u_i^l=\frac{2}{L}\int_0^L u^l(x)\sin\frac{i\pi x}{L}\,dx,\ \ l=0,1,\quad f_i(t)=\frac{2}{L}\int_0^L f(x,t)\sin\frac{i\pi x}{L}\,dx.$$

To solve problem (2.1), (2.2) we use the difference method. For this, on the time interval $[0,T]$ we put the grid with step $\tau=\frac{T}{M}$, $0<\tau<1$, and nodes $t_m=m\tau$, $m=0,1,\dots,M$. On the layer m, e.g. for $t=t_m$, the approximate value of $u_{ni}(t_m)$ is denoted by u_{ni}^m. Let us apply the Crank–Nicolson type scheme

$$\Big(1+h\Big(\frac{i\pi}{L}\Big)^2\Big)\frac{u_{ni}^{m+1}-2u_{ni}^m+u_{ni}^{m-1}}{\tau^2}+\Big\{\Big(\frac{i\pi}{L}\Big)^4$$
$$+\Big(\frac{i\pi}{L}\Big)^2\Big(\lambda+\frac{L}{4}\sum_{j=1}^{n}\Big(\frac{j\pi}{L}\Big)^2\Big[\Big(\frac{u_{nj}^{m+1}+u_{nj}^m}{2}\Big)^2+\Big(\frac{u_{nj}^m+u_{nj}^{m-1}}{2}\Big)^2\Big]\Big)\Big\}$$
$$\times\frac{u_{ni}^{m+1}+2u_{ni}^m+u_{ni}^{m-1}}{4}=f_i^m, \tag{2.3}$$
$$m=1,2,\dots,M-1,\ \ i=1,2,\dots,n,$$
$$u_{ni}^0=u_i^0,\quad \frac{u_{ni}^1-u_{ni}^0}{\tau}=u_i^1,\ \ i=1,2,\dots,n. \tag{2.4}$$

Here

$$f_i^m=f_i(t_m).$$

The approximate solution of (2.3), (2.4) at the node t_m is defined by the sum

$$u_n^m(x)=\sum_{i=1}^{n}u_{ni}^m\sin\frac{i\pi x}{L}\,.$$

Note that in [7], the approximate solution of system (2.3), (2.4) is obtained by Newton's iteration method, the error of which is estimated.

3. Difference scheme error

Under the error of difference scheme (2.3), (2.4) we understand the difference between the functions $u_n^m(x)$ and $u_n(x,t_m)$

$$\Delta u_n^m(x)=u_n^m(x)-u_n(x,t_m).$$

Denote by $\|\cdot\|$ the norm in the space $L_2(0,L)$. Let us formulate the main result.

Theorem. *Suppose that for functions $u_{ni}(t)$, $i=1,2,\dots,n$, the condition*

$$u_{ni}(t)\in C_4[0,T]$$

is fulfilled and the grid step τ satisfies the restriction

$$0<\tau<\frac{2}{\alpha}\,\theta_0\Big(1-\frac{1}{p}\Big)(\theta_0+\overline{\theta}_0),$$

where $\forall\,p>1$. Then the error of difference scheme (2.3), (2.4) *is estimated by the inequality*

$$\|\Delta u_n^m(x)\|\le C\tau^2,\ \ m=2,3,\dots,M.$$

The definition formulas of α, θ_0, $\overline{\theta}_0$ and C are given in the Appendix.

Appendix

$$C=(c_1+c_2)c_3,$$

$$c_1=\frac{1}{4}\,m_3n,\quad c_2=\frac{1}{4}\,(1+\tau pp_2)(1+\tau p_3)T$$

$$\times\Bigg[\frac{1}{3}\,m_4n\Big(1+h\Big(\frac{n\pi}{L}\Big)^2\Big)+m_2\Big(\lambda+\theta_1+\Big(\frac{n\pi}{L}\Big)^2\Big)\Big(\frac{n\pi}{L}\Big)^2$$

$$+\Big(\frac{2}{L}\,\theta_0\Big)^{1/2}\Big(m_2n^{1/2}+\Big(\frac{2}{L}\,\theta_0\Big)^{1/2}\Big(\frac{n\pi}{L}\Big)^2\Big)\Big(\theta_0+\frac{L}{8}\,\Big(1+\tau^2m_2\Big(\frac{n\pi}{L}\Big)^2\Big)\Big)\Bigg],$$

$$c_3=\Big(\frac{L}{2}\Big)^{1/2}\max\Big(1,\frac{L}{\pi}\Big)e^{(p_1+p_2+\tau pp_2)T},$$

$$m_k=\max_{t,i}\Big|\frac{d^ku_{ni}(t)}{dt^k}\Big|,\ \ 0\le t\le T,\ \ i=1,2,\dots,n,\ \ k=2,3,4,$$

$$\alpha=\Big(\frac{1}{1+h(\frac{\pi}{2})}\Big)^{\frac{1}{2}},$$

$$p_1=\max\Big(\frac{1}{h}\,,\lambda+\overline{\theta}_1\Big)\,\frac{n\pi}{L}\,,\quad p_2=p_{21}+p_{22}n^{1/2},$$

$$p_{21}=\frac{1}{4}\,(\theta_1+\overline{\theta}_1+2L),\quad p_{22}=\frac{2}{L}\,\Big(1+\Big(\frac{1}{2}\,\max{}^{\backprime}\Big(1,\frac{L}{\pi}\Big)\Big)^2\Big),$$

$$p_3=\max(p_3^1,p_3^2),$$

$$p_3^1=\frac{1}{2}\,\min\Big(\max\Big(\frac{1}{4}\,,\frac{1}{h}\Big),\alpha^2\,\frac{n\pi}{L}\Big),\quad p_3^2=\frac{1}{2}\,\min\Big(\frac{1}{h},\lambda+\overline{\theta}_1\Big)\,\frac{n\pi}{L}\,,$$

$$\theta_0=\Bigg(\|u^1(x)\|^2+h\|u^{1\prime}(x)\|^2+\|u^{0\prime\prime}(x)\|^2+\frac{1}{2}\,(\lambda+\|u^{0\prime}(x)\|^2)^2$$

$$+\frac{1}{1+h(\frac{i\pi}{L})^2}\int\limits_0^T\|\pi_nf(x,t)\|^2\Bigg)e^T,$$

$$\theta_1=\Big(\Big(\frac{\pi}{2}\Big)^4+2\lambda\Big(\frac{\pi}{2}\Big)^2+2\theta_0\Big)^{1/2}-\Big(\Big(\frac{\pi}{2}\Big)^2+\lambda\Big),$$

$$\overline{\theta}_0=\Bigg(\frac{L}{2}\sum_{i=1}^n\Big(1+h\Big(\frac{i\pi}{L}\Big)^2\Big)\Big(\frac{u_{ni}^1-u_{ni}^0}{\tau}\Big)^2+\frac{L}{2}\sum_{i=1}^n\Big(\frac{i\pi}{L}\Big)^4\Big(\frac{u_{ni}^1+u_{ni}^0}{2}\Big)^2$$

$$+\frac{1}{2}\left(\lambda+\frac{L}{2}\sum_{i=1}^{n}\Big(\frac{i\pi}{L}\Big)^2\Big(\frac{u_{ni}^1+u_{ni}^0}{2}\Big)^2\right)^2+\frac{\tau}{1-\frac{\tau}{2}}\sum_{l=1}^{m}\|f(x,t_l)\|^2\Big)e^{T/(1-\tau/2)},$$

$$\overline{\theta}_1=\Big(\Big(\frac{\pi}{L}\Big)^4+2\lambda\Big(\frac{\pi}{L}\Big)^2+2\overline{\theta}_0\Big)^{1/2}-\Big(\Big(\frac{\pi}{L}\Big)^2+\lambda\Big),$$

$$\pi_n f(x,t)=\sum_{i=1}^{n}f_i(t)\sin\frac{i\pi x}{L}\,.$$

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Received 04.05.2023; revised 31.05.2023; accepted 09.06.2023

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