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ON THE EXISTENCE OF AN OPTIMAL INITIAL DATA FOR ONE CLASS OF NEUTRAL FUNCTIONAL DIFFERENTIAL EQUATION WITH SEVERAL DELAY AND TWO TYPES CONTROL

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Abstract. For an optimal problem containing neutral functional differential equation with the two type controls, whose right-hand side is linear with respect to prehistory of the phase velocity, existence theorems of an optimal initial data are provided. Under the initial data we imply the collection of the initial and final moments, delay parameters, initial vector and control functions.

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1. Introduction

For an optimal problem containing neutral functional differential equation with the two type controls, whose right-hand side is linear with respect to prehistory of the phase velocity, the existence theorem of an optimal initial data are provided. Moreover, here the general existence theorem is concretized for an optimal control problem with the integral functional and for an optimization problem corresponding to economic growth model.

Under the initial data we imply the collection of the initial and final moments, delay parameters, initial vector and control functions.

The controlled neutral functional differential equation is the mathematical model of a controlled system whose behavior at a given moment of time depends on the velocity and the state of this system in the past. Many real processes in different areas of natural sciences and economics are described by the neutral functional differential equations [1-4,8].

To illustrate this, here we consider a simple theoretic model of economic growth. Let $g(t)$ be a quantity of a product, produced at the moment t which is expressed in money unites. The fundamental principle of the economic growth has the form

$$g(t) = h(t) + i(t), \tag{1.1}$$

where $h(t)$ is a quantity money for the salaries and social programs etc; $i(t)$ is a quantity money for the induced investment (purchase of new technologies and etc). We consider the case, where the functions $h(t)$ and $i(t)$ have the form

$$h(t) = \alpha(t, g(t), u(t)), \tag{1.2}$$

$$i(t) = \beta\Big(t, g(t-\tau_1), \dots, g(t-\tau_p), \dot g(t), \dot g(t-\tau_1), \dots, \dot g(t-\tau_p), u(t)\Big)$$

$$+\gamma \ddot g(t) + \sum_{i=1}^{m} \vartheta_i(t, v(t)) \ddot g(t-\sigma_i), \tag{1.3}$$

where $u(t)$ and $v(t)$ are scalar control functions (investment from the government or from private firms) with $\sigma_i > 0$, $i = 1, \ldots, m$, $\tau_i > 0$, $i = 1, \ldots, p$, are so-called delays.

The formula (1.3) shows that the value of investment at the moment t depends: on the quantity of money at the moments $t - \tau_i, i = 1, \ldots, p$ (in the past); on the velocity (product current) at the moments t and $t - \tau_i, i = 1, \ldots, p$; on the acceleration at the moments t and $t - \sigma_i, i = 1, \ldots, m$.

From formulas (1.1)-(1.3) we get the equation

$$\ddot{g}(t) = \frac{1}{\gamma}\Big[g(t) - \alpha(t, g(t), u(t)) - \beta\Big(t, g(t-\tau_1), \ldots, g(t-\tau_p), \dot{g}(t)$$
$$, \dot{g}(t-\tau_1), \ldots, \dot{g}(t-\tau_p), u(t)\Big) - \sum_{i=1}^{m} \vartheta_i(t, v(t))\ddot{g}(t-\sigma_i)\Big],$$

which is equivalent to the following controlled neutral functional differential equation:

$$\begin{cases} \dot{x}^1(t) = x^2(t), \\ \dot{x}^2(t) = \frac{1}{\gamma}\Big[x^1(t) - \alpha(t, x^1(t), u(t)) - \beta\Big(t, x^1(t-\tau_1), \ldots, x^1(t-\tau_p), x^2(t) \\ , x^2(t-\tau_1), \ldots, x^2(t-\tau_p), u(t)\Big) - \sum_{i=1}^{m} \vartheta_i(t, v(t))\dot{x}^2(t-\sigma_i)\Big]; \end{cases} \tag{1.4}$$

here $x^1(t) = g(t)$. The equation (1.4) is called a quasi-linear controlled neutral functional differential equation because the right-hand side is linear with respect to $\dot{x}^2(t - \sigma_i)$, $i = 1, \ldots, m$.

2. Problem statement and existence theorems

Let $I = [a, b] \subset \mathbb{R}^1$ be a finite interval and let

$$a < \zeta_{01} < \zeta_{02} < \zeta_{11} < \zeta_{12} < b, \sigma_{i2} > \sigma_{i1} > 0, i = 1, \ldots, m,$$

$$\tau_{i2} > \tau_{i1} > 0, i = 1, \ldots, p,$$

be given numbers; suppose that $O \subset \mathbb{R}^n$ is a open set and $V \subset \mathbb{R}^s$ and $U \subset \mathbb{R}^r$ are compact sets.

The $n \times n$-dimensional matrix functions $A_i(t, v)$, $i = 1, \ldots, m$ and the n-dimensional vector function $f(t, x_1, \ldots, x_{p+1}, u)$ satisfy the standard conditions, i.e. $A_i(t, v)$ is continuous on the set $I \times V$; $f(t, x_1, \ldots, x_{p+1}, u)$ is continuous on the set $I \times O^{p+1} \times U$ and continuously differentiable with respect to $x_1, \ldots, x_{p+1}$.

Further, denote by $\Delta_s = \Delta_s(I, V, k, L)$ the set of piecewise-continuous functions $v : I \to V$ satisfying the condition: for each function $v(\cdot) \in \Delta_s$ there exists a partition $a = \xi_0 < \cdots < \xi_{k+1} = b$ such that the restriction of the function $v(t)$ satisfies the Lipschitz condition on the open intervals (ξ_i, ξ_{i+1}), $i = 0, \ldots, k$, i.e.

$$|v(t') - v(t'')| \le L|t' - t''|, \ \ \forall\, t', t'' \in (\xi_i, \xi_{i+1}), \ \ i = 0, \ldots, k,$$

where the numbers k and L do not depend on $v(\cdot)$.

By $\Omega_r = \Omega_r(I,U)$ we denote the set of measurable functions $u : I \to U$. Let

$$q^i : [\zeta_{01}, \zeta_{02}] \times [\zeta_{11}, \zeta_{12}] \times [\sigma_{11}, \sigma_{12}] \times \cdots \times [\sigma_{m1}, \sigma_{m2}] \times [\tau_{11}, \tau_{12}] \times \cdots$$

$$\times [\tau_{p1}, \tau_{p2}] \times X_0 \times O \to \mathbb{R}^1, \;\; i = 0, \dots, l,$$

be continuous functions, where $X_0 \subset O$ is a compact set.

To each element **(initial data)**

$$w = \big(t_0, t_1, \sigma_1, \dots, \sigma_m, \tau_1, \dots, \tau_p, x_0, v(\cdot), u(\cdot)\big) \in W = [\zeta_{01}, \zeta_{02}] \times [\zeta_{11}, \zeta_{12}] \times [\sigma_{11}, \sigma_{12}]$$

$$\times \cdots \times [\sigma_{m1}, \sigma_{m2}] \times [\tau_{11}, \tau_{12}] \times \cdots \times [\tau_{p1}, \tau_{p2}] \times X_0 \times \Delta_s \times \Omega_r$$

we assign the neutral differential equation

$$\dot{x}(t) = \sum_{i=1}^{m} A_i(t, v(t)) \dots x(t - \sigma_i) + f\big(t, x(t), x(t - \tau_1), \dots, x(t - \tau_p), u(t)\big) \tag{2.1}$$

with the initial condition

$$x(t) = \varphi(t), \;\; t \in [\widehat{\tau}, t_0), \;\; x(t_0) = x_0, \tag{2.2}$$

where $\varphi : I_1 = [\widehat{\tau}, \zeta_{02}] \to O$ is a given absolutely continuous initial function with

$$|\dot{\varphi}(t)| \le const, \widehat{\tau} = \inf\{a - \sigma_{12}, \dots, a - \sigma_{m2}, a - \tau_{12}, \dots, a - \tau_{p2}\}$$

.

Definition 2.1. Let $w = (t_0, t_1, \sigma_1, \dots, \sigma_m, \tau_1, \dots, \tau_p, x_0, v(\cdot), u(\cdot)) \in W$. A function $x(t) = x(t; w) \in O$, $t \in [\widehat{\tau}, t_1]$, is called a solution corresponding to the element w, if it satisfies condition (2.2) and is absolutely continuous on the interval $[t_0, t_1]$ and satisfies equation (2.1) almost everywhere on $[t_0, t_1]$.

Definition 2.2. An element $w \in W$ is said to be admissible, if there exists the corresponding solution $x(t) = x(t; w)$ satisfying the conditions

$$q^i\big(t_0, t_1, \sigma_1, \dots, \sigma_m, \tau_1, \dots, \tau_p, x_0, x(t_1)\big) = 0, \;\; i = 1, \dots, l. \tag{2.3}$$

We denote the set of admissible elements by W_0.

Now we consider the functional

$$J(w) = q^0\big(t_0, t_1, \sigma_1, \dots, \sigma_m, \tau_1, \dots, \tau_p, x_0, x(t_1)\big).$$

Definition 2.3. An element $w_0 = (t_{00}, t_{10}, \sigma_{10}, \dots, \sigma_{m0}, \tau_{10}, \dots, \tau_{p0}, x_{00}, v_0(\cdot), u_0(\cdot)) \in W_0$ is said to be optimal, if

$$J(w_0) = \inf_{w \in W_0} J(w). \tag{2.4}$$

(2.1)-(2.4) is called the quasi-linear neutral optimal problem.

Theorem 2.1. *There exists an optimal element $w_0 \in W_0$ if the following conditions hold:*

(1) $W_0 \neq \varnothing$;

(2) *there exists a compact set* $K \subset O$ *such that for an arbitrary* $w \in W_0$

$$x(t; w) \in K, \ \ t \in I;$$

(3) *for each fixed* $(t, x_1, \dots, x_{p+1}) \in I \times K^{p+1}$ *the set*

$$\{f(t, x_1, \dots, x_{p+1}, u) : \ \ u \in U\}$$

is convex.

The existence theorems for various classes of neutral optimal problems are given in [2, 5-10].

Remark 2.1. Let U be a convex set and

$$f(t, x_1, \dots, x_{p+1}, u) = B(t, x_1, \dots, x_{p+1}) + C(t, x_1, \dots, x_{p+1})u.$$

Then the condition (3) of Theorem 2.1 holds. Theorem 2.1 is proved by schemes given in [4] and [9].

Now we consider the optimal problem with the integral functional and with fixed ends

$$\dot{x}(t) = \sum_{i=1}^{m} A_i(t, v(t))\dot{x}(t - \sigma_i) + f\big(t, x(t), x(t - \tau_1), \dots, x(t - \tau_p), u(t)\big),$$

$$x(t) = \varphi(t), \ \ t \in [\widehat{\tau}, t_0), \ \ x(t_0) = x_0, \ \ x(t_1) = x_1,$$

$$\int_{t_0}^{t_1} \Big[\sum_{i=1}^{m} a_i^0(t, v(t))\dot{x}(t - \sigma_i) + f^0\big(t, x(t), x(t - \tau_1), \dots, x(t - \tau_p), u(t)\big)\Big]\, dt \to \min.$$

Here

$$a_i^0(t, v) : I \times V \to \mathbb{R}^n, i = 1, \dots, m, f^0(t, x_1, \dots, x_{p+1}, u) : I \times O^{p+1} \times U \to \mathbb{R}^1$$

are continuous functions, $x_0, x_1 \in O$ are fixed points.

Evidently, this problem is equivalent to the following problem

$$\dot{x}^0(t) = \sum_{i=1}^{m} a_i^0(t, v(t))\dot{x}(t - \sigma_i) + f^0\big(t, x(t), x(t - \tau_1), \dots, x(t - \tau_p), u(t)\big),$$

$$\dot{x}(t) = \sum_{i=1}^{m} A_i(t, v(t))\dot{x}(t - \sigma_i) + f\big(t, x(t), x(t - \tau_1), \dots, x(t - \tau_p), u(t)\big),$$

$$x^0(t_0) = 0, x(t) = \varphi(t), \ \ t \in [\widehat{\tau}, t_0), \ \ x(t_0) = x_0, \ \ x(t_1) = x_1,$$

$$x^0(t_1) \to \min,$$

which is a particular case to similar problem (2.1)–(2.4) in the space $\mathbb{R}^{1+n}$. For the last posed neutral optimal problem by Z_0 we denote the set of admissible elements

$$z = (t_0, t_1, \sigma_1, \dots, \sigma_m, \tau_1, \dots, \tau_p, v(\,\cdot\,), u(\,\cdot\,)) \in Z = [\zeta_{01}, \zeta_{02}] \times [\zeta_{11}, \zeta_{12}]$$

$$\times [\sigma_{11}, \sigma_{12}] \times \cdots \times [\sigma_{m1}, \sigma_{m2}] \times [\tau_{11}, \tau_{12}] \times \cdots \times [\tau_{p1}, \tau_{p2}] \times \Delta_s \times \Omega_r$$

and by

$$z_0 = (t_{00}, t_{10}, \sigma_{10}, \dots, \sigma_{m0}, \tau_{10}, \dots, \tau_{p0}, v_0(\,\cdot\,), u_0(\,\cdot\,))$$

we denote an optimal element (see Definitions 2.2 and 2.3. Let us introduce the function $F = (f^0, f)^T$.

Theorem 2.2. *There exists an optimal element $z_0 \in Z_0$ if the following conditions hold:*

(4) $Z_0 \neq \varnothing$;

(5) *there exists a compact set $K_0 \subset \mathbb{R}^1 \times O$ such that for an arbitrary $z \in Z_0$*

$$\big(x^0(t; z), x(t; z)\big)^T \in K_0, \;\; t \in [t_0, t_1];$$

(6) *for each fixed $(t, x_1, \dots, x_{p+1}) \in I \times K_0^{p+1}$ the set*

$$\big\{F(t, x_1, \dots, x_{p+1}, u) : \;\; u \in U\big\}$$

is convex.

Theorem 2.2 it follows from a Theorem similar to Theorem 2.1 formulated in the space $\mathbb{R}^{1+n}$.

On the fixed interval $[t_0, t_1]$, where $t_0 \in [\zeta_{01}, \zeta_{02}]$, $t_1 \in [\zeta_{11}, \zeta_{12}]$, now we consider the optimal problem for the economic growth model (1.4) with the initial condition

$$x^1(t) = \varphi^1(t), \;\; x^2(t) = \varphi^2(t), \;\; t \in [\widehat{\tau}, t_0), \;\; x^1(t_0) = x_0^1, \;\; x^2(t_0) = x_0^2 \tag{2.5}$$

and with the functional

$$-x^1(t_1) \to \min. \tag{2.6}$$

Here $\varphi^1(t)$ and $\varphi^2(t)$ are scalar absolutely continuous functions with $|\dot\varphi^1(t)| + |\dot\varphi^2(t)| \leq cons$, $t \in I$; x_0^1 and x_0^2 are fixed numbers. It is assumed that the functions involving in the equation (1.4) satisfy the standard conditions on the corresponding sets. In this case by E_0 we denote the set of admissible elements

$$e = \big(\sigma_1, \dots, \sigma_m, \tau_1, \dots, \tau_p, v(\,\cdot\,), u(\,\cdot\,)\big) \in E = [\sigma_{11}, \sigma_{12}] \times \cdots \times [\sigma_{m1}, \sigma_{m2}] \times [\tau_{11}, \tau_{12}]$$

$$\times [\tau_{p1}, \tau_{p2}] \times \Delta_1([t_0, t_1], [v_1, v_2], k, L) \times \Omega_1([t_0, t_1], [u_1, u_2]),$$

where $v_2 > v_1 > 0$, $u_2 > u_1 > 0$ are given numbers and by

$$e_0 = \big(\sigma_{10}, \dots, \sigma_{m0}, \tau_{10}, \dots, \tau_{p0}, v_0(\,\cdot\,), u_0(\,\cdot\,)\big)$$

we denote an optimal element (see Definitions 1.2 and 1.3. Evidently, the problem (1.4), (2,4), (2.5) is a particular case of the problem (2.1)-(2.4).

Theorem 2.3. *There exists an optimal element* $e_0 \in E_0$ *if the following conditions hold:*

(7) $E_0 \neq \varnothing$;

(8) *there exists a compact set* $K_1 \subset \mathbb{R}^1$ *such that for an arbitrary* $e \in E_0$

$$x^2(t;e) \in K_1, \quad t \in I;$$

(9) *for each fixed* $(t, x^1, x^2, y^1, \ldots, y^{2p}) \in [t_0, t_1] \times K_1^{2p+2}$ *the set*

$$\Big\{\alpha(t, x^1, u) + \beta\big(t, y^1, \ldots, y^p, x^2, y^{p+1}, \ldots, y^{2p}, u\big) : \quad u \in [u_1, u_2]\Big\}$$

is convex.

It is clear that Theorem 1.3 is a simple corollary of Theorem 1.1.

REFERENCES

1. Driver R. D. A functional-differential system of neutral type arising in a two-body problem of classical electrodynamics. 1963 *Internat. Sympos. Nonlinear Differential Equations and Nonlinear Mechanics*, pp. 474–484, Academic Press, New York.

2. Bokhua E. Dvalishvili Ph., N. Gorgodze N., Tadumadze T. Existence of an optimal element for the quasi-linear neutral control problem. *Proceedings of the 8 th International Conference on Control and Optimization with Industrial Applications (August 24-26, 2022)*, **1** (2022), 132-134, Baku, Azerbaijan,

3. Hadeler K. P. Neutral delay equations from and for population dynamics. The 8th Colloquium on the Qualitative Theory of Differential Equations, **11**, 18 pp., *Proc. Colloq. Qual. Theory Differ. Equ., 8, Electron. J. Qual. Theory Differ. Equ., Szeged,* 2008.

4. Hale J. *Theory of Functional Differential Equations.* Second edition. Applied Mathematical Sciences, *Springer-Verlag, New York–Heidelberg*, **3**, 1977.

5. Mordukhovich B. Sh. Approximation Methods in Problems of Optimization and Control (Russian). *Nauka, Moscow*

6. Tadumadze T. A. Existence of a solution in optimal problems of neutral type (Russian). *Soobshch. Akad. Nauk Gruzin. SSR*, **97**, 1 (1980), 33-36.

7. Tadumadze T. Existence of optimal initial data and continuity of the integral functional minimum with respect to perturbations for a class of neutral differential equation. *Semin. I. Vekua Inst. Appl. Math. Rep.* **36-37** (2010/11), 37-40, 109.

8. Tadumadze T., Gorgodze N. Variation formulas of a solution and initial data optimization problems for quasi-linear neutral functional differential equations with discontinuous initial condition. *Mem. Differ. Equ. Math. Phys.* **63** (2014), 1–77.

9. Tadumadze T., Nachaoui A. On the existence of an optimal element in quasi-linear neutral optimal problems. *Semin. I. Vekua Inst. Appl. Math. Rep.*, **40** (2014), 50-67.

10. Shavadze T., Tadumadze T. Existence of an optimal element for a class of neutral optimal problems. *Mem.Differ. Eq. Math. Phys.*, **86** (2022), 127-138.

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