Seminar of I. Vekua Institute of Applied Mathematics REPORTS, Vol. 49, 2023

ON EXISTENCE OF BOUNDED SOLUTIONS ON NONNEGATIVE REAL SEMIAXIS OF LINEAR SYSTEMS OF GENERALIZED ORDINARY DIFFERENTIAL EQUATIONS

Ashordia M.

Abstract. Effective sufficient conditions are established for the existence of bounded solutions on nonnegative real semiaxis linear systems of generalized ordinary differential equations.

Keywords and phrases: Linear systems of generalized ordinary differential equations, the Kurzweil integral, bonded solutions, nonnegative real semiaxis.

AMS subject classification (2010): 34K06, 34K10.

1. Statement of the problem. Basic notation and definitions

For the linear system of the generalized differential equations

$$dx = dA(t) \cdot x + df(t) \quad \text{for} \quad t \in \mathbb{R}_+, \tag{1.1}$$

consider the problem on the bounded on $\mathbb{R}_+$ solution

$$\sup\{\|x(t)\| : t \in \mathbb{R}_+\} < +\infty, \tag{1.2}$$

where $\mathbb{R}_+ = [0, +\infty[$, $A = (a_{ik})_{i,k=1}^n \in \mathrm{BV}_{loc}(\mathbb{R}_+, \mathbb{R}^{n\times n})$, and $f = (f_i)_{i=1}^n \in \mathrm{BV}_{loc}(\mathbb{R}_+, \mathbb{R}^n)$.

The generalized ordinary differential equations were introduced by J. Kurzweil [10]. To a considerable extent, the interest to the theory has also been stimulated by the fact that this theory enabled one to investigate ordinary differential, impulsive differential and difference equations from a unified point of view (see [1]–[7], [9, 11] and references therein).

Therefore, we can consider the ordinary differential, impulsive differential and difference equations as equations of the same type.

In this paper effective sufficient conditions are established for the existence of solutions of problem (1.1), (1.2). Analogous results are contained in [8] (see also references therein) for the problem for systems of ordinary differential equations.

In the paper the use will be made of the following notation and definitions

$\mathbb{R} =]-\infty, +\infty[$. $[a, b], [a, b[$ are, standard intervals.

$\mathbb{R}^{n\times m}$ is the space of all real $n \times m$ matrices X with the standard norm. $\mathbb{R}^n = \mathbb{R}^{n\times 1}$ is the space of all real column n-vectors x.

If $X \in \mathbb{R}^{n\times n}$, then X^{-1}, $\det(X)$ and $r(X)$ are, respectively, the matrix inverse to X, the determinant of X and the spectral radius of X; I_n is the identity $n \times n$-matrix.

The inequalities between the matrices are understood componentwise.

$\bigvee\limits_a^b(X)$ is the sum of total variations of the components of the matrix-function $X : [a, b] \to \mathbb{R}^{n\times m}$. If $X = (x_{ij})_{i,j=1}^{n,m} : \mathbb{R} \to \mathbb{R}^{n\times m}$, then $V(X)(t) = (\bigvee\limits_0^t(x_{ij}))_{i,j=1}^{n,m}$.

$X(t-)$ and $X(t+)$ are, respectively, the left and the right limits of $X :]\alpha, \beta[\to \mathbb{R}^{n\times m}$ at the point t. X is defined by continuity outside of the interval. $d_1X(t) = X(t) - X(t-)$, $d_2X(t) = X(t+) - X(t)$.

$\mathrm{BV}([a,b];\mathbb{R}^{n\times m})$ is the set of all matrix-functions $X:[a,b]\to\mathbb{R}^{n\times m}$ such that $\operatorname*{V}\limits_a^b(X)<\infty$. $\mathrm{BV}_{loc}(\mathbb{R};\mathbb{R}^{n\times m})$ is the set of all matrix-functions $X:\mathbb{R}\to\mathbb{R}^{n\times m}$ whose restrictions on every closed interval $[a,b]$ belong to $\mathrm{BV}([a,b],\mathbb{R}^{n\times n})$.

A matrix-function is said to be continuous, integrable, nondecreasing, etc., if each of its components is such.

s_1,s_2 and $s_c:\mathrm{BV}_{loc}(\mathbb{R};\mathbb{R})\to\mathrm{BV}_{loc}(\mathbb{R};\mathbb{R})$ are the operators, defined by

$$s_1(x)(0)=s_2(x)(0)=0,\quad s_c(x)(0)=x(0)$$
$$s_1(x)(t)=s_1(x)(s)+\sum_{s<\tau\le t}d_1x(\tau),\quad s_2(x)(t)=s_2(x)(s)+\sum_{s\le\tau<t}d_2x(\tau),$$
$$s_c(x)(t)=s_c(x)(s)+x(t)-x(s)-\sum_{j=1}^{2}(s_j(x)(t)-s_j(x)(s))\ \text{ for }\ s<t.$$

If $g\in\mathrm{BV}([a,b];\mathbb{R})$, $f:[a,b]\to\mathbb{R}$ and $a\le s<t\le b$, then we assume

$$\int\limits_s^t x(\tau)\,dg(\tau)=(L-S)\int\limits_{]s,t[}x(\tau)\,dg(\tau)+f(t)d_1g(t)+f(s)d_2g(s).$$

where $(L-S)\int\limits_{]s,t[}f(\tau)\,dg(\tau)$ is the Lebesgue–Stieltjes integral over the open interval $]s,t[$. It is known (see, [11]) that if the integral exists, then the right side of the integral equality equals to the Kurzeil–Stieltjes integral $(K-S)\int\limits_s^t f(\tau)\,dg(\tau)$ and, therefore, $\int\limits_s^t f(\tau)\,dg(\tau)=(K-S)\int\limits_s^t f(\tau)\,dg(\tau)$. If $a=b$, then we assume $\int\limits_a^b x(t)\,dg(t)=0$.

$$\int\limits_{-\infty}^{a}f(\tau)dg(\tau)=\lim_{t\to-\infty}\int\limits_t^a f(\tau)dg(\tau)\ \text{and}\ \int\limits_a^{+\infty}f(\tau)dg(\tau)=\lim_{t\to+\infty}\int\limits_a^t f(\tau)dg(\tau)$$

if the last limits exist (finite or infinite).

If $G=(g_{ik})_{i,k=1}^n\in\mathrm{BV}([a,b];\mathbb{R}^{n\times n})$ and $x=(x_k)_k^n:[a,b]\to\mathbb{R}^n$, then

$$\int_a^b dG(\tau)\cdot x(\tau)=\left(\sum_{k=1}^n\int_a^b x_k(\tau)\,dg_{ik}(\tau)\right)_i^n.$$

We introduce the operator $\mathcal{A}(X,Y)$ in the following way:

if $X\in\mathrm{BV}_{loc}(\mathbb{R};\mathbb{R}^{n\times n})$, $\det(I_n+(-1)^jd_jX(t))\neq 0$ for $t\in\mathbb{R}$ $(j=1,2)$, and $Y\in\mathrm{BV}_{loc}(\mathbb{R};\mathbb{R}^{n\times m})$, then

$$\begin{aligned}\mathcal{A}(X,Y)(0)&=O_{n\times m},\\ \mathcal{A}(X,Y)(t)&=\mathcal{A}(X,Y)(s)+Y(t)-Y(s)\\ &+\sum_{s<\tau\le t}d_1X(\tau)(I_n-d_1X(\tau))^{-1}d_1Y(\tau)\\ &-\sum_{s\le\tau<t}d_2X(\tau)(I_n+d_2X(\tau))^{-1}\,d_2Y(\tau)\ \ (s<t).\end{aligned}$$

By a solution of system (1.1) we mean a vector-function $x \in \mathrm{BV}_{loc}(\mathbb{R}, \mathbb{R}^n)$ if

$$x(t) = x(s) + \int_s^t dA(\tau) \cdot x(\tau) + f(t) - f(s)) \text{ for } s < t, \ s, t \in \mathbb{R}.$$

If $\alpha \in \mathrm{BV}_{loc}(\mathbb{R}, \mathbb{R})$ and $t_0 \in \mathbb{R}$ are such that $1 + (-1)^j d_j \alpha(t) \neq 0$ for $t \in \mathbb{R}$, $t \neq t_0$ $(j = 1, 2)$. Then it is known that (see [7, 9] the initial problem

$$d\xi = \xi d\alpha(t), \ \ \xi(0) = 1$$

has the unique solution ξ_α and it is defined by

$$\xi_\alpha(t) = \begin{cases} \exp(s_c(\alpha)(t) - s_c(\alpha)(0)) \prod\limits_{0<\tau\leq t} (1 - d_1\alpha(\tau))^{-1} \prod\limits_{0\leq\tau<t} (1 + d_2\alpha(\tau)) \\ \qquad \text{for } t > 0, \\ \exp(s_c(\alpha)(t) - s_c(\alpha)(0)) \prod\limits_{t<\tau\leq 0} (1 - d_1\alpha(\tau)) \prod\limits_{t\leq\tau<0} (1 + d_2\alpha(\tau))^{-1} \\ \qquad \text{for } t < 0. \end{cases}$$

Let $\gamma_\alpha(t, s) \equiv \xi_\alpha(t)\xi_\alpha^{-1}(s)$ be the Cauchy function of the problem.

Note that the following equality holds (see, [1, 3]

$$d\xi_\alpha^{-1}(t, t_0) \equiv -\xi_\alpha^{-1}(t, t_0) d\mathcal{A}(\alpha, \alpha).(\sqcup) \tag{1.3}$$

We introduce the operator

$$\nu(\zeta)(t) = \sup\Big\{\tau \geq t: \ \zeta(\tau) \leq \zeta(t+) + 1\Big\}$$

if $\xi : \mathbb{R} \to \mathbb{R}$ is a nondecreasing function, and

$$\nu(\zeta)(t) = \inf\Big\{\tau \leq t: \ \zeta(\tau) \leq \zeta(t-) + 1\Big\},$$

if $\zeta : \mathbb{R} \to \mathbb{R}$ is a non-increasing function.

2. Formulation of the results

For every $t_i \in \mathbb{R}_+ \cup \{+\infty\}$ $(i = 1, \dots, n)$ we put $\mathcal{N}_0(t_1, \dots, t_n) = \{i : t_i \in \mathbb{R}_+\}$. It is evident that $\mathcal{N}_0(t_1, \dots, t_n) = \{1, \dots, n\}$ if $t_i \in \mathbb{R}_+$ $(i = 1, \dots, n)$, and $\mathcal{N}_0(t_1, \dots, t_n) = \varnothing$ if $t_i \in \{+\infty\}$ $(i = 1, \dots, n)$.

In the case, where $t_i = +\infty$, we assume $\operatorname{sgn}(t - t_i) = -1$ for $t \in \mathbb{R}_+$.

Theorem 1. *Let*

$$1 + (-1)^j d_j a_{ii}(t) \neq 0 \ \textit{for} \ t \in \mathbb{R}_+ \ \ (j = 1, 2; \ i = 1, \dots, n) \tag{2.1}$$

and let there exist $t_i \in \mathbb{R}_+ \cup \{+\infty\}$ $(i = 1, \dots, n)$ *such that*

$$s_{ik} = \sup\left\{\left|\int_{t_i}^t |\gamma_i(t, \tau)| dV\big(\mathcal{A}(a_{ii}, a_{ik})\big)(\tau)\right| : t \in \mathbb{R}_+\right\} < +\infty$$

$$(i \neq k; \ i, k = 1, \dots, n), \tag{2.2}$$

$$\sup\left\{\left|\int_{t_i}^t |\gamma_i(t, \tau)| dV\big(\mathcal{A}(a_{ii}, f_i)\big)(\tau)\right| : t \in \mathbb{R}_+\right\} < +\infty \ \ (i = 1, \dots, n) \tag{2.3}$$

and

$$\sup\{|\gamma_i(t,t_i)| : t \in \mathbb{R}_+\} < +\infty \ \ for \ \ i \in \mathcal{N}_0(t_1,\dots,t_n), \tag{2.4}$$

where $\gamma_i(t,\tau) \equiv \gamma_{a_{ii}}(t,\tau)$ $(i = 1,\dots,n)$. *Let, moreover, the matrix* $S = (s_{ik})_{i,k=1}^n$, *where* $s_{ii} = 0$ $(i = 1,\dots,n)$, *be such that*

$$r(S) < 1. \tag{2.5}$$

Then for every $c_i \in \mathbb{R}_+$ $(i \in \mathcal{N}_0(t_1,\dots,t_n))$ *system* (1.1) *has at last one a bounded on* $\mathbb{R}_+$ *solution satisfying the condition*

$$x_i(t_i) = c_i \ \ for \ \ i \in \mathcal{N}_0(t_1,\dots,t_n)). \tag{2.6}$$

If the case, where $\mathcal{N}_0(t_1,\dots,t_n) = \varnothing$, conditions (2.4) and (2.6) are eliminated and the theorem has the following form.

Theorem 1′. *Let conditions* (2.1), (2.2) *and* (2.3) *hold for* $t_i = +\infty$ $(i = 1,\dots,n)$, *where* $\gamma_i(t,\tau) \equiv \gamma_{a_{ii}}(t,\tau)$ $(i = 1,\dots,n)$, *and the matrix* $S = (s_{ik})_{i,k=1}^n$, *where* $s_{ii} = 0$ $(i = 1,\dots,n)$, *satisfy condition* (2.5). *Then system* (1.1) *has at least one solution bounded on* $\mathbb{R}_+$.

Corollary 1. Let

$$1 + (-1)^j d_j a_{ii}(t) > 0 \ \ \text{for} \ \ t \in \mathbb{R}_+ \ \ (j = 1,2;\ i = 1,\dots,n) \tag{2.7}$$

and let there exist $t_i \in \mathbb{R}_+ \cup \{+\infty\}$ $(i = 1,\dots,n)$ such that conditions (2.2), (2.3), (2.4) and (2.5) hold, where $S = (s_{ik})_{i,k=1}^n$, $s_{ii} = 0$ $(i = 1,\dots,n)$ and $\gamma_i(t,\tau) \equiv \gamma_{a_{ii}}(t,\tau)$ $(i = 1,\dots,n)$. Let, moreover, the functions

$$\mathcal{A}(a_{ii},a_{ik})(t)\,\operatorname{sgn}(t-t_i), \ \ \mathcal{A}(a_{ii},f_i)(t)\,\operatorname{sgn}(t-t_i) \ \ (i \neq k;\ i,k = 1,\dots,n) \\ \text{are nondecreasing on } \mathbb{R}_+. \tag{2.8}$$

Then for every $c_i \in \mathbb{R}_+$ $(i \in \mathcal{N}_0(t_1,\dots,t_n))$ system (1.1) has at last one nonnegative and bounded on $\mathbb{R}$ solution satisfying condition (2.6).

If $\mathcal{N}_0(t_1,\dots,t_n) = \varnothing$ then Corollary 1 has the following form.

Corollary 1′. Let conditions (2.7) and (2.8) hold and let there exist $t_i = +\infty$ $(i = 1,\dots,n)$ such that conditions (2.2), (2.3) and (2.5) hold, where $S = (s_{ik})_{i,k=1}^n$, $s_{ii} = 0$ $(i = 1,\dots,n)$ and $\gamma_i(t,\tau) \equiv \gamma_{a_{ii}}(t,\tau)$ $(i = 1,\dots,n)$. Then system (1.1) has at least one nonnegative and bounded on $\mathbb{R}_+$ solution.

Theorem 2. *Let* (2.1) *hold and let there exist* $t_i \in \mathbb{R}_+ \cup \{+\infty\}$ $(i = 1,\dots,n)$ *such that conditions* (2.2), (2.3), (2.4) *and* (2.5) *hold, where* $S = (s_{ik})_{i,k=1}^n$, $s_{ii} = 0$ $(i = 1,\dots,n)$ *and* $\gamma_i(t,\tau) \equiv \gamma_{a_{ii}}(t,\tau)$ $(i = 1,\dots,n)$. *Let, moreover,*

$$\liminf_{t\to t_i} \gamma_i(0,t) = 0 \ \ for \ \ i \in \{1,\dots,n\} \setminus \mathcal{N}_0(t_1,\dots,t_n). \tag{2.9}$$

Then for every $c_i \in \mathbb{R}_+$ $(i \in \mathcal{N}_0(t_1,\dots,t_n))$ *system* (1.1) *has the unique and bounded on* $\mathbb{R}_+$ *solution* $(x_i)_{i=1}^n$ *satisfying condition* (2.6) *and*

$$\sum_{i=1}^n |x_i(t) - x_{im}(t)| \le \rho_0 \alpha^m \ \ for \ \ t \in \mathbb{R}_+ \ \ (m = 1,2,\dots)), \tag{2.10}$$

where ρ_0 *and* α *are the positive numbers independent of* m, $(x_{im})_{i=1}^n$ $(m=0,1,\dots)$ *is the sequence of the vector-functions the components of which are defined by*

$$x_{i0}(t)\equiv 0,\ x_{im}(t)\equiv u_i(t)+\sum_{k=1,\,k\neq i}^{n}\int_{t_i}^{t}\gamma_i(t,\tau)d\mathcal{A}(a_{ii},a_{ik}x_{k\,m-1})(\tau) \tag{2.11}$$

$(i=1,\dots,n;\ m=1,2,\dots)$, *and the functions* u_i $(i=1,\dots,n)$ *are defined due to*

$$u_i(t)\equiv c_i\gamma_i(t,t_i)+\int_{t_i}^{t}\gamma_i(t,\tau)d\mathcal{A}(a_{ii},f_i)(\tau)\ \ \textit{for}\ \ i\in\mathcal{N}_0(t_1,\dots,t_n), \tag{2.12}$$

$$u_i(t)\equiv\int_{t_i}^{t}\gamma_i(t,\tau)d\mathcal{A}(a_{ii},f_i)(\tau)\ \ \textit{for}\ \ i\in\{1,\dots,n\}\setminus\mathcal{N}_0(t_1,\dots,t_n). \tag{2.13}$$

Corollary 2. Let (2.7) hold and let there exist $t_i\in\mathbb{R}_+\cup\{+\infty\}$ $(i=1,\dots,n)$ such that the functions $a_{ii}(t)\,\mathrm{sgn}(t-t_i)$ $(i=1,\dots,n)$ are non-increasing on $\mathbb{R}_+$,

$$\liminf_{t\to t_i} a_{ii}(t)=+\infty\ \ \text{for}\ \ i\in\{1,\dots,n\}\setminus\mathcal{N}_0(t_1,\dots,t_n), \tag{2.14}$$

$$V(\mathcal{A}(a_{ii},a_{ik}))(t)\le -h_{ik}\ \mathrm{sgn}(t-t_i)\mathcal{A}(a_{ii},a_{ii})(t)\ \ \text{for}\ \ t\in\mathbb{R}_+ \\ (i\neq k;\ i,k=1,\dots,n) \tag{2.15}$$

and

$$r(\mathcal{H})<1, \tag{2.16}$$

where h_{ik} $(i,k=1,\dots,n)$ are such that $\mathcal{H}=\big((1-\delta_{ik})h_{ik}\big)_{i,k=1}^n$. Let, moreover,

$$\rho_i=\sup\left\{\Big|\bigvee_{t}^{\nu(\zeta_i)(t)}\big(\mathcal{A}(a_{ii},f_i)\big)\Big|:\ t\in\mathbb{R}_+\right\}<\infty\ \ (i=1,\dots,n), \tag{2.17}$$

where $\zeta_i(t)\equiv\xi_{a_{ii}}\,\mathrm{sgn}(t-t_i)$ $(i=1,\dots,n)$. Then conclusion of Theorem 2 is true.

Corollary 3. Let there exist the points $t_i\in\mathbb{R}_+\cup\{+\infty\}$ $(i=1,\dots,n)$, the functions $\alpha_i:\mathbb{R}_+\to\mathbb{R}$ $(i=1,\dots,n)$ and the numbers $\eta_{ik}\in\mathbb{R}$ $(i,k=1,\dots,n)$ such that the functions $\alpha_i(t)\,\mathrm{sgn}(t-t_i)$ $(i=1,\dots,n)$ are nondecreasing on $\mathbb{R}_+$ and conditions

$$\big(s_c(a_{ii})(t)-s_c(a_{ii})(s)\big)\ \mathrm{sgn}(t-s)\le\eta_{ii}\big(s_c(\alpha_i)(t)-s_c(\alpha_i)(s)\big) \\ \text{for}\ \ (t-s)(s-t_i)>0\ \ (i=1,\dots,n), \tag{2.18}$$

$$d_1a_{ii}(t)\le\eta_{ii}d_1\alpha_i(t)<1,\ \ -1<d_2a_{ii}(t)\le\eta_{ii}d_2\alpha_i(t)\ \ (i=1,\dots,n), \tag{2.19}$$

$$\big|s_c(a_{ik})(t)-s_c(a_{ik})(s)\big|\ \mathrm{sgn}(t-s)\le\eta_{ik}\big(s_c(\alpha_i)(t)-s_c(\alpha_i)(s)\big) \\ \text{for}\ \ (t-s)(s-t_i)>0\ \ (i\neq k;\ i,k=1,\dots,n) \tag{2.20}$$

and

$$|d_ja_{ik}(t)|\le\eta_{ik}|d_j\alpha_i(t)|\ \ (j=1,2;\ i\neq k;\ i,k=1,\dots,n) \tag{2.21}$$

hold on $\mathbb{R}_+$, where $\mathcal{H} = \big((1-\delta_{ik})\eta_{ik}|\eta_{ii}|^{-1}\big)_{i,k=1}^n$. Let, moreover,

$$\rho_i = \sup\left\{\Big|\bigvee_{t}^{\nu(\vartheta_i)(t)}\big(\mathcal{A}(\eta_i\alpha_i, f_i)\big)\Big| : \ t\in\mathbb{R}_+\right\} < \infty \ \ (i=1,\dots,n), \tag{2.22}$$

where $\vartheta_i(t) \equiv \xi_{\eta_i a_i}\operatorname{sgn}(t-t_i)$ $(i=1,\dots,n)$. Then the conclusion of Theorem 2 is true.

Theorem 2′. *Let* (2.1) *hold and let there exist* $t_i = +\infty$ $(i=1,\dots,n)$ *such that conditions* (2.2), (2.3) *and* (2.5) *hold, where* $S = (s_{ik})_{i,k=1}^n$, $s_{ii} = 0$ $(i=1,\dots,n)$ *and* $\gamma_i(t,\tau) \equiv \gamma_{a_{ii}}(t,\tau)$ $(i=1,\dots,n)$.*'Let, moreover,*

$$\liminf_{t\to t_i}\gamma_i(0,t) = 0 \ \ for \ \ i\in\{1,\dots,n\}. \tag{2.23}$$

Then system (1.1) *has the unique and bounded on* $\mathbb{R}_+$ *solution* $(x_i)_{i=1}^n$ *and*

$$\sum_{i=1}^n |x_i(t) - x_{im}(t)| \le \rho_0\alpha^m \ \ for \ \ t\in\mathbb{R}_+ \ \ (m=1,2,\dots),$$

where ρ_0 *and* α *are the positive numbers independent of* m, $(x_{im})_{i=1}^n$ $(m=0,1,\dots)$ *is the sequence of the vector-functions the components of which are defined by*

$$x_{i0}(t)\equiv 0, \ x_{im}(t) \equiv \int_{t_i}^{t}\gamma_i(t,\tau)d\mathcal{A}(a_{ii}, f_i)(\tau)$$

$$+\sum_{k=1,\,k\ne i}^{n}\int_{t_i}^{t}\gamma_i(t,\tau)d\mathcal{A}(a_{ii}, a_{ik}x_{k\,m-1})(\tau) \ \ (i=1,\dots,n; \ m=1,2,\dots).$$

Corollary 2′. Let (2.7) hold and let there exist $t_i = +\infty$ $(i=1,\dots,n)$ such that the functions $a_{ii}(t)\operatorname{sgn}(t-t_i)$ $(i=1,\dots,n)$ are non-increasing on $\mathbb{R}_+$, conditions (2.15), (2.16), (2.17) and

$$\liminf_{t\to t_i} a_{ii}(t) = +\infty \ \ \text{for} \ \ i\in\{1,\dots,n\} \tag{2.24}$$

hold, where $\zeta_i(t) \equiv \xi_{a_{ii}}\operatorname{sgn}(t-t_i)$ $(i=1,\dots,n)$, and the numbers h_{ik} $(i,k=1,\dots,n)$ are such that $\mathcal{H} = \big((1-\delta_{ik})h_{ik}\big)_{i,k=1}^n$. Then the conclusion of Theorem 2′ is true.

Corollary 3′. Let there exist $t_i = +\infty$ $(i=1,\dots,n)$ and the functions $\alpha_i : \mathbb{R}_+ \to \mathbb{R}$ $(i=1,\dots,n)$ such that the functions $\alpha_i(t)\operatorname{sgn}(t-t_i)$ $(i=1,\dots,n)$ are nondecreasing on $\mathbb{R}_+$ and conditions (2.16), (2.18) – (2.22) hold on $\mathbb{R}_+$, where $\vartheta_i(t) \equiv \xi_{\eta_i a_i}\operatorname{sgn}(t-t_i)$ $(i=1,\dots,n)$, and the numbers η_{ik}, $\eta_{ii} < 0$ $(i,k=1,\dots,n)$ are such that $\mathcal{H} = \big((1-\delta_{ik})\eta_{ik}|\eta_{ii}|^{-1}\big)_{i,k=1}^n$. Then the conclusion of Theorem 2′ is true.

Corollary 4. Let the conditions of Theorem 2 or Corollary 2 (Corollary 3) are fulfilled. Let, in addition, condition (2.8) hold. Then for every $c_i\in\mathbb{R}_+$ $(i\in\mathcal{N}_0(t_1,\dots,t_n))$ system (1.1) has the unique and bounded on $\mathbb{R}_+$ solution satisfying condition (2.6) and it is nonnegative.

Corollary 4′. Let the conditions of Theorem 2′ or Corollary 2′ (Corollary 3′) are fulfilled. Let, in addition, condition (2.8) hold. Then system (1.1) has the unique and bounded on $\mathbb{R}_+$ solution and it is nonnegative.

3. Proof of the results

We use the following Lemma for the proofs of the results.

Lemma 1. *Let $x \in \mathrm{BV}_{loc}(\mathbb{R}_+, \mathbb{R}^{n\times m})$ be a solution of system* (1.1)*. Then x will be the solution of the system*

$$dx = dA(t)\cdot x + df(t) \quad \textit{for} \quad t\in\mathbb{R} \tag{3.1}$$

under the condition

$$\sup\{\|x(t)\| : t\in\mathbb{R}\} < +\infty, \tag{3.2}$$

as well, if

$$a_{ik}(t) = f_k(t) = 0 \ \ (i,k=1,\dots,n) \ \ \textit{for} \ \ t<0.$$

The lemma immediately follows from the definitions of solutions of systems (1.1) and (3.1).

So that, problem (1.1), (1.2) is a particular case to problem (3.1), (3.2).

Problem (3.1), (3.2) is investigated in [6]. We give the results obtained in [1].

Differing to section 1, we introduce the set $\mathcal{N}_0$ in such a way.

For every $t_i \in \mathbb{R}\cup\{-\infty,+\infty\}$ $(i=1,\dots,n)$ we put $\mathcal{N}_0(t_1,\dots,t_n) = \{i : t_i\in\mathbb{R}\}$. It is evident that $\mathcal{N}_0(t_1,\dots,t_n) = \{1,\dots,n\}$ if $t_i\in\mathbb{R}$ $(i=1,\dots,n)$, and $\mathcal{N}_0(t_1,\dots,t_n)=\varnothing$ if $t_i\in\{-\infty,+\infty\}$ $(i=1,\dots,n)$.

In the case, where $t_i=-\infty$ $(t_i=+\infty)$, we assume $\mathrm{sgn}(t-t_i)=1$ for $t\in\mathbb{R}$ $(\mathrm{sgn}(t-t_i)=-1$ for $t\in\mathbb{R})$.

Theorem 3. *Let*

$$1+(-1)^j d_j a_{ii}(t) \neq 0 \ \ \textit{for} \ \ t\in\mathbb{R} \ \ (j=1,2;\ i=1,\dots,n) \tag{3.3}$$

and let there exist $t_i\in\mathbb{R}\cup\{-\infty,+\infty\}$ $(i=1,\dots,n)$ such that

$$s_{ik} = \sup\left\{\left|\int_{t_i}^{t} |\gamma_i(t,\tau)|dV\big(\mathcal{A}(a_{ii},a_{ik})\big)(\tau)\right| : t\in\mathbb{R}\right\} < +\infty$$
$$(i\neq k;\ i,k=1,\dots,n), \tag{3.4}$$

$$\sup\left\{\left|\int_{t_i}^{t} |\gamma_i(t,\tau)|dV\big(\mathcal{A}(a_{ii},f_i)\big)(\tau)\right| : t\in\mathbb{R}\right\} < +\infty \ \ (i=1,\dots,n) \tag{3.5}$$

and

$$\sup\{|\gamma_i(t,t_i)| : t\in\mathbb{R}\} < +\infty \ \ \textit{for} \ \ i\in\mathcal{N}_0(t_1,\dots,t_n), \tag{3.6}$$

where $\gamma_i(t,\tau)\equiv\gamma_{a_{ii}}(t,\tau)$ $(i=1,\dots,n)$. Let, moreover, the matrix $S=(s_{ik})_{i,k=1}^n$, where $s_{ii}=0$ $(i=1,\dots,n)$, be such that condition (2.5) *holds. Then for every $c_i\in\mathbb{R}$ $(i\in\mathcal{N}_0(t_1,\dots,t_n))$ system* (3.1) *has at least one bounded on $\mathbb{R}$ solution, are satisfying the condition*

$$x_i(t_i) = c_i \ \ \textit{for} \ \ i\in\mathcal{N}_0(t_1,\dots,t_n). \tag{3.7}$$

If the case, where $\mathcal{N}_0(t_1,\dots,t_n)=\varnothing$, conditions (2.5) and (3.6) are eliminated and the theorem has the following form.

Theorem 3′. *Let conditions* (3.3), (3.4) *and* (3.5) *hold for some $t_i\in\{-\infty,+\infty\}$ $(i=1,\dots,n)$, where $\gamma_i(t,\tau)\equiv\gamma_{a_{ii}}(t,\tau)$ $(i=1,\dots,n)$, and the matrix $S=(s_{ik})_{i,k=1}^n$, where*

$s_{ii} = 0$ $(i = 1, \dots, n)$, *satisfy condition* (2.5). *Then system* (3.1) *has at least one solution bounded on* $\mathbb{R}$.

Corollary 5. Let

$$1 + (-1)^j d_j a_{ii}(t) > 0 \ \text{ for } \ t \in \mathbb{R}, \ \ (j = 1, 2;\ i = 1, \dots, n) \tag{3.8}$$

and let there exist $t_i \in \mathbb{R} \cup \{-\infty, +\infty\}$ $(i = 1, \dots, n)$ such that conditions (2.5), (3.4), (3.5) and (3.6) hold, where $S = (s_{ik})_{i,k=1}^n$, $s_{ii} = 0$ $(i = 1, \dots, n)$ and $\gamma_i(t, \tau) \equiv \gamma_{a_{ii}}(t, \tau)$ $(i = 1, \dots, n)$. Let, moreover, the functions

$$\begin{gathered}\mathcal{A}(a_{ii}, a_{ik})(t)\ \operatorname{sgn}(t - t_i), \ \ \mathcal{A}(a_{ii}, f_i)(t)\ \operatorname{sgn}(t - t_i) \ \ (i \neq k;\ i, k = 1, \dots, n) \\ \textit{are nondecreasingon } \mathbb{R}.\end{gathered} \tag{3.9}$$

Then for every $c_i \in \mathbb{R}_+$ $(i \in \mathcal{N}_0(t_1, \dots, t_n))$ system (3.3) has at least one nonnegative and bounded on $\mathbb{R}$ solution, satisfying condition (3.7).

If $\mathcal{N}_0(t_1, \dots, t_n) = \varnothing$ then Corollary 5 has the following form.

Corollary 5′. Let conditions (3.8) and (3.9) hold and let there exist $t_i \in \{-\infty, +\infty\}$ $(i = 1, \dots, n)$ such that conditions (2.5), (3.4) and (3.5) hold, where $S = (s_{ik})_{i,k=1}^n$, $s_{ii} = 0$ $(i = 1, \dots, n)$ and $\gamma_i(t, \tau) \equiv \gamma_{a_{ii}}(t, \tau)$ $(i = 1, \dots, n)$. Then system (3.1) has at least one nonnegative and bounded on $\mathbb{R}$ solution.

Theorem 4. *Let* (3.3) *hold and let there exist* $t_i \in \mathbb{R} \cup \{-\infty, +\infty\}$ $(i = 1, \dots, n)$ *such that conditions* (2.5), (3.4), (3.5) *and*(3.6) *hold, where* $S = (s_{ik})_{i,k=1}^n$, $s_{ii} = 0$ $(i = 1, \dots, n)$ *and* $\gamma_i(t, \tau) \equiv \gamma_{a_{ii}}(t, \tau)$ $(i = 1, \dots, n)$.*'Let, moreover, condition* (2.9) *hold. Then for every* $c_i \in \mathbb{R}$ $(i \in \mathcal{N}_0(t_1, \dots, t_n))$ *system* (3.1) *has the unique and bounded on* $\mathbb{R}$ *solution* $(x_i)_{i=1}^n$ *satisfying condition* (3.7) *and*

$$\sum_{i=1}^n |x_i(t) - x_{im}(t)| \leq \rho_0 \alpha^m \ \ \textit{for} \ \ t \in \mathbb{R} \ \ (m = 1, 2, \dots)),$$

where ρ_0 *and* α *are the positive numbers independent of* m, $(x_{im})_{i=1}^n$ $(m = 0, 1, \dots)$ *is the sequence of the vector-functions the components of which are defined by* (2.11), (2.12) *and* (2.13).

Corollary 6. Let (3.8) hold and let there exist $t_i \in \mathbb{R} \cup \{-\infty, +\infty\}$ $(i = 1, \dots, n)$ such that the functions $a_{ii}(t) \operatorname{sgn}(t - t_i)$ $(i = 1, \dots, n)$ are non-increasing on $\mathbb{R}$, and conditions (2.9),

$$\begin{gathered}V(\mathcal{A}(a_{ii}, a_{ik}))(t) \leq -h_{ik}\ \operatorname{sgn}(t - t_i)\mathcal{A}(a_{ii}, a_{ii})(t) \ \text{ for } \ t \in \mathbb{R} \\ (i \neq k;\ i, k = 1, \dots, n)\end{gathered} \tag{3.10}$$

and (2.16) hold, where $\mathcal{H} = \big((1 - \delta_{ik})h_{ik}\big)_{i,k=1}^n$. Let, moreover,

$$\rho_i = \sup\left\{\Big|\bigvee_t^{\nu(\zeta_i)(t)} \big(\mathcal{A}(a_{ii}, f_i)\big)\Big| :\ t \in \mathbb{R}\right\} < \infty \ \ (i = 1, \dots, n), \tag{3.11}$$

where $\zeta_i(t) \equiv \xi_{a_{ii}} \operatorname{sgn}(t - t_i)$ $(i = 1, \dots, n)$. Then the conclusion of Theorem 4 is true.

Corollary 7. Let there exist the points $t_i \in \mathbb{R} \cup \{-\infty, +\infty\}$ $(i = 1, \dots, n)$, the functions $\alpha_i : \mathbb{R} \to \mathbb{R}$ $(i = 1, \dots, n)$ and the numbers $\eta_{ik} \in \mathbb{R}$ $(i, k = 1, \dots, n)$ such that the functions

$\alpha_i(t)\operatorname{sgn}(t-t_i)$ $(i=1,\dots,n)$ are nondecreasing on $\mathbb{R}$ and conditions (2.18), (2.19), (2.20) and (2.21) hold on $\mathbb{R}_+$, where $\mathcal{H}=\big((1-\delta_{ik})\eta_{ik}|\eta_{ii}|^{-1}\big)_{i,k=1}^n$. Let, moreover,

$$\rho_i=\sup\left\{\Big|\bigvee_t^{\nu(\vartheta_i)(t)}\big(\mathcal{A}(\eta_i\alpha_i,f_i)\big)\Big|:\ t\in\mathbb{R}\right\}<\infty\ \ (i=1,\dots,n), \tag{3.12}$$

where $\vartheta_i(t)\equiv\xi_{\eta_i a_i}\operatorname{sgn}(t-t_i)$ $(i=1,\dots,n)$. Then the conclusion of Theorem 4 is true.

Theorem 4′. *Let* (3.3) *hold and let there exist* $t_i\in\{-\infty,+\infty\}$ $(i=1,\dots,n)$ *such that conditions* (2.5), (3.4) *and* (3.5) *hold, where* $S=(s_{ik})_{i,k=1}^n$, $s_{ii}=0$ $(i=1,\dots,n)$ *and* $\gamma_i(t,\tau)\equiv\gamma_{a_{ii}}(t,\tau)$ $(i=1,\dots,n)$. *Let, moreover, condition* (2.23) *hold. Then system* (3.1) *has the unique and bounded on* $\mathbb{R}$ *solution* $(x_i)_{i=1}^n$ *and*

$$\sum_{i=1}^n|x_i(t)-x_{im}(t)|\le\rho_0\alpha^m\ \ \textit{for}\ \ t\in\mathbb{R}\ \ (m=1,2,\dots),$$

where ρ_0 *and* α *are the positive numbers independent of* m, $(x_{im})_{i=1}^n$ $(m=0,1,\dots)$ *is the sequence of the vector-functions the components of which are defined as in Theorem 2′.*

Corollary 7′. Let (3.8) hold and let there exist $t_i\in\{-\infty,+\infty\}$ $(i=1,\dots,n)$ such that the functions $a_{ii}(t)\operatorname{sgn}(t-t_i)$ $(i=1,\dots,n)$ are non-increasing on $\mathbb{R}$, conditions (2.16), (3.10), (3.11) and (2.24) hold, where $\zeta_i(t)\equiv\xi_{a_{ii}}\operatorname{sgn}(t-t_i)$ $(i=1,\dots,n)$, and the numbers h_{ik} $(i,k=1,\dots,n)$ are such that $\mathcal{H}=\big((1-\delta_{ik})h_{ik}\big)_{i,k=1}^n$. Then the conclusion of Theorem 3′ is true.

Corollary 8. Let there exist the points $t_i\in\{-\infty,+\infty\}$ $(i=1,\dots,n)$ and the functions $\alpha_i:\mathbb{R}\to\mathbb{R}$ $(i=1,\dots,n)$ such that the functions $\alpha_i(t)\operatorname{sgn}(t-t_i)$ $(i=1,\dots,n)$ are nondecreasing on $\mathbb{R}$ and conditions (2.16), (2.18) – (2.21) and (3.12) hold on $\mathbb{R}$, where $\vartheta_i(t)\equiv\xi_{\eta_i a_i}\operatorname{sgn}(t-t_i)$ $(i=1,\dots,n)$, and the numbers η_{ik}, $\eta_{ii}<0$ $(i,k=1,\dots,n)$ are such that $\mathcal{H}=\big((1-\delta_{ik})\eta_{ik}|\eta_{ii}|^{-1}\big)_{i,k=1}^n$. Then the conclusion of Theorem 3′ is true.

Corollary 8′. Let the conditions of Theorem 4 or Corollary 6 (Corollary 7) are fulfilled. Let, in addition, condition (3.9) hold. Then for every $c_i\in\mathbb{R}_+$ $(i\in\mathcal{N}_0(t_1,\dots,t_n))$ system (3.1) has the unique and bounded on $\mathbb{R}$ solution satisfying condition (3.7) and it is nonnegative.

Corollary 9. Let the conditions of Theorem 4′ or Corollary 6′ (Corollary 7′) are fulfilled. Let, in addition, condition (3.9) hold. Then system (3.1) has the unique and bounded on $\mathbb{R}$ solution and it is nonnegative.

REFERENCES

1. Ashordia M. On the general and multipoint boundary value problem for linear systems of generalized ordinary differential equations, linear impulsive and linear difference systems. *Mem. Differ. Equ. Math. Phys.*, **36** (2005), 1-80.

2. Ashordia M. T. On some boundary value problems for linear generalized differential systems with singularities (Russian). *D*iffer. Uravn., **46**, 2 (2010), 163-177; English transl.: *Differ. Equ.*, **46**, 2 (2010), 167-181.

3. Ashordia M. T. The Ienitial Problem for Linear Systems of Generalized Ordinary Differential Equations, Linear Impulsive and Ordinary Differential Systems. Numerical Solvability.*Mem. Differ. Equ. Math. Phys.*, **78** (2019), 1-162.

4. Ashordia M. T. The General boundary value Problems for Linear Systems of Generalized Ordinary Differential Equations, Linear Impulsive differential and Ordinary Differential Systems. Numerical Solvability. *Mem. Differ. Equ. Math. Phys.*, **81** (2020), 1-184.

5. Ashordia M., Gabisonia I. and Talakhadze M. On the solvability of the Cauchy problem for linear systems of generalized ordinary differential equations with singularities. *Georgian Math. J.*, **28**, 1 (2021), 29-47.

6. Ashordia Malkhaz. On Existence of Bounded Solutions on Real Axis $\mathbb{R}$ of Linear Systems of Generalized Ordinary Differential Equations. *Mishkolc Math. Notes.*, **24**, 1 (2023), 63-79.

7. Groh J. A nonlinear Volterra-Stieltjes integral equation and a Gronwall inequality in one dimension. *Illinois J. Math.* **24**, 2 (1980), 244-263.

8. Kiguradze I. The Initial Value Problem and Boundary Value Problems for Systems of Ordinary Differential Equations (Russian). Vol. I. Linear Theory, Metsniereba, Tbilisi, 1997.

9. Hildebrandt T. H. On systems of linear differentio-Stieltjes-integral equations. *Illinois J. Math.*, **3** (1959), 352-373.

10. Kurzweil J. Generalized ordinary differential equations and continuous dependence on a parameter. *Czechoslovak Math. J.*, **7(82)**, 3 (1957), 418-449.

11. Schwabik Š., Tvrdý M. and Vejvoda O. Differential and integral equations. Boundary value problems and adjoints. *D. Reidel Publishing Co., Dordrecht–Boston, Mass.–London,* 1979.

Received 10.09.2023; revised 08.10.2023; accepted 01.11.2023

Author's address:

M. Ashordia
A. Razmadze Mathematical Institute
6, Tamarashvili St., Tbilisi 0177

Muskhelishvili Institute of Computational Mathematics
4, Grigol Peradze St., Tbilisi 0159

Sokhumi State University
61, Politkovskaia St., Tbilisi 0177
Georgia
E-mail: ashord@rmi.ge, malkhaz.ashordia@tsu.ge