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ON THE ASYMPTOTIC ESTIMATIONS FOR GENERALIZED FOURIER COEFFICIENTS

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Abstract. An asymptotic estimations for generalized Fourier coefficients is obtained. In particular, under specific circumstances this result coincides with the theorem of A. I. Stepanets[1].

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1. Introduction

Let f be a 2π-periodic locally integrable function and let

$$S_n(x,f) = \frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos kx + b_k \sin kx$$

be the particular sums of Fourier series of f with respect to the trigonometric system (see [2]). If f is a continuous function on $[a,b]$ then

$$\omega(\delta, f) = \sup\{|f(x_1) - f(x_2)| : |x_1 - x_2| \leq \delta, \quad x_1, x_2 \in [a,b]\}$$

is called the modulus of continuity of f. For a modulus of continuity ω denote by $H_\omega[a,b]$ the class of functions f with property $|f(x) - f(x')| \leq \omega(|x - x'|), x, x' \in [a,b]$. N. P. Korneichuk [3] proved the statement, which in the sequel was named as Korneichuk-Stechkin lemma. In particular N. P. Korneichuk received the estimation of the following value

$$\mathcal{E}_\omega(\psi) \overset{def}{=} \sup_{f \in H_\omega[a,b]} \left| \int_a^b f(t)\psi(t)\, dt \right|, \tag{1.1}$$

where ψ is an integrable function with the average mean 0 on $[a,b]$. In addition, sign of ψ on (a,c) and (c,b), $a < c < b$, maintains almost everywhere (in this case we write $\psi \in V_{a,b}^c$). The estimation of (1.1) for a convex modulus of continuity ω is exact and explicitly is given. In this work we use Lemma Korneichuk-Stechkin in the following form.

Lemma 1.1 (Korneichuk-Stechkin) *Let ω be any modulus of continuity, $\psi(t) \in V_{a,b}^c$, $c = \frac{a+b}{2}$ and $\psi(t) = -\psi(2c - t)$, then*

$$\mathcal{E}_\omega(\psi) \leq \int_a^c |\psi(t)|\, \omega(2(c-t))\, dt = \int_c^b |\psi(t)|\, \omega(2(t-c))\, dt. \tag{1.2}$$

If modulus of continuity ω is convex, then the equality in (1.2) is achieved for the function from $H_\omega[a,b]$ like $K \pm f_*(x)$, where K is a constant and

$$f_*(x) = \begin{cases} -\frac{1}{2}\omega(2(c-x)), x \in [a,c], \\ \\ \frac{1}{2}\omega(2(c-x)), x \in [c,b]. \end{cases}$$

Based on the above Korneichuk-Stechkin lemma, A. I. Stepanets proved many statements. We present one of them.

Proposition 1.2 (Stepanets) *For any ω modulus of continuity we have an asymptotic inequality*

$$\sup_{f\in H_\omega}|a_n| = \sup_{f\in H_\omega}|b_n| \le \frac{2}{\pi}\left|\int_a^b \omega\left(\frac{2t}{n}\right)\sin(t)\,dt\right|. \tag{1.3}$$

In the case when ω is a convex continuous function (1.3) *becomes the equality.*

In 1974 Taberski [4] considered the following quantities:

$$a_k^l = \frac{1}{l}\int_{-l}^{l} f(t)\cos\frac{k\pi t}{l}\,dt, \quad b_k^l = \frac{1}{l}\int_{-l}^{l} f(t)\sin\frac{k\pi t}{l}\,dt,$$

$$S_n^l(x,f) = \frac{a_0}{2} + \sum_{k=1}^{n} a_k^l\cos kx + b_k^l\sin kx,$$

where f is a locally integrable function on $x\in(-\infty,\infty)$, $l>0$, and $n=1,2,3,\dots$. The last sum can be represented by Dirichlets integrals as follows

$$S_n^l(x,f) = \frac{1}{l}\int_{-l}^{l} f(u)D_n^l(u-x)\,du,$$

where

$$D_n^l(t) = \frac{1}{2} + \sum_{k=1}^{n}\cos\frac{k\pi t}{l} = \frac{\sin(2n+1)\pi t/2l}{2\sin\pi t/2l}.$$

If f is a locally integrable periodic function with period 2π then the last equality for $l=\pi$ coincides with the partial sums of the trigonometric Fourier series.

2. An asymptotic estimation for generalized Fourier coefficients

Definition 2.1 Let f be an uniform continuous function on $\mathbb{R}$. We say $f\in H_\omega$ if for any $t_1,t_2\in\mathbb{R}$

$$|f(t_1)-f(t_2)| \le \omega(|t_1-t_2|)$$

where ω is a modulus of continuity.

Theorem 2.2 *For any modulus of continuity $\omega=\omega(t)$ we have:*

$$\sup_{f\in H_\omega}|a_n| \le \frac{2}{\pi}\int_0^{\frac{\pi}{2}}\sin t\omega\left(\frac{2l}{\pi n}t\right)dt. \tag{2.1}$$

Besides, if ω is bounded then

$$\sup_{f\in H_\omega}|b_n| \le \frac{2}{\pi}\int_0^{\frac{\pi}{2}}\sin t\omega\left(\frac{2l}{\pi n}t\right)dt + O\left(\frac{1}{n}\right). \tag{2.2}$$

In the case where ω is a convex continuous function, inequalities (2.1) *and* (2.2) *become equalities.*

Proof. Let

$$\sup_{f\in H_\omega}\left|a_n^l\right| = \sup_{f\in H_\omega}\frac{1}{l}\left|\int_{-l}^{l} f(t)\cos\frac{n\pi t}{l}dt\right|.$$

The function f can be represented as a sum of an even and odd functions: $f(x) = f_1(x)+f_2(x)$ where $f_1(x) = \frac{f(x)+f(-x)}{2}$, $f_2(x) = \frac{f(x)-f(-x)}{2}$. If $f(x) \in H_\omega$ then $f_1(x) \in H_\omega$ and $f_2(x) \in H_\omega$. If we take into account that $a_k^l(f_2) = 0$, we get

$$\sup_{f\in H_\omega}|a_n| = \sup_{f\in H_{\omega,r}}\frac{2}{l}\left|\int_0^l f(t)\cos\frac{n\pi t}{l}dt\right| =: e_n^l,$$

where $H_{\omega,r}$ is a subset of H_ω which contains only even functions. We have

$$e_n^l \le \frac{2}{l}\sum_{i=0}^{n-1}\sup_{f\in H_{\omega,r}}\left|\int_{\frac{il}{n}}^{(i+1)\frac{l}{n}} f(t)\cos\frac{n\pi t}{l}dt\right| = \frac{2}{l}\sum_{i=0}^{n-1}e_{n,l}^{(i)}.$$

For the estimations of $e_{n,l}^{(i)}$ we use Lemma 1.1 in the case $\psi(t) = \cos\frac{n\pi t}{l}$, $a_i = \frac{il}{n}$, $b_i = \frac{(i+1)l}{n}$, $c_i = \frac{(2i+1)l}{2n}$. $\psi(t) = -\psi(2c_i - t)$. Indeed,

$$-\psi(2c_i - t) = -\cos\left[\frac{n\pi}{l}\left(2\frac{(2i+1)l}{2n} - t\right)\right] = \cos\frac{n\pi t}{l}.$$

Thus,

$$e_{n,l}^{(i)} \le \int_{\frac{il}{n}}^{(2i+1)\frac{l}{2n}}\left|\cos\frac{n\pi t}{l}\right|\omega\left(2\left(\frac{2i+1}{2n}l - t\right)\right)dt.$$

For the last integral we have

$$-\int_{\frac{l}{2n}}^{0}\left|\cos\frac{n\pi}{l}\left(\left(\frac{2i+1}{2n}l - x\right)\right)\right|\omega(2x)\,dx = \int_0^{\frac{l}{2n}}\sin\frac{n\pi t}{l}\omega(2t)\,dt.$$

Therefore,

$$e_n^l \le \frac{2}{l}\sum_{i=0}^{n-1}\frac{l}{n\pi}\int_0^{\frac{\pi}{2}}\sin t\omega\left(\frac{2lt}{n\pi}\right)dt = \frac{2}{\pi}\int_0^{\frac{\pi}{2}}\sin t\omega\left(\frac{2lt}{n\pi}\right)dt.$$

Thus,

$$\sup_{f\in H_\omega}\left|a_n^l\right| = e_n^l \le \frac{2}{\pi}\int_0^{\frac{\pi}{2}}\sin t\omega\left(\frac{2lt}{n\pi}\right)dt.$$

Now let us show that in the case of a convex modulus of continuity, the last inequality can be replaced by the equality. For each fixed i consider the following function on $\left[\frac{il}{n}, \frac{i+1}{n}l\right]$

$$\varphi_l^i(x) = \begin{cases} -\frac{1}{2}\omega\left(2\left(\frac{2i+1}{2n}l - x\right)\right), x \in \left(\frac{il}{n}, \frac{2i+1}{2n}l\right], \\ \frac{1}{2}\omega\left(2\left(x - \frac{2i+1}{2n}l\right)\right), x \in \left(\frac{2i+1}{2n}l, \frac{i+1}{n}l\right]. \end{cases}$$

It easy to see that

$$\left| \int_{\frac{i}{n}l}^{\frac{i+1}{n}l} \varphi_i^l(x) \cos\frac{n\pi t}{l} dt \right|$$

$$= \left| - \int_{\frac{i}{n}l}^{\frac{2i+1}{2n}l} \frac{1}{2}\omega\left(2\left(\frac{2i+1}{2n}l - x\right)\right) \cos\frac{n\pi x}{l} dx \right.$$

$$\left. + \int_{\frac{2i+1}{2n}l}^{\frac{i+1}{n}l} \frac{1}{2}\omega\left(2\left(x - \frac{2i+1}{2n}l\right)\right) \cos\frac{n\pi x}{l} dx \right|.$$

For the first term we have

$$- \int_{\frac{i}{n}l}^{\frac{2i+1}{2n}l} \frac{1}{2}\omega\left(2\left(\frac{2i+1}{2n}l - x\right)\right) \cos\frac{n\pi}{l} x dx$$

$$= \int_{\frac{l}{2n}}^{0} \frac{1}{2}\omega(2t) \cos\left[\frac{\pi}{2}(2i+1) - \frac{n\pi}{l}t\right] dt.$$

For the second term we get

$$\int_{0}^{\frac{l}{2n}} \frac{1}{2}\omega(2t) \cos\frac{n\pi}{l}\left(t + \frac{2i+1}{2n}l\right) dt$$

$$= \int_{0}^{\frac{l}{2n}} \frac{1}{2}\omega(2t) \cos\left[\frac{\pi}{2}(2i+1) + \frac{n\pi}{l}t\right] dt.$$

Hence,

$$\left| \int_{\frac{i}{n}l}^{\frac{i+1}{n}l} \varphi_i^l(x) \cos\frac{n\pi t}{l} dt \right|$$

$$= \left| \int_{0}^{\frac{l}{2n}} \frac{1}{2}\omega(2t) \left(\cos\left[\frac{\pi}{2}(2i+1) - \frac{n\pi}{l}t\right] - \cos\left[\frac{\pi}{2}(2i+1) + \frac{n\pi}{l}t\right]\right) dt \right|$$

$$=\left|\int_{0}^{\frac{l}{2n}}\omega\left(2t\right)\sin\frac{n\pi}{l}t\sin\frac{\pi}{2}\left(2i+1\right)dt\right|=\int_{0}^{\frac{l}{2n}}\omega\left(2t\right)\sin\frac{n\pi}{l}tdt$$

$$=\frac{l}{n\pi}\int_{0}^{\frac{\pi}{2}}\omega\left(\frac{2l}{n\pi}t\right)\sin tdt,$$

i.e.

$$\left|\int_{\frac{i}{n}l}^{\frac{i+1}{n}l}\varphi_i^l\left(x\right)\cos\frac{n\pi t}{l}dt\right|=\frac{l}{n\pi}\int_{0}^{\frac{\pi}{2}}\omega\left(\frac{2l}{n\pi}t\right)\sin tdt.$$

Let $f^*\left(x\right)=\left(-1\right)^{i+1}\varphi_i^l\left(x\right)$, where $x\in\left[\frac{il}{n},\frac{i+1}{n}l\right]$, $i=0,...,n-1$. Besides, $f^*\left(x\right)\cos\frac{n\pi}{l}x$ is positive, for each $i=0,...,n-1$. So

$$\left|\frac{2}{l}\int_{0}^{l}f^*\left(t\right)\cos\frac{n\pi t}{l}dt\right|=\frac{2}{l}\sum_{i=0}^{n-1}\left(-1\right)^{i+1}\int_{\frac{il}{n}}^{\frac{i+1}{n}l}\varphi_i^l\left(t\right)\cos\frac{n\pi t}{l}dt$$

$$=\frac{2}{l}\frac{l}{n\pi}\sum_{i=0}^{n-1}\int_{0}^{\frac{\pi}{2}}\omega\left(\frac{2l}{n\pi}t\right)\sin tdt=\frac{2}{\pi}\int_{0}^{\frac{\pi}{2}}\sin t\omega\left(\frac{2lt}{n\pi}\right)dt$$

and $f^*\left(x\right)\in H_\omega$. Therefore, for a convex modulus of continuity ω

$$\sup_{f\in H_\omega}\left|a_n^l\left(f\right)\right|=\frac{2}{\pi}\int_{0}^{\frac{\pi}{2}}\sin t\omega\left(\frac{2lt}{n\pi}\right)dt.$$

Now consider the coefficients b_n^l. We have

$$\sup_{f\in H_\omega}\left|b_n^l\right|=\sup_{f\in H_\omega}\frac{1}{l}\left|\int_{-l}^{l}f\left(t\right)\sin\frac{n\pi t}{l}dt\right|.$$

Similarly as we derived for a_n^l, we can write:

$$\sup_{f\in H_\omega}\left|b_n^l\right|=\sup_{f\in H_{\omega,odd}}\frac{2}{l}\left|\int_{0}^{l}f\left(t\right)\sin\frac{n\pi t}{l}dt\right|\stackrel{def}{=}\sup_{f\in H_{\omega,odd}}\left|I\left(f\right)\right|,$$

where $H_{\omega,odd}$ is a subset of H_ω which contains only odd functions. Let

$$s\left(t\right)=\begin{cases}\sin\frac{n\pi t}{l},\ t\in\left[\frac{l}{2n},l-\frac{l}{2n}\right],\\ 0,\ t\in\left[0,\frac{l}{2n}\right]\cup\left[l-\frac{l}{2n},l\right].\end{cases}\tag{2.3}$$

Thus,

$$\frac{2}{l}\int_{0}^{l}f\left(t\right)\sin\frac{n\pi t}{l}dt=\frac{2}{l}\int_{0}^{l}f\left(t\right)s\left(t\right)dt+$$

$$+\frac{2}{l}\int\limits_{0}^{\frac{l}{2n}} f(t)\sin\frac{n\pi t}{l}dt+\frac{2}{l}\int\limits_{\frac{l(2n-1)}{2n}}^{l} f(t)\sin\frac{n\pi t}{l}dt.$$

Since f is an odd function and $\omega(t)$ is bounded, we can estimate the last two terms as follows:

$$\left|\frac{2}{l}\int\limits_{0}^{\frac{l}{2n}} f(t)\sin\frac{n\pi t}{l}dt\right|$$

$$\leq\frac{2}{l}\int\limits_{0}^{\frac{l}{2n}}\left|f(t)-f(0)\right|dt\leq\frac{1}{n}\omega\left(\frac{l}{2n}\right)=O\left(\frac{1}{n}\right)$$

and

$$\left|\frac{2}{l}\int\limits_{\frac{l(2n-1)}{2n}}^{l} f(t)\sin\frac{n\pi t}{l}dt\right|$$

$$\leq\frac{2}{l}\int\limits_{\frac{l(2n-1)}{2n}}^{l}\left|f(t)-f(0)\right|dt\leq\frac{1}{n}\omega(l)=O\left(\frac{1}{n}\right).$$

Thus,

$$I(f)=\frac{2}{l}\int\limits_{0}^{l} f(t)\,s(t)\,dt+O\left(\frac{1}{n}\right).$$

Therefore,

$$\sup_{f\in H_{\omega,odd}}\left|I(f)\right|=\sup_{f\in H_{\omega,odd}}\left|\frac{2}{l}\int\limits_{0}^{l} f(t)\,s(t)\,dt+O\left(\frac{1}{n}\right)\right|$$

$$\leq\frac{2}{l}\sum_{i=0}^{n-2}\sup_{f\in H_{\omega,odd}}\left|\int\limits_{(2i+1)\frac{l}{2n}}^{(2i+3)\frac{l}{2n}} f(t)\sin\frac{n\pi t}{l}dt\right|+O\left(\frac{1}{n}\right).$$

Let's define

$$u_{n,l}^{(i)}\overset{def}{=}\sup_{f\in H_{\omega,odd}}\left|\int\limits_{(2i+1)\frac{l}{2n}}^{(2i+3)\frac{l}{2n}} f(t)\sin\frac{n\pi t}{l}dt\right|,$$

$i=0,...,n-1$. For the estimations of $u_{n,l}^{(i)}$ we use Lemma 1.1 in the case $\psi(t)=\sin\frac{n\pi t}{l}$, $a_i=(2i+1)\frac{l}{2n}$, $b_i=(2i+3)\frac{l}{2n}$, $c_i=(i+1)\frac{l}{n}$, $i=0,..,n-2$, $\psi(t)=-\psi(2c_i-t)$. Indeed,

$$-\psi(2c_i-t)=-\sin\left[2\pi(i+1)-\frac{n\pi t}{l}\right]=\sin\frac{n\pi t}{l}.$$

Therefore,

$$u_{n,l}^{(i)} \leq \int\limits_{(2i+1)\frac{l}{2n}}^{(i+1)\frac{l}{n}} \left|\sin\frac{n\pi t}{l}\right| \omega\left(2\left((i+1)\frac{l}{n}-t\right)\right)dt.$$

So we have:

$$\int\limits_{(2i+1)\frac{l}{2n}}^{(i+1)\frac{l}{n}} \left|\sin\frac{n\pi t}{l}\right| \omega\left(2\left((i+1)\frac{l}{n}-t\right)\right)dt$$

$$=-\int\limits_{\frac{l}{2n}}^{0} \left|\sin\frac{n\pi}{l}t\right| \omega(2t)\,dt = \int\limits_{0}^{\frac{l}{2n}} \sin\frac{n\pi}{l}t\omega(2t)\,dt.$$

Thus,

$$\frac{2}{l}\sum_{i=0}^{n-2}\sup_{f\in H_{\omega,odd}}\left|\int\limits_{(2i+1)\frac{l}{2n}}^{(2i+3)\frac{l}{2n}} f(t)\sin\frac{n\pi t}{l}dt\right| + O\left(\frac{1}{n}\right)$$

$$\leq \left(1-\frac{1}{n}\right)\frac{2}{\pi}\int\limits_{0}^{\frac{\pi}{2}}\sin t\omega\left(\frac{2lt}{n\pi}\right)dt + O\left(\frac{1}{n}\right)$$

$$\leq \frac{2}{\pi}\int\limits_{0}^{\frac{\pi}{2}}\sin t\omega\left(\frac{2lt}{n\pi}\right)dt + O\left(\frac{1}{n}\right).$$

Therefore,

$$\sup_{f\in H_\omega}\left|b_n^l\right| \leq \frac{2}{\pi}\int\limits_{0}^{\frac{\pi}{2}}\sin t\omega\left(\frac{2lt}{n\pi}\right)dt + O\left(\frac{1}{n}\right).$$

Let

$$\psi_i^l(x) = \begin{cases} -\frac{1}{2}\omega(2x)\,,\ x\in\left[0,\frac{l}{2n}\right], \\ -\frac{1}{2}\omega\left(2\left(\frac{i+1}{n}l-x\right)\right),\ x\in\left[\frac{2i+1}{2n}l,\frac{i+1}{n}l\right], \\ \frac{1}{2}\omega\left(2\left(x-\frac{i+1}{n}l\right)\right),\ x\in\left[\frac{i+1}{n}l,\frac{2i+3}{2n}l\right], \\ \frac{1}{2}\omega\left(2\left(l-x\right)\right),\ x\in\left[l-\frac{l}{2n},l\right], \end{cases}$$

for $i=0,...,n-2$. It easy to see that

$$\left|\int\limits_{(2i+1)\frac{l}{2n}}^{(2i+3)\frac{l}{2n}} \psi_i^l(x)\sin\frac{n\pi t}{l}dt\right| = \frac{l}{n\pi}\int\limits_{0}^{\frac{\pi}{2}}\sin t\omega\left(\frac{2lt}{n\pi}\right)dt.$$

Let $f^*(x) = (-1)^{i+1}\psi_i^l(x)$, where $x \in \left[\frac{2i+1}{2n}l, \frac{2i+3}{2n}l\right]$, $i = 0, .., n-2$. $f^*(x)\sin\frac{n\pi x}{l}$ is positive, for $i = 0, .., n-2$ and $f^*(x) \in H_\omega$. Hence, by definition of function $s(t)$ (see (2.3))

$$|I(f^*)| = \left|\frac{2}{l}\int_0^l f^*(t)\, s(t)\, dt + O\left(\frac{1}{n}\right)\right|$$

$$= \left|\frac{2}{l}\int_{\frac{l}{2n}}^{l-\frac{l}{2n}} (-1)^{i+1}\psi_i^l(x)\sin\frac{n\pi t}{l}dt + O\left(\frac{1}{n}\right)\right|$$

$$= \frac{2}{l}\sum_{i=0}^{n-2}\int_{\frac{2i+1}{2n}l}^{\frac{2i+3}{2n}l} (-1)^{i+1}\psi_i^l(x)\sin\frac{n\pi t}{l}dt + O\left(\frac{1}{n}\right)$$

$$= \left(1 - \frac{1}{n}\right)\frac{2}{\pi}\int_0^{\frac{\pi}{2}} \sin t\,\omega\left(\frac{2lt}{n\pi}\right)dt + O\left(\frac{1}{n}\right)$$

$$= \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \sin t\,\omega\left(\frac{2lt}{n\pi}\right)dt + O\left(\frac{1}{n}\right).$$

R E F E R E N E S

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